Transactions of NAS of Azerbaijan, Issue Mathematics, **38** (1), 52–61 (2018)
Series of Physical-Technical and Mathematical Sciences

Transfer of some differential geometric structures from $(1,1)$-tensor bundle to the $(0,2)$-tensor bundle over a Riemannian manifold

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Received: 28.09.2017 / Revised: 15.02.2018/ Accepted: 23.04.2018

Abstract. *In this paper, transfers (g-lifts) of certain differential geometric structures are studied using the natural diffeomorphism between the (1,1) -tensor bundle and the (0,2) -tensor bundle over a Riemannian manifold. Some problems concerning $g-$lifts of tensor fields (vector fields, affinor fields, vector 2-forms) and the Sasaki type metric are investigated.*

Keywords. Tensor bundle · Killing vector field · complete lift · Yano-Ako operator · Kahler-Norden manifold · Sasaki metric

Mathematics Subject Classification (2010): 55R10, 53B35, 53C15

1 Introduction

Let M be a Riemannian manifold of dimension n with a Riemannian metric g. We denote by $T_1^1 M = \bigcup_{x\in M} T_1^1(x)$ the $(1,1)$ - tensor bundle (or an affinor bundle) over M with local coordinates (x^i, t_i^j), where $t_x = t_i^j \frac{\partial}{\partial x^j} \otimes dx^i \in T_1^1(x), \forall x \in M$. Also we denote by $T_2^0 M = \bigcup_{x\in M} T_2^0(x)$ the $(0,2)$ tensor bundle over M with local coordinates $(x^i, t_{i_1 i_2})$, where $t_x = t_{i_1 i_2} dx^{i_1} \otimes dx^{i_2} \in T_2^0(x), \forall x \in M$. Throughout this paper we assume the manifolds, tensor fields and connections to be differentiable of class C^∞. We always use the ranges of the index i being $\{1,...,n\}$ and the index $\bar{i}$ being $\left\{n+1,...,n+n^2\right\}$.

Any Riemannian metric g is that it supplies a diffeomorphisms $g^1 : T_1^1 M \to T_2^0 M$ from the $(1,1)$- tensor bundle onto $(0,2)$- tensor bundle and $g^2 : T_2^0 M \to T_1^1 M$ from $(0,2)$- tensor bundle onto $(1,1)$- tensor bundle. Some properties of differential geometric structures on $(0,2)$- tensor bundle with respect to the difeomorphism g^1 are studied in [1].

The natural diffeomorphisms g^1 and g^2 are expressed by

$$g^1 : x^I = (x^i, x^{\bar{\bar{i}}}) = (x^i, t_i^j) \to \tilde{x}^K = (x^k, \tilde{x}^k) = (\delta_i^k x^i, t_{k_1 k_2} = g_{k_1 j} t_{k_2}^j)$$

and

$$g^2 : \tilde{x}^K = (x^k, \tilde{x}^{\bar{k}}) = (x^k, t_{k_1 k_2}) \to x^I = (x^i, x^i) = (\delta_k^i x^k, t_i^j = g^{jk} t_{ki})$$

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with respect to the local coordinates, respectively. The Jacobian matrices of g^1 and g^2 are given by

$$\left(g_*^1\right) = \left(\frac{\partial \tilde{x}^K}{\partial x^I}\right) = \begin{pmatrix} \delta_i^k & 0 \\ t_{k_2}^m \partial_i g_{k_1 m} & g_{k_1 j}\delta_{k_2}^i \end{pmatrix} \tag{1.1}$$

and

$$\left(g_*^2\right) = \left(\frac{\partial x^I}{\partial \tilde{x}^K}\right) = \begin{pmatrix} \delta_k^i & 0 \\ t_{mi} \partial_k g^{jm} & g^{jk_1}\delta_i^{k_2} \end{pmatrix} \tag{1.2}$$

respectively, where δ is the Kronecker delta.

We denote by $\Im_q^p(M)$ the set of all differentiable tensor fields of type (p,q) on M. Let ${}^C X_{T_1^1} \in \Im_0^1\left(T_1^1 M\right)$, ${}^C \varphi_{T_1^1} \in \Im_1^1\left(T_1^1 M\right)$ and ${}^C S_{T_1^1} \in \Im_2^1\left(T_1^1 M\right)$ be complete lifts of tensor fields $X \in \Im_0^1(M)$, $\varphi \in \Im_1^1(M)$ and $S \in \Im_2^1(M)$ to the $(1,1)$- tensor bundle $T_1^1 M$, respectively.

When we transferred the complete lifts of the tensor fields from the $(1,1)$- tensor bundle to $(0,2)$- tensor bundle we obtained the *g-lift* of tensor fields to $(0,2)$- tensor bundle. The aim of this paper is to study the *g-lift* problems of $(0,2)$- tensor bundles of Riemannian manifolds.

2 Transfer of vector fields

Let $X = X^a \partial_a$ be the local expression in $U \subset M$ of a vector field $X \in \Im_0^1(M)$. Then the complete lift ${}^C X_{T_1^1}$ of X to the $(1,1)$- tensor bundle $T_1^1 M$is given by [3]

$${}^C X_{T_1^1} = X^a \partial_a + \left(t_a^m \partial_m X^j - t_m^j \partial_a X^m\right) \partial_{\bar{a}} \tag{2.1}$$

with respect to the natural frame $\{\partial_a, \partial_{\bar{a}}\}$.

Using (1.1) and (2.1), we have

$$g_*^{1C} X_{T_1^1} = \begin{pmatrix} \delta_a^k & 0 \\ t_{k_2}^m \frac{\partial g_{k_1 m}}{\partial x^a} & g_{k_1 j}\delta_{k_2}^a \end{pmatrix} \begin{pmatrix} X^a \\ t_a^m \partial_m X^j - t_m^j \partial_a X^m \end{pmatrix}$$

$$= \begin{pmatrix} X^k \\ t_{k_2}^m \left(L_X g\right)_{k_1 m} - t_{mk_2} \partial_{k_1} X^m - t_{k_1 m} \partial_{k_2} X^m \end{pmatrix}, \tag{2.2}$$

where $t_i^m g_{km} = t_{ki}$ and L_X is the Lie derivation of g with respect to the vector field X :

$$\left(L_X g\right)_{km} = X^a \partial_a g_{km} + \left(\partial_k X^a\right) g_{am} + \left(\partial_m X^a\right) g_{ka}.$$

The vector field in the form (2.2) is called a $g-lift$ of X to $(0,2)$- tensor bundle $T_2^0 M$.

The complete lift ${}^C X_{T_2^0}$ of X to the $(0,2)$ tensor bundle $T_2^0 M$ is given by [2]:

$${}^C X_{T_2^0} = X^k \partial_k + \left(-t_{mk_2} \partial_{k_1} X^m - t_{k_1 m} \partial_{k_2} X^m\right) \partial_{\bar{k}}.$$

From (2.2) we find $g-lift$ $g_*^{1C} X_{T_1^1}$ of X:

$$g_*^{1C} X_{T_1^1} = {}^C X_{T_2^0} + \gamma\left(L_X g\right),$$

where the vertical vector field $\gamma(L_X g)$ in $T_2^0 M$ is defined by

$$\gamma(L_X g) = \begin{pmatrix} 0 \\ t_{k_2}^m (L_X g)_{k_1 m} \end{pmatrix}.$$

In a Riemannian manifold (M, g), a vector field X is called a Killing vector field if $L_X g = 0$. Thus we have

Theorem 2.1 *Let (M, g) be a Riemannian manifold. Let ${}^C X_{T_1^1}$ and ${}^C X_{T_2^0}$ be complete lifts of a vector field $X$to the $(1,1)$ and $(0,2)$- tensor bundles, respectively. In order the image of ${}^C X_{T_1^1}$ by g^1, i.e. a $g - lift$ $g_*^1 {}^C X_{T_1^1}$ coincides with ${}^C X_{T_2^0}$, it is necessary and sufficient that a vector X is a Killing vector field.*

Now using (1.1) and (2.1) we have

$$\left(g_*^1\left[{}^C X_{T_1^1}, {}^C Y_{T_1^1}\right]\right)^i$$
$$= {}^C X^a \partial_a {}^C Y^i - {}^C Y^a \partial_a {}^C X^i + {}^C X^{\bar a} \partial_{\bar a} {}^C Y^i - {}^C Y^{\bar a} \partial_{\bar a} {}^C X^i$$
$$= X^a \partial_a Y^i - Y^a \partial_a X^i = [X, Y]^i,$$
$$\left(g_*^1\left[{}^C X_{T_1^1}, {}^C Y_{T_1^1}\right]\right)^{\bar i}$$
$$= {}^C X^a \partial_a {}^C Y^{\bar i} - {}^C Y^a \partial_a {}^C X^{\bar i} + {}^C X^{\bar a} \partial_{\bar a} {}^C Y^{\bar i} - {}^C Y^{\bar a} \partial_{\bar a} {}^C X^{\bar i}$$
$$= X^a \partial_a \left(t_{i_2}^m (L_Y g)_{i_1 m} - t_{m i_2}(\partial_{i_1} Y^m) - t_{i_1 m}(\partial_{i_2} Y^m)\right)$$
$$+ \left(t_{a_2}^s (L_X g)_{a_1 s} - t_{s a_2}(\partial_{a_1} X^s) - t_{a_1 s}(\partial_{a_2} X^s)\right) \partial_{\bar a} \left(t_{i_2}^m (L_Y g)_{i_1 m} - t_{m i_2}(\partial_{i_1} Y^m)\right.$$
$$\left.-t_{\ i_1 m}(\partial_{i_2} Y^m)\right) - Y^a \partial_a \left(t_{i_2}^m (L_X g)_{i_1 m} - t_{m i_2}(\partial_{i_1} X^m) - t_{i_1 m}(\partial_{i_2} X^m)\right)$$
$$\left(t_{a_2}^s (L_Y g)_{a_1 s} - t_{s a_2}(\partial_{a_1} Y^s) - t_{a_1 s}(\partial_{a_2} Y^s)\right) \partial_{\bar a} \left(t_{i_2}^m (L_X g)_{i_1 m} - t_{m i_2}(\partial_{i_1} X^m)\right.$$
$$\left.-t_{\ i_1 m}(\partial_{i_2} X^m)\right) = t_{i_2}^m X^a \partial_a (L_Y g)_{i_1 m} - t_{m i_2} X^a \partial_a (\partial_{i_1} Y^m) - t_{i_1 m} X^a \partial_a (\partial_{i_2} Y^m)$$
$$+ \left(t_{a_2}^s (L_X g)_{a_1 s} - t_{s a_2}(\partial_{a_1} X^s) - t_{a_1 s}(\partial_{a_2} X^s)\right) \partial_{\bar a} \left(g^{ml} t_{l i_2} (L_Y g)_{i_1 m} - t_{m i_2}(\partial_{i_1} Y^m)\right.$$
$$\left.-t_{\ i_1 m}(\partial_{i_2} Y^m)\right) - t_{i_2}^m Y^a \partial_a (L_X g)_{i_1 m} + t_{m i_2} Y^a \partial_a (\partial_{i_1} X^m) + t_{i_1 m} Y^a \partial_a (\partial_{i_2} X^m)$$
$$+ \left(t_{a_2}^s (L_Y g)_{a_1 s} - t_{s a_2}(\partial_{a_1} Y^s) - t_{a_1 s}(\partial_{a_2} Y^s)\right) \partial_{\bar a} \left(g^{ml} t_{l i_2} (L_X g)_{i_1 m} - t_{m i_2}(\partial_{i_1} X^m)\right.$$
$$\left.-t_{\ i_1 m}(\partial_{i_2} X^m)\right) = t_{i_2}^m X^a \partial_a (L_Y g)_{i_1 m} - t_{m i_2} X^a \partial_a (\partial_{i_1} Y^m)$$
$$-t_{i_1 m} X^a \partial_a (\partial_{i_2} Y^m) + \left(t_{a_2}^s \ (L_X g)_{a_1 s} - t_{s a_2}(\partial_{a_1} X^s)\right.$$
$$\left.-\ t_{a_1 s}(\partial_{a_2} X^s)\right) \left(g^{ml} \delta_l^{a_1} \delta_{i_2}^{a_2} (L_Y g)_{i_1 m} - \delta_m^{a_1} \delta_{i_2}^{a_2} (\partial_{i_1} Y^m) - \delta_{i_1}^{a_1} \delta_m^{a_2} \ (\partial_{i_2} Y^m)\right)$$
$$-t_{i_2}^m Y^a \partial_a (L_X g)_{i_1 m} + t_{m i_2} Y^a \partial_a (\partial_{i_1} X^m) + t_{i_1 m} Y^a \partial_a (\partial_{i_2} X^m)$$
$$-\left(t_{a_2}^s \ (L_Y g)_{a_1 s} - t_{s a_2}(\partial_{a_1} Y^s) - \ t_{a_1 s}(\partial_{a_2} Y^s)\right) \left(g^{ml} \delta_l^{a_1} \delta_{i_2}^{a_2} (L_X g)_{i_1 m}\right.$$
$$\left.-\delta_m^{a_1} \delta_{i_2}^{a_2} (\partial_{i_1} X^m) - \delta_{i_1}^{a_1} \delta_m^{a_2} \ (\partial_{i_2} X^m)\right) = t_{i_2}^m X^a \partial_a (L_Y g)_{i_1 m}$$
$$-t_{i_2}^m Y^a \partial_a (L_X g)_{i_1 m} - t_{m i_2} X^a \partial_a (\partial_{i_1} Y^m) - t_{i_1 m} X^a \partial_a (\partial_{i_2} Y^m)$$
$$+t_{m i_2} Y^a \partial_a (\partial_{i_1} X^m) + t_{i_1 m} Y^a \partial_a (\partial_{i_2} X^m) + t_{i_2}^s g^{ml} (L_X g)_{ls} (L_Y g)_{i_1 m}$$

$$-t_{i_2}^{s} g^{ml} (L_X g)_{ms} (\partial_{i_1} Y^m) - t_m^s (L_X g)_{i_1 s} (\partial_{i_2} Y^m) - t_{si_2} (\partial_l X^s) g^{ml} (L_Y g)_{i_1 m}$$
$$+t_{si_2} (\partial_m X^s) (\partial_{i_1} Y^m) + t_{sm} (\partial_{i_1} X^s) (\partial_{i_2} Y^m) - t_{ls} g^{ml} (\partial_{i_2} X^s) (L_Y g)_{i_1 m}$$
$$+t_{ms} (\partial_{i_2} X^s) (\partial_{i_1} Y^m) + t_{i_1 s} (\partial_m X^s) (\partial_{i_2} Y^m) - t_{i_2}^{s} g^{ml} (L_Y g)_{ls} (L_X g)_{i_1 m}$$
$$+t_{i_2}^{s} (L_Y g)_{ms} (\partial_{i_1} X^m) + t_m^s (L_Y g)_{i_1 s} (\partial_{i_2} X^m) + t_{si_2} (\partial_l Y^s) g^{ml} (L_X g)_{i_1 m}$$
$$-t_{si_2} (\partial_m Y^s) (\partial_{i_1} X^m) - t_{sm} (\partial_{i_1} Y^s) (\partial_{i_2} X^m) + t_{ls} g^{ml} (\partial_{i_2} Y^s) (L_X g)_{i_1 m}$$
$$-t_{ms} (\partial_{i_2} Y^s) (\partial_{i_1} X^m) - t_{i_1 s} (\partial_m Y^s) (\partial_{i_2} X^m) = -t_{mi_2} \partial_{i_1} [X, Y]^m$$
$$-t_{i_1 m} \partial_{i_2} [X, Y]^m + t_{i_2}^{m} X^a \partial_a (L_Y g)_{i_1 m} - t_{i_2}^{m} Y^a \partial_a (L_X g)_{i_1 m}$$
$$+t_{i_2}^{s} g^{ml} (L_X g)_{ls} (L_Y g)_{i_1 m} - t_{i_2}^{s} (L_X g)_{ms} (\partial_{i_1} Y^m)$$
$$-t_m^s (L_X g)_{i_1 s} (\partial_{i_2} Y^m) - t_{si_2} (\partial_l X^s) g^{ml} (L_Y g)_{i_1 m} - t_{ls} g^{ml} (\partial_{i_2} X^s) (L_Y g)_{i_1 m}$$
$$-t_{i_2}^{s} g^{ml} (L_Y g)_{ls} (L_X g)_{i_1 m} + t_{i_2}^{s} (L_Y g)_{ms} (\partial_{i_1} X^m)$$
$$+t_m^s (L_Y g)_{i_1 s} (\partial_{i_2} X^m) + t_{si_2} (\partial_l Y^s) g^{ml} (L_X g)_{i_1 m}$$
$$+t_{ls} g^{ml} (\partial_{i_2} Y^s) (L_X g)_{i_1 m} + t_{sm} (\partial_{i_1} X^s) (\partial_{i_2} Y^m) + t_{ms} (\partial_{i_2} X^s) (\partial_{i_1} Y^m)$$
$$-t_{sm} (\partial_{i_1} Y^s) (\partial_{i_2} X^m) - t_{ms} (\partial_{i_2} Y^s) (\partial_{i_1} X^m)$$
$$= -t_{mi_2} \partial_{i_1} ([X, Y]^m)\, t_{i_1 m} \partial_{i_2} ([X, Y]^m)\, t_{i_2}^{m} \left(X^a \partial_a (L_Y g)_{i_1 m} + \partial_{i_1} X^s (L_Y g)_{sm}\right.$$
$$\left.+\partial_m X^s (L_Y g)_{i_1 s}\right) - t_{i_2}^{m} \partial_{i_1} X^s (L_Y g)_{sm} - t_{i_2}^{m} \left(Y^a \partial_a (L_X g)_{i_1 m}\right.$$
$$\left.+\partial_{i_1} Y^s (L_X g)_{sm} + \partial_m Y^s (L_X g)_{i_1 s}\right) + t_{i_2}^{m} \partial_{i_1} Y^s (L_X g)_{sm}$$
$$+t_{i_2}^{s} g^{ml} (L_X g)_{ls} (L_Y g)_{i_1 m} - t_m^s (L_X g)_{i_1 s} (\partial_{i_2} Y^m) - t_{si_2} (\partial_l X^s) g^{ml} (L_Y g)_{i_1 m}$$
$$-t_{ls} g^{ml} (\partial_{i_2} X^s) (L_Y g)_{i_1 m} - t_{i_2}^{s} g^{ml} (L_Y g)_{ls} (L_X g)_{i_1 m}$$
$$+t_m^s (L_Y g)_{i_1 s} (\partial_{i_2} X^m) + t_{si_2} (\partial_l Y^s) g^{ml} (L_X g)_{i_1 m}$$
$$+t_{ls} g^{ml} (\partial_{i_2} Y^s) (L_X g)_{i_1 m} = -t_{mi_2} \partial_{i_1} ([X, Y]^m) - t_{i_1 m} \partial_{i_2} ([X, Y]^m)$$
$$+t_{i_2}^{m} [L_X, L_Y]\, g_{i_1 m} - t_{i_2}^{m} \partial_{i_1} X^s (L_Y g)_{sm} + t_{i_2}^{m} \partial_{i_1} Y^s (L_X g)_{sm}$$
$$+t_{i_2}^{s} g^{ml} (L_X g)_{ls} (L_Y g)_{i_1 m} - t_m^s (L_X g)_{i_1 s} (\partial_{i_2} Y^m) - t_{si_2} (\partial_l X^s) g^{ml} (L_Y g)_{i_1 m}$$
$$-t_{ls} g^{ml} (\partial_{i_2} X^s) (L_Y g)_{i_1 m} - t_{i_2}^{s} g^{ml} (L_Y g)_{ls} (L_X g)_{i_1 m}$$
$$+t_m^s (L_Y g)_{i_1 s} (\partial_{i_2} X^m) + t_{si_2} (\partial_l Y^s) g^{ml} (L_X g)_{i_1 m} + t_{ls} g^{ml} (\partial_{i_2} Y^s) (L_X g)_{i_1 m}.$$

Thus we have

Theorem 2.2 *Let (M, g) be a Riemannian manifold, and let ${}^C X_{T_1^1}, {}^C Y_{T_1^1}$ and ${}^C X_{T_2^0}, {}^C Y_{T_2^0}$ be complete lifts of the vector fields $X$and $Y$to the $(1,1)$ and $(0,2)$- tensor bundles, respectively. Then the differential (pushforward) of $\left[{}^C X_{T_1^1}, {}^C Y_{T_1^1}\right]$ by g^1, i.e. a $g-lift$ $g_*^1 \left[{}^C X_{T_1^1}, {}^C Y_{T_1^1}\right]$ in the $(0,2)$-tensor bundle $T_2^0 M$coincides with a $\left[{}^C X_{T_2^0}, {}^C Y_{T_2^0}\right]$ if $X$and Y are Killing vector fields.*

Since ${}^C [X, Y]_{T_2^0} = \left[{}^C X_{T_2^0}, {}^C Y_{T_2^0}\right]$ (see [3]), from Theorem 2.2 we obtain

Theorem 2.3 *If X, Y are Killing vector fields, then $\left[{}^C X_{T_1^1}, {}^C Y_{T_1^1}\right]$ is g^1-related to $\left[{}^C X_{T_2^0}, {}^C Y_{T_2^0}\right]$, where ${}^C X_{T_1^1}, {}^C Y_{T_1^1}$ and ${}^C X_{T_2^0}, {}^C Y_{T_2^0}$ are complete lifts of $X$and $Y$to the $(1,1)$ and $(0,2)$-tensor bundles, respectively.*

A similar result for the tangent and cotangent bundles was obtained in [5].

3 Transfer of affinor fields

Let M_{2n} be a Riemannian manifold with a neutral metric, i.e. with a pseudo-Riemannian metric g of signature (n,n). We say (M_{2n},φ) is an almost complex manifold if M_{2n} can be endowed with an affinor field $\varphi\in\Im_1^1(M_{2n})$ such that $\varphi^2=-I$, where I is a field of identity endomorphhisms. If the Nijenhuis tensor field $N_\varphi\in\Im_2^1(M_{2n})$ vanishes, then φ is a complex structure and moreover M_{2n} is a $C-$holomorphic manifold $X_n(C)$ whose transition functions are $C-$holomorphic mappings. A metric g is a Norden metric if

$$g(\varphi X,Y)=g(X,\varphi Y)$$

for any $X,Y\in\Im_0^1(M_{2n})$, i.e. g is a pure with respect to φ [5].

Let $\varphi=\varphi_j^i\partial_i\otimes dx^j$ be the local expression in $U\in M$ of an affinor field φ. We assume that T_1^1M and T_2^0M are pure subbundles with respect to φ [3]. Then the complete lift ${}^C\varphi_{T_1^1}$ of φ to the to $(1,1)$ - pure tensor subbundle $T_1^1M$is given by [3]

$$^C\varphi_{T_1^1}=\left({}^C\varphi_J^I\right)=\begin{pmatrix}\varphi_j^i & 0\\ t_i^m\partial_m\varphi_j^p-t_m^p\partial_i\varphi_j^m & \varphi_l^p\delta_i^j\end{pmatrix}\tag{3.1}$$

with respect to the induced coordinates $(x^i,x^{\bar{i}})=(x^i,t_i^p)$ in T_1^1M. It is well known that ${}^C\varphi_{T_1^1}$ defines an almost complex structure on T_1^1M, if and only if so does φ on M [3].

Using (1.1), (1.2) and (3.1), we have

$$g_*^{1C}\varphi_{T_1^1}=(\tilde{\varphi}_L^J)=\left(\frac{\partial\tilde{x}^J}{\partial x^I}\frac{\partial x^K}{\partial\tilde{x}^L}{}^C\varphi_K^I\right)$$

$$\begin{pmatrix}\varphi_l^j & 0\\ t_{j_2}^m\Phi_lg_{j_1m}+t_{mj_2}(\partial_l\varphi_{j_1}^m-\partial_{j_1}\varphi_l^m)-t_{j_1m}\partial_{j_2}\varphi_l^m & g^{rl_1}\delta_k^{l_2}g_{j_1p}\delta_{j_2}^i\varphi_r^p\delta_i^k\end{pmatrix},\tag{3.2}$$

where

$$\Phi_lg_{j_1m}=\varphi_l^s\partial_sg_{j_1m}+g_{j_1s}\partial_m\varphi_l^p+g_{ms}\partial_{j_1}\varphi_l^s-g_{ms}\partial_l\varphi_{j_1}^s-\varphi_{j_1}^s\partial_lg_{ms}.$$

The expression (3.2) is called a $g-lift\ \ g_*^1\ {}^C\varphi_{T_1^1}$ of φ to (0,2)-pure tensor subbundle T_2^0M. It is well known that the complete lift ${}^C\varphi_{T_2^0}$ of $\varphi\in\Im_1^1(M)$ to the (0,2)- pure tensor subbundle T_2^0M is given by [2]

$$^C\varphi_{T_2^0}=\begin{pmatrix}\varphi_l^j & 0\\ t_{mj_2}(\partial_l\varphi_{j_1}^m-\partial_{j_1}\varphi_l^m)-t_{j_1m}\partial_{j_2}\varphi_l^m & \varphi_{j_1}^{l_1}\delta_{j_2}^{l_2}\end{pmatrix}$$

with respect to the induced coordinates in T_2^0M.

From (3.2) we find $g-lift\ \ g_*^1\ {}^C\varphi_{T_1^1}$ of φ

$$g_*^1\ {}^C\varphi_{T_1^1}={}^C\varphi_{T_2^0}+\gamma(\Phi_\varphi g),$$

where $\gamma(\Phi_\varphi g)$ is defined by

$$\gamma(\Phi_\varphi g)=\begin{pmatrix}0 & 0\\ t_{j_2}^m\Phi_lg_{j_1m} & 0\end{pmatrix}.$$

From here, we have

Theorem 3.1 *Let (M,g) be a Riemannian manifold, and let ${}^C\varphi_{T_1^1}$ and ${}^C\varphi_{T_2^0}$ be complete lifts of an affinor field φ to the $(1,1)$ and $(0,2)$- pure tensor subbundles, respectively. In order the image of ${}^C\varphi_{T_1^1}$ by g^1, i.e. a $g-lift$ $g_*^{1C}\varphi_{T_1^1}$ coincides with ${}^C\varphi_{T_2^0}$, it is necessary and sufficient that (M,g,φ) is a Kahler-Norden manifold.*

Let ${}^C X_{T_1^1}$ be a complete lift of a vector field X and ${}^C\varphi_{T_1^1}$ be a complete lift of an affinor field φ to the $(1,1)$- pure tensor subbundle. We have

$$g_*^{1C}\varphi_{T_1^1}{}^C X_{T_1^1} = \tilde{\varphi}_L^J \tilde{X}^L,$$

$$\tilde{\varphi}_L^j \tilde{X}^L = \tilde{\varphi}_l^j \tilde{X}^l + \tilde{\varphi}_{\bar{l}}^j \tilde{X}^{\bar{l}} = \varphi_l^j X^l = (\varphi X)^j, \tag{3.3}$$

$$\begin{aligned}
\tilde{\varphi}_L^{\bar{j}} \tilde{X}^L &= \tilde{\varphi}_l^{\bar{j}} \tilde{X}^l + \tilde{\varphi}_{\bar{l}}^{\bar{j}} \tilde{X}^{\bar{l}} = \big(t_{j_2}^m \,\Phi_l g_{j_1 m} + t_{mj_2}(\partial_l \varphi_{j_1}^m - \partial_{j_1}\varphi_l^m) \\
&-t_{j_1 m}\,\partial_{j_2}\varphi_l^m\big) X^l + \varphi_{j_1}^{l_1}\delta_{j_2}^{l_2}\left(t_{l_2}^m\,(L_X g)_{l_1 m} - t_{ml_2}\partial_{l_1}X^m - t_{l_1 m}\,\partial_{l_2}X^m\right) \\
&= t_{j_2}^m\left(\Phi_l g_{j_1 m} + \varphi_j^l{}_1 (L_X g)_{lm}\right) + \left(t_{mj_2}X^l\partial_l\varphi_{j_1}^m - t_{mj_2}\varphi_{j_1}^l\partial_l X + t_{mj_2}\varphi_l^m\,\partial_{j_1}X^l\right) \\
&-t_{mj_2}\varphi_l^m\partial_{j_1}X^l - t_{mj_2}(\partial_{j_1}\varphi_l^m)X^l - t_{j_1 m}(\partial_{j_2}\varphi_l^m)X^l - t_{lm}\varphi_{j_1}^l\partial_{j_2}X^m \\
&= t_{j_2}^m\left(\Phi_l g_{j_1 m} + \varphi_j^l{}_1 (L_X g)_{lm}\right) + t_{mj_2}(L_X\varphi)_{j_1}^m \\
&+ \left(-t_{mj_2}\,\partial_{j_1}(\varphi_l^m X^l) - t_{j_1 m}\partial_{j_2}(\varphi_l^m X^l)\right).
\end{aligned} \tag{3.4}$$

From (3.3) and (3.4) follows

$$g_*^{1C}\varphi_{T_1^1}{}^C X_{T_1^1} = \tilde{\varphi}_L^J\tilde{X}^L = \begin{pmatrix} 0 \\ t_{j_2}^m\left(\Phi_l g_{j_1 m} + \varphi_{j_1}^l (L_X g)_{lm}\right)\end{pmatrix}$$

$$+\begin{pmatrix} 0 \\ t_{mj_2}(L_X\varphi)_{j_1}^m\end{pmatrix} + \begin{pmatrix} \varphi_l^j X^l \\ (-t_{mj_2}\partial_{j_1}\,(\varphi_l^m X^l) - t_{j_1 m}\partial_{j_2}(\varphi_l^m X^l))\end{pmatrix}.$$

So we have

Theorem 3.2 *Let (M,g) be a Riemannian manifold, and let ${}^C\varphi_{T_1^1}$ and ${}^C\varphi_{T_2^0}$ be complete lifts of an affinor field φ to the $(1,1)$ and $(0,2)$-pure tensor subbundles, let ${}^C X_{T_1^1}$ and ${}^C X_{T_2^0}$ be complete lifts of a vector field X to the $(1,1)$ and $(0,2)$ pure tensor subbundles, respectively. Then the differential (pushforward) of ${}^C\varphi_{T_1^1}{}^C X_{T_1^1}$ by g^1, i.e. a $g-lift$ $g_*^{1C}\varphi_{T_1^1}{}^C X_{T_1^1}$ in the $(0,2)$-pure tensor subbundle $T_2^0 M$is a complete lift ${}^C\varphi_{T_2^0}{}^C X_{T_2^0}$ if (M,g,φ) is a Kahler-Norden manifold and X is a Killing vector field.*

Using (1.1), (1.2) and (3.1) we obtain

$$g_*^1\left({}^C\varphi\right)_{T_1^1}^2 = \tilde{\varphi}_L^J\tilde{\varphi}_K^L$$

$$=\begin{pmatrix} \varphi_l^j & 0 \\ t_{j_2}^m\Phi_l g_{j_1 m} + t_{mj_2}(\partial_l\varphi_{j_1}^m - \partial_{j_1}\varphi_l^m) - t_{j_1 m}\partial_{j_2}\varphi_l^m & \varphi_{j_1}^{l_1}\delta_{j_2}^{l_2}\end{pmatrix}\times$$

$$\times\begin{pmatrix} \varphi_k^l & 0 \\ t_{l_2}^s\Phi_k g_{l_1 s} + t_{sl_2}(\partial_k\varphi_{l_1}^s - \partial_{l_1}\varphi_k^s) - t_{l_1 s}\partial_{l_2}\varphi_k^s & \varphi_{l_1}^{k_1}\delta_{l_2}^{k_2}\end{pmatrix},$$

$$\tilde{\varphi}_L^j \tilde{\varphi}_k^L = \varphi_l^j \varphi_k^l,$$
$$\tilde{\varphi}_L^j \tilde{\varphi}_{\bar{k}}^L = 0,$$
$$\tilde{\varphi}_L^{\bar{j}} \tilde{\varphi}_{\bar{k}}^L = \varphi_j^l \varphi_l^k,$$
$$\tilde{\varphi}_L^{\bar{j}} \tilde{\varphi}_k^L = \tilde{\varphi}_l^{\bar{j}} \tilde{\varphi}_k^l + \tilde{\varphi}_{\bar{l}}^{\bar{j}} \tilde{\varphi}_k^{\bar{l}}$$
$$= \left(t_{j_2}^m \Phi_{\,l}\, g_{j_1 m} + t_{mj_2}(\partial_l \varphi_{j_1}^m - \partial_{j_1}\varphi_l^m) - t_{j_1 m}\partial_{\,j_2}\varphi_l^m\right)\varphi_k^l$$
$$+\varphi_{j_1}^{l_1}\delta_{j_2}^{l_2}\left(t_{l_2}^m \Phi_k\, g_{l_1 m} + t_{ml_2}(\partial_k \varphi_{l_1}^m - \partial_{l_1}\varphi_k^m) - t_{l_1 m}\partial_{\,l_2}\varphi_k^m\right)$$
$$+t_{j_2}^m \varphi_k^l \Phi_l g_{j_1 m} + t_{j_2}^m \varphi_{j_1}^l \Phi_k g_{lm} + t_{mj_2}\varphi_k^l \partial_l \varphi_{j_1}^m - t_{mj_2}\varphi_k^l \partial_{j_1}\varphi_l^m$$
$$-t_{j_1 m}\varphi_k^l \partial_{j_2}\varphi_l^m + t_{mj_2}\varphi_{j_1}^l \partial_k \varphi_l^m - t_{mj_2}\varphi_{j_1}^l \partial_l \varphi_k^m - t_{lm}\varphi_{j_1}^l \partial_{j_2}\varphi_k^m$$
$$= t_{j_2}^m \varphi_k^l \Phi_l g_{j_1 m} + t_{j_2}^m \varphi_{j_1}^l \Phi_k g_{lm} + (t_{mj_2}\,\partial_k(\varphi_{j_1}^l \varphi_l^m) - t_{mj_2}\partial_{j_1}(\varphi_k^l \varphi_l^m)$$
$$-t_{j_1 m}\partial_{j_2}(\varphi_k^l \varphi_{\ l}^{m})) + t_{mj_2}(\varphi_k^l \partial_l \varphi_{j_1}^m - \varphi_{j_1}^l \partial_l \varphi_k^m - \varphi_l^m \partial_k \varphi_{j_1}^l + \varphi_l^m \partial_{j_1}\varphi_k^l)$$
$$= {}^C\left(\varphi^2\right)_{T_2^0} + (\gamma N_\varphi)_{T_2^0} + t_{j_2}^m \varphi_k^l \Phi_l g_{j_1 m} + t_{j_2}^m \varphi_{j_1}^l \Phi_k g_{lm}.$$

Since in Kahler-Norden manifold $N_\varphi = 0$, we have

Theorem 3.3 *Let (M,g) be a Riemannian manifold, let ${}^C\varphi_{T_1^1}$ and ${}^C\varphi_{T_2^0}$ be complete lifts of an affinor field φ to the $(1,1)$ and $(0,2)$- pure tensor subbundles, respectively. Then the differential (pushforward) of $\left({}^C\varphi\right)^2_{T_1^1}$ by g^1, i.e. a $g-lift\ g_*^1 \left({}^C\varphi\right)^2_{T_1^1}$ in the $(0,2)$ pure tensor subbundle $T_2^0 M$ is an affinor field $\left({}^C\varphi\right)^2_{T_2^0}$ if (M,g,φ) is a Kahler-Norden manifold.*

4 Transfer of vector-valued 2-forms

Let S be a vector-valued 2-form on M. A semi-Riemannian metric g is called pure with respect to S if

$$g(S_Y X_1, X_2) = g(X_1, S_Y X_2)$$

for any $X_1, X_2, Y \in \Im_0^1(M)$, where $S_Y$denotes a tensor field of type $(1,1)$ such that $S_Y(Z) = S(Y,Z) = -S(Z,Y) = -S_Z(Y)$ for any $Y, Z \in \Im_0^1(M)$.The condition of purity of g may be expressed in terms of the local components as follows:

$$g_{mi_2} S_{i_1 l}^m = g_{i_1 m} S_{i_2 l}^m.$$

Now we consider the Yano-Ako operator $\Phi_S : \Im_2^0(M) \to \Im_4^0(M)$ connected with S and applied to a pure tensor field g (see [3], [8]). The Yano-Ako operator has following components with respect to the natural coordinate system:

$$(\Phi_S g)_{jihs} = S_{ji}^m \partial_m g_{hs} - (\partial_j S_{hi}^m) g_{ms} - (\partial_j g_{ms}) S_{hi}^m - (\partial_i S_{ji}^m) g_{ms}$$
$$-(\partial_i g_{ms}) S_{jh}^m + (\partial_h S_{ji}^m) g_{ms} + (\partial_s S_{ji}^m) g_{hm}. \tag{4.1}$$

We assume that $T_1^1 M$ and $T_2^0 M$ are pure subbundles with respect to S [3]. The non-zero components of the complete lift ${}^C S_{T_1^1}$ of S to the $(1,1)$- pure tensor subbundle $T_1^1 M$ are given by [3]

$${}^C S_{k_1 k_2}^i = S_{k_1 k_2}^i, {}^C S_{\bar{k}_1 k_2}^{\bar{i}} = S_{a_1 k_2}^j \delta_i^{k_1}, {}^C S_{k_1 \bar{k}_2}^{\bar{i}} = S_{k_1 a_2}^j \delta_i^{k_2},$$

$$^{C}S^{\bar{i}}_{k_1k_2} = t^m_i \partial_m S^j_{k_1k_2} - t^j_m \partial_i S^m_{k_1k_2}. \tag{4.2}$$

Using (1.1), (1.2) and (4.2), we can easily verify that $g - lift\ g^{1C}_{*} S_{T^1_1}$ $= (\tilde{S}^J_{KL}) = \left(\frac{\partial \tilde{x}^J}{\partial x^I} \frac{\partial x^R}{\partial \tilde{x}^K} \frac{\partial x^M}{\partial \tilde{x}^L} {}^C S^I_{RM} \right)$, $J, L_1, L_2, ... = 1, ..., n + n^2$ has non-zero components of the form

$$\tilde{S}^j_{kl} = \delta^j_i \delta^r_k \delta^{mC}_l S^i_{rm} = S^j_{kl},$$

$$\tilde{S}^{\bar{j}}_{\bar{k}l} = g_{j_1p} \delta^i_{j_2} g^{sk_1} \delta^{k_2}_r \delta^m_l S^p_{sm} \delta^r_i = g_{j_1p} g^{sk_1} S^p_{sl} \delta^{k_2}_{j_2} = S^{k_1}_{j_1l} \delta^{k_2}_{j_2},$$

$$\tilde{S}^{\bar{j}}_{k\bar{l}} = g_{j_1p} \delta^i_{j_2} \delta^r_k g^{ql_1} \delta^{l_2}_m S^p_{rq} \delta^m_i = g_{j_1p} g^{ql_1} S^p_{rq} \delta^{l_2}_{j_2} = S^{l_1}_{j_1l} \delta^{l_2}_{j_2},$$

$$\begin{aligned}
\tilde{S}^{\bar{j}}_{kl} &= t^p_{j_2} (\partial_i g_{j_1p}) \delta^r_k \delta^m_l S^i_{rm} + g_{j_1p} \delta^i_{j_2} \delta^r_k \delta^m_l (t^s_i \partial_s S^p_{rm} - t^p_s \partial_i S^s_{rm}) + \\
&\quad + g_{j_1p} \delta^i_{j_2} t_{ar} (\partial_k g^{qa}) \delta^m_l S^p_{qm} \delta^r_i + g_{j_1p} \delta^i_{j_2} \delta^r_k t_{bm} (\partial_l g^{qb}) S^p_{rq} \delta^m_i \\
&= t^p_{j_2} (\partial_i g_{j_1p}) S^i_{kl} + g_{j_1p} t^s_{j_2} \partial_s S^p_{kl} - g_{j_1p} t^p_s \partial_{j_2} S^s_{kl} + g_{j_1p} t_{aj_2} (\partial_k g^{qa}) S^p_{ql} \\
&\quad + g_{j_1p} t_{bj_2} (\partial_l g^{qb}) S^p_{kq} = t^p_{j_2} (\Phi_S g)_{klj_1p} + t^p_{j_2} g_{mp} (\partial_k S^m_{j_1l}) \\
&\quad + t^p_{j_2} g_{mp} (\partial_l S^m_{kj_1}) - t^p_{j_2} g_{mp} (\partial_{j_1} S^m_{kl}) - g_{j_1p} t^p_s (\partial_{j_2} S^m_{kl}) = t^p_{j_2} (\Phi_S g)_{klj_1p} \\
&\quad + t_{mj_2} (\partial_k S^m_{j_1l}) + t_{mj_2} (\partial_l S^m_{kj_1}) - t_{mj_2} (\partial_{j_1} S^m_{kl}) - t_{j_1m} (\partial_{j_2} S^m_{kl})
\end{aligned} \tag{4.3}$$

From equation (4.3) we find $g - lift\ g^{1C}_{*} S_{T^1_1}$ of S :

$$g^1_{*}\ {}^C S_{T^1_1} = {}^C S_{T^0_2} + \gamma(\Phi_S g), \tag{4.4}$$

Using (4.4), we have

Theorem 4.1 *Let (M, g) be a Riemannian manifold, and let ${}^C S_{T^1_1}$ and ${}^C S_{T^0_2}$ be complete lifts of a vector-valued 2-form S to the $(1,1)$ and $(0,2)$- pure tensor subbundles, respectively. In order the image of ${}^C S_{T^1_1}$ by g^1, i.e. a $g - lift\ g^{1C}_{*} S_{T^1_1}$ coincides with ${}^C S_{T^0_2}$ is necessary and sufficient that $\Phi_S g = 0$.*

5 Transfer of Sasaki type metrics

Let ${}^S g$ be a Sasaki type metric in the (1,1)- tensor bundle $T^1_1 M$ (see [7]):

$$^S g = \begin{pmatrix} g_{ij} + g_{ab} g^{pq} (\tilde{\Gamma}^a_{ip} - \bar{\Gamma}^a_{ip})(\tilde{\Gamma}^b_{jq} - \bar{\Gamma}^b_{jq}) & g_{bm} g^{aj} (\tilde{\Gamma}^b_{ia} - \bar{\Gamma}^b_{ia}) \\ g_{pr} g^{iq} (\tilde{\Gamma}^p_{jq} - \bar{\Gamma}^p_{jq}) & g_{rm} g^{ij} \end{pmatrix}, \tag{5.1}$$

where $\tilde{\Gamma}^a_{ip} = \Gamma^a_{im} t^m_p$, $\bar{\Gamma}^a_{ip} = \Gamma^m_{ip} t^a_m$. On the other hend the Sasaki type metric in the (0,2) -tensor bundle $T^0_2 M$ has components of the form (see [6])

$$^S \tilde{g} = \begin{pmatrix} g_{ij} + g^{ap} g^{bq} (-\tilde{\Gamma}_{iab} - \bar{\Gamma}_{iba})(-\tilde{\Gamma}_{jpq} - \bar{\Gamma}_{jqp}) & g^{aj_1} g^{bj_2} (\tilde{\Gamma}_{iab} - \bar{\Gamma}_{iba}) \\ g^{i_1p} g^{i_2q} (\tilde{\Gamma}_{jpq} - \bar{\Gamma}_{jqp}) & g^{i_1j_1} g^{i_2j_2} \end{pmatrix} \tag{5.2}$$

with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}\}$, where $\tilde{\Gamma}_{iab} = \Gamma^m_{ia} t_{mb}$, $\bar{\Gamma}_{iba} = \Gamma^m_{ib} t_{am}$.

Using (1.2) and (5.1) we see that the pullback of ${}^Sg$ by g^2

$$(((g^2_*)\ {}^Sg)_{KL}) = \left(\frac{\partial x^I}{\partial\tilde{x}^K}\frac{\partial x^J}{\partial\tilde{x}^L}{}^Sg_{IJ}\right),\ \ I,J,K,L,...=1,...,n+n^2$$

is the (0,2) tensor field on the T^0_2M and has the following components:

$$\begin{aligned}
\left((g^2_*)\ {}^Sg\right)_{kl} &= \frac{\partial x^I}{\partial\tilde{x}^k}\frac{\partial x^J}{\partial\tilde{x}^l}{}^Sg_{IJ} = \frac{\partial x^i}{\partial\tilde{x}^k}\frac{\partial x^j}{\partial\tilde{x}^l}{}^Sg_{ij} + \frac{\partial x^{\bar{i}}}{\partial\tilde{x}^k}\frac{\partial x^j}{\partial\tilde{x}^l}{}^Sg_{\bar{i}j} + \frac{\partial x^i}{\partial\tilde{x}^k}\frac{\partial x^{\bar{j}}}{\partial\tilde{x}^l}{}^Sg_{i\bar{j}} \\
&+\frac{\partial x^{\bar{i}}}{\partial\tilde{x}^k}\frac{\partial x^{\bar{j}}}{\partial\tilde{x}^l}{}^Sg_{\bar{i}\bar{j}} = \delta^i_k\delta^j_l\left[g_{ij}+g_{ab}g^{pq}(\Gamma^a_{im}t^m_p-\Gamma^m_{ip}t^a_m)(\Gamma^b_{jr}t^r_q-\Gamma^r_{jq}t^b_r)\right] \\
&+t_{mi}(\partial_kg^{sm})\delta^j_lg_{ps}g^{iq}(\Gamma^p_{jc}t^c_q-\Gamma^c_{jq}t^p_c)+\delta^i_kt_{mj}(\partial_lg^{rm})\delta^j_lg_{br}g^{aj}(\Gamma^b_{ic}t^c_a \\
&-\Gamma^c_{ia}t^b_c)+t_{mi}(\partial_kg^{sm})t_{aj}(\partial_lg^{ra})g_{sr}g^{ij}=g_{kl} \\
&+g_{ab}g^{pq}\Gamma^a_{km}t^m_p\Gamma^b_{lr}t^r_q-g_{ab}g^{pq}\Gamma^a_{km}t^m_p\Gamma^r_{lq}t^b_r-g_{ab}g^{pq}\Gamma^m_{kp}t^a_m\Gamma^b_{lr}t^r_q \\
&+g_{ab}g^{pq}\Gamma^m_{kp}t^a_m\Gamma^r_{lq}t^b_r+t_{mi}(-\Gamma^s_{kr}g^{rm}-\Gamma^m_{kr}g^{sr})g_{ps}g^{iq}(\Gamma^p_{lc}t^c_q-\Gamma^c_{lq}t^p_c) \\
&+t_{mj}(-\Gamma^r_{ls}g^{sm}-\Gamma^m_{ls}g^{rs})g_{br}g^{aj}(\Gamma^b_{kc}t^c_a-\Gamma^c_{ka}t^b_c) \\
&+t_{mi}(-\Gamma^s_{kb}g^{bm}-\Gamma^m_{kb}g^{sb})t_{aj}(-\Gamma^r_{lp}g^{pa}-\Gamma^a_{lp}g^{pr})g_{sr}g^{ij} \\
&=g_{kl}+g_{ab}g^{pq}\Gamma^a_{km}g^{ms}t_{sp}\Gamma^b_{lr}g^{rc}t_{cq}-g_{ab}g^{pq}\Gamma^a_{km}g^{ms}t_{sp}\Gamma^r_{lq}g^{bh}t_{hr} \\
&-g_{ab}g^{pq}\Gamma^m_{kp}g^{as}t_{sm}\Gamma^b_{lr}g^{rh}t_{hq}+g_{ab}g^{pq}\Gamma^m_{kp}g^{as}t_{sm}\Gamma^r_{lq}g^{bh}t_{hr} \\
&-t_{mi}\Gamma^s_{kr}g^{rm}g_{ps}g^{iq}\Gamma^p_{lc}g^{ch}t_{hq}+t_{mi}\Gamma^s_{kr}g^{rm}g_{ps}g^{iq}\Gamma^c_{lq}g^{ph}t_{hc} \\
&-t_{mi}\Gamma^m_{kr}g^{sr}g_{ps}g^{iq}\Gamma^p_{lc}g^{ch}t_{hq}+t_{mi}\Gamma^m_{kr}g^{sr}g_{ps}g^{iq}\Gamma^c_{lq}g^{ph}t_{hc} \\
&-t_{mj}\Gamma^r_{ls}g^{sm}g_{br}g^{aj}\Gamma^b_{kc}g^{ch}t_{ha}+t_{mj}\Gamma^r_{ls}g^{sm}g_{br}g^{aj}\Gamma^c_{ka}g^{bh}t_{hc} \\
&-t_{mj}\Gamma^m_{ls}g^{rs}g_{br}g^{aj}\Gamma^b_{kc}g^{ch}t_{ha}+t_{mj}\Gamma^m_{ls}g^{rs}g_{br}g^{aj}\Gamma^c_{ka}g^{bh}t_{hc} \\
&+t_{mi}\Gamma^s_{kb}g^{bm}t_{aj}\Gamma^r_{lp}g^{pa}g_{sr}g^{ij}+t_{mi}\Gamma^s_{kb}g^{bm}t_{aj}\Gamma^a_{lp}g^{pr}g_{sr}g^{ij} \qquad (5.3) \\
&+t_{mi}\Gamma^m_{kb}g^{sb}t_{aj}\Gamma^r_{lp}g^{pa}g_{sr}g^{ij}+t_{mi}\Gamma^m_{kb}g^{sb}t_{aj}\Gamma^a_{lp}g^{pr}g_{sr}g^{ij} \\
&=g_{kl}+g^{pq}\Gamma^m_{kp}t_{bm}\Gamma^r_{lq}g^{bh}t_{hr}+t_{mi}\Gamma^m_{kp}g^{iq}\Gamma^c_{lq}g^{ph}t_{hc} \\
&+t_{mj}\Gamma^m_{ls}g^{rs}\Gamma^c_{ka}g^{aj}t_{rc}+t_{mi}\Gamma^m_{kb}g^{sb}\Gamma^a_{ls}g^{ij}t_{aj}={}^S\tilde{g}_{kl};
\end{aligned}$$

$$\begin{aligned}
\left((g^2_*)\ {}^Sg\right)_{\bar{k}l} &= \frac{\partial x^I}{\partial\tilde{x}^{\bar{k}}}\frac{\partial x^J}{\partial\tilde{x}^l}{}^Sg_{IJ} = \frac{\partial x^i}{\partial\tilde{x}^{\bar{k}}}\frac{\partial x^j}{\partial\tilde{x}^l}{}^Sg_{ij} + \frac{\partial x^{\bar{i}}}{\partial\tilde{x}^{\bar{k}}}\frac{\partial x^j}{\partial\tilde{x}^l}{}^Sg_{\bar{i}j} + \frac{\partial x^i}{\partial\tilde{x}^{\bar{k}}}\frac{\partial x^{\bar{j}}}{\partial\tilde{x}^l}{}^Sg_{i\bar{j}} \\
&+\frac{\partial x^{\bar{i}}}{\partial\tilde{x}^{\bar{k}}}\frac{\partial x^{\bar{j}}}{\partial\tilde{x}^l}{}^Sg_{\bar{i}\bar{j}} = g^{rk_1}\delta^{k_2}_i\delta^j_lg_{pr}g^{iq}(\Gamma^p_{jm}t^m_q-\Gamma^m_{jq}t^p_m) \\
&+g^{rk_1}\delta^{k_2}_it_{mj}(\partial_lg^{sm})g_{rs}g^{ij}=g^{k_2q}(\Gamma^{k_1}_{lm}g^{ms}t_{sq}-\Gamma^m_{lq}g^{k_1s}t_{sm}) \\
&+t_{mj}(\partial_lg^{km})g^{ij}=g^{k_2q}\Gamma^{k_1}_{lm}g^{ms}t_{sq}-g^{k_2q}\Gamma^m_{lq}g^{k_1s}t_{sm} \\
&+t_{mj}(-\Gamma^{k_1}_{lr}g^{rm}-\Gamma^m_{lr}g^{k_1r})g^{k_2j}=g^{k_2q}\Gamma^{k_1}_{lm}g^{ms}t_{sq}-g^{k_2q}\Gamma^m_{lq}g^{k_1s}t_{sm} \qquad (5.4) \\
&-t_{mj}\Gamma^{k_1}_{lr}g^{rm}g^{k_2j}-t_{mj}\Gamma^m_{lr}g^{k_1r}g^{k_2j}=g^{k_2q}g^{k_1s}(-\Gamma^m_{lq}t_{sm}-\Gamma^m_{ls}t_{mq})={}^S\tilde{g}_{\bar{k}l};
\end{aligned}$$

$$\left((g_*^2)\,{}^Sg\right)_{k\bar{l}} = \frac{\partial x^I}{\partial \tilde{x}^k}\frac{\partial x^J}{\partial \tilde{x}^{\bar{l}}}{}^Sg_{IJ} = \frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial x^j}{\partial \tilde{x}^{\bar{l}}}{}^Sg_{ij} + \frac{\partial x^{\bar{i}}}{\partial \tilde{x}^k}\frac{\partial x^j}{\partial \tilde{x}^{\bar{l}}}{}^Sg_{\bar{i}j} + \frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial x^{\bar{j}}}{\partial \tilde{x}^{\bar{l}}}{}^Sg_{i\bar{j}}$$
$$+\frac{\partial x^{\bar{i}}}{\partial \tilde{x}^k}\frac{\partial x^{\bar{j}}}{\partial \tilde{x}^{\bar{l}}}{}^Sg_{\bar{i}\bar{j}} = \delta_k^i g^{rl_1}\delta_j^{j_2} g_{br} g^{aj}(\Gamma_{im}^b t_a^m - \Gamma_{ia}^m t_m^b)$$
$$+t_{mi}(\partial_k g^{sm})g^{rl_1}\delta_j^{l_2} g_{sr} g^{ij} = g^{al_2}\Gamma_{km}^{l_1} t_a^m - g^{al_2}\Gamma_{ka}^m t_m^{l_1}$$
$$+t_{mi}(\partial_k g^{l_1 m})g^{il_2} = g^{al_2}\Gamma_{km}^{l_1} g^{ms} t_{sa} - g^{al_2}\Gamma_{ka}^m g^{l_1 s} t_{sm}$$
$$+t_{mi} g^{il_2}(-\Gamma_{kp}^{l_1} g^{pm} - \Gamma_{kp}^m g^{l_1 p}) = g^{al_2}\Gamma_{km}^{l_1} g^{ms} t_{sa} - g^{al_2}\Gamma_{ka}^m g^{l_1 s} t_{sm} \tag{5.5}$$
$$-t_{mi} g^{il_2}\Gamma_{kp}^{l_1} g^{pm} - t_{mi} g^{il_2}\Gamma_{kp}^m g^{l_1 p}) = g^{al_2} g^{l_1 s}(-\Gamma_{ka}^m t_{sm} - \Gamma_{ks}^m t_{ma}) = {}^S\tilde{g}_{k\bar{l}};$$
$$\left((g_*^2)\,{}^Sg\right)_{\bar{k}\bar{l}} = \frac{\partial x^I}{\partial \tilde{x}^{\bar{k}}}\frac{\partial x^J}{\partial \tilde{x}^{\bar{l}}}{}^Sg_{IJ} = \frac{\partial x^i}{\partial \tilde{x}^{\bar{k}}}\frac{\partial x^j}{\partial \tilde{x}^{\bar{l}}}{}^Sg_{ij} + \frac{\partial x^{\bar{i}}}{\partial \tilde{x}^{\bar{k}}}\frac{\partial x^j}{\partial \tilde{x}^{\bar{l}}}{}^Sg_{\bar{i}j} + \frac{\partial x^i}{\partial \tilde{x}^{\bar{k}}}\frac{\partial x^{\bar{j}}}{\partial \tilde{x}^{\bar{l}}}{}^Sg_{i\bar{j}}$$
$$+\frac{\partial x^{\bar{i}}}{\partial \tilde{x}^{\bar{k}}}\frac{\partial x^{\bar{j}}}{\partial \tilde{x}^{\bar{l}}}{}^Sg_{\bar{i}\bar{j}} = g^{rk_1}\delta_i^{k_2} g^{sl_1}\delta_j^{l_2} g_{rs} g^{ij} = g^{k_1 l_1} g^{k_2 l_2} = {}^S\tilde{g}_{\bar{k}\bar{l}}. \tag{5.6}$$

Thus, from (5.2)-(5.6) follows

$$(g_*^2)^S g = {}^S\tilde{g}.$$

From here, we have

Theorem 5.1 *The Sasaki type metric* ${}^S\tilde{g} \in \Im_2^0(T_2^0 M)$ *coincides with* $g - lift\ (g_*^2)^S g$ *on (0,2)- tensor bundle* $T_2^0 M$.

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