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Spectral properties of a problem of vibrations of a loaded string in Lebesgue spaces

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Abstract. *Basis properties of the eigenfunctions of a spectral problem for a second order discontinuous differential operator with a spectral parameter in discontinuity (conjugation) conditions are studied for Lebesgue spaces. This kind of problem arises when one tries to solve the problem of vibrations of a loaded string with fixed ends. New method is suggested to prove the basicity of eigenfunctions in the spaces* $L_p \oplus C$ *and* L_p.

Keywords. eigenvalue · eigenfunction · biorthogonal system · basicity

Mathematics Subject Classification (2010): 34B05 · 34B24 · 34L10 · 34L20 · 46E30

1 Introduction

Consider the following spectral problem with a point of discontinuity:

$$y''(x) + \lambda y(x) = 0 \quad , x \in (0, \frac{1}{3}) \cup (\frac{1}{3}, 1), \tag{1.1}$$

$$\left.\begin{array}{l} y(0) = y(1) = 0, \\ y(\frac{1}{3} - 0) = y(\frac{1}{3} + 0), \\ y'(\frac{1}{3} - 0) - y'(\frac{1}{3} + 0) = \lambda m y(\frac{1}{3}), \end{array}\right\} \tag{1.2}$$

where λ is the spectral parameter, $m$is a non-zero complex number. Such spectral problems arise when the problem of vibrations of a loaded string with fixed ends is solved by applying the Fourier method [19, 1]. In case when the load is placed at the middle of the string, some aspects of this spectral problem have been studied in [9, 8]. Using different methods, similar aspects have been treated in [12–15] for a spectral problem corresponding to the problem of vibrations of a loaded string in case when the string is loaded at one or both ends. In [10], the completeness of eigenfunctions for the problem (1.1),(1.2) has been proved in the spaces $L_p\,(0, 1) \oplus C$ and $L_p\,(0, 1)$. Spectral problems with a point of discontinuity and a spectral

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parameter in boundary conditions have been considered in [16,11,18,17]. Note that these spectral problems strongly differ from the usual ones. The study of basis properties of the systems consisting of eigenfunctions of the spectral problems with a point of discontinuity sometimes requires completely new research methods, different from the known ones. In [3,4], new method for exploring basis properties of discontinuous differential operators has been suggested. The present paper is an extension of the method of [3,4].

2 Necessary information

Let us give some results from [10], which we will need throughout the paper.

Theorem 2.1 *[10] The spectral problem* (1.1), (1.2) *has two series of eigenvalues:* $\lambda_{1,n} = (\rho_{1,n})^2$, $n = 1, 2, ...$, $\lambda_{2,n} = (\rho_{2,n})^2$, $n = 0, 1, 2, ...$, *where*

$$\left.\begin{array}{l} \rho_{1,n} = 3\pi n, \\ \rho_{2,n} = \frac{3\pi n}{2} + \frac{2+(-1)^n}{\pi m n} + O\left(\frac{1}{n^2}\right). \end{array}\right\} \tag{2.1}$$

The corresponding eigenfunctions are given by the following expressions

$$y_{1,n}(x) = \sin 3\pi n x, x \in [0,1], \qquad n = 1, 2, ...,$$

$$y_{2,n}(x) = \begin{cases} \sin \rho_{2,n}\left(x - \frac{1}{3}\right) + \sin \rho_{2,n}\left(x + \frac{1}{3}\right), \ x \in \left[0, \frac{1}{3}\right], \\ \sin \rho_{2,n}(1 - x), \ x \in \left[\frac{1}{3}, 1\right], \qquad n = 0, 1, 2, ... \end{cases} . \tag{2.2}$$

Let us construct an operator L, linearizing the problem (1.1),(1.2) in the direct sum $L_p(0,\ 1) \oplus C$, where C is the complex plane. Denote by $W_p^2\left(0, \frac{1}{3}\right) \oplus W_p^2\left(\frac{1}{3}, 1\right)$ the space of functions whose restrictions to intervals $\left(0,\ \frac{1}{3}\right)$ and $\left(\frac{1}{3},\ 1\right)$ belong to Sobolev spaces $W_p^2\left(0, \frac{1}{3}\right)$ and $W_p^2\left(\frac{1}{3}, 1\right)$, respectively, where $1 < p < \infty$. Let us define the operator L in the following way. As the domain $D(L)$ we take the manifold

$$\begin{array}{r} D(L) = \left\{\hat{y} = \left(y(x), my\left(\frac{1}{3}\right)\right) : y(x) \in W_p^2\left(0, \frac{1}{3}\right) \oplus W_p^2\left(\frac{1}{3}, 1\right), \right. \\ \left. y(0) = y(1) = 0, \quad y\left(\frac{1}{3} - 0\right) = y\left(\frac{1}{3} + 0\right)\right\}, \end{array} \tag{2.3}$$

and for $\hat{y} \in D(L)$ the operator L is defined by the relation

$$L\hat{y} = \left(-y'';\ y'\left(\frac{1}{3} - 0\right) - y'\left(\frac{1}{3} + 0\right)\right). \tag{2.4}$$

The following lemma is true.

Lemma 2.1 *[10] The operator L defined by expressions* (2.3), (2.4) *is a densely defined closed operator with completely continuous resolution. The eigenvalues of the operator L and the problem* (1.1), (1.2) *coincide. If* $y(x)$ *is the eigenfunction (associated function) of problem* (1.1), (1.2), *then* $\hat{y} = \left(y(x);\ my\left(\frac{1}{3}\right)\right)$ *is the eigenvector (associated vector) of the operator* L.

When obtaining the main results, we need the following easily proved statement
Statement 1. *Let the system* $\{u_n\}_{n\in N}$ *form a basis with parentheses for a Banach space* X. *If the sequence* $\{n_{k+1} - n_k\}_{k\in N}$ *is bounded and the condition*

$$\sup_n \|u_n\| \, \|\vartheta_n\| < \infty,$$

holds, where $\{\vartheta_n\}_{n\in N}$ *is a biorthogonal system, then the system* $\{u_n\}_{n\in N}$ *forms an ordinary basis for* X.

The following statement is also true.

Statement 2. *Let the system* $\{u_n\}_{n\in N}$ $\{u_n\}_{n\in N}$ *form a Riesz basis with parentheses for a Hilbert space* X*. If the sequence* $\{n_{k+1}-n_k\}_{k\in N}$ *is bounded and the following condition*

$$\sup_n \{\|u_n\|\,;\|\vartheta_n\|\} < \infty,$$

holds, where $\{\vartheta_n\}_{n\in N}$ *is a biorthogonal system, then* $\{u_n\}_{n\in N}$ *forms a usual Riesz basis for* X.

We also need the following definition.

Definition 2.1 *The bases* $\{u_n\}_{n\in N}$ *of Banach space* X *is called a p-basis, if for any* $x \in X$

$$\left(\sum_{n=1}^{\infty} |\langle x, \vartheta_n\rangle|^p\right)^{\frac{1}{p}} \le M\,\|x\|\,,$$

where $\{\vartheta_n\}_{n\in N}$ *is a biorthogonal system to* $\{u_n\}_{n\in N}$.

Definition 2.2 *The sequences* $\{u_n\}_{n\in N}$ *and* $\{\varphi_n\}_{n\in N}$ *of Banach space*X *is called a* p*-close, if*

$$\sum_{n=1}^{\infty} \|u_n - \varphi_n\|^p < \infty.$$

We will also use the following results from [5–7].

Theorem 2.2 *[5] Let* $\{x_n\}_{n\in N}$ *form a q-basis for the space* X*, and the system* $\{y_n\}_{n\in N}$ *is p-close to* $\{x_n\}_{n\in N}$*, where* $\frac{1}{p}+\frac{1}{q}=1$*. Then the following properties are equivalent:*

i) $\{y_n\}_{n\in N}$ *is complete in* X*;*
ii) $\{y_n\}_{n\in N}$ *is minimal in* X*;*
iii) $\{y_n\}_{n\in N}$ *forms an isomorphic basis to* $\{x_n\}_{n\in N}$ *for* X.

Let $X_1 = X \oplus C^m$ and $\{\hat{u}_n\}_{n\in N} \subset X_1$ be some minimal system, and $\left\{\hat{\vartheta}_n\right\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ be its biorthogonal system:

$$\hat{u}_n = (u_n; \alpha_{n1}, ..., \alpha_{nm})\,; \quad \hat{\vartheta}_n = (\vartheta_n; \beta_{n1}, ..., \beta_{nm})\,.$$

Let $J = \{n_1, ..., n_m\}$ be some set of m natural numbers. Suppose

$$\delta = \det \|\beta_{n_i j}\|_{i,j=\overline{1,m}}\,.$$

The following theorem is true.

Theorem 2.3 *[6, 7] Let the system* $\{\hat{u}_n\}_{n\in N}$ *form a basis for* X_1*. In order for the system* $\{u_n\}_{n\in N_j}$*, where* $N_j = N \setminus J$*, form a basis for* X *it is necessary and sufficient that the condition* $\delta \neq 0$ *be satisfied. In this case the biorthogonal system to* $\{u_n\}_{n\in N_j}$ *is defined by*

$$\vartheta_n^* = \frac{1}{\delta}\begin{vmatrix} \vartheta_n & \vartheta_{n1} & ... & \vartheta_{nm} \\ \beta_{n1} & \beta_{n_1 1} & ... & \beta_{n_m 1} \\ ... & ... & ... & ... \\ \beta_{nm} & \beta_{n_1 m} & ... & \beta_{n_m m} \end{vmatrix}.$$

In particular, if X *is a Hilbert space and the system* $\{u_n\}_{n\in N}$ *forms a Riesz basis for* X_1*, then under the condition* $\delta \neq 0$*, the system* $\{u_n\}_{n\in N_J}$ *also forms a Riesz basis for* X.

For $\delta = 0$ *the system* $\{u_n\}_{n\in N_J}$ *is not complete and is not minimal in* X.

3 Main results

Let X be a Banach space and $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ be some system in X. Let $a_{ik}^{(n)}$, $i,k=\overline{1,m}$, $n\in N$, be some complex number. Let

$$A_n=\left(a_{ik}^{(n)}\right)_{i,k=\overline{1,m}} \quad \text{and} \quad \Delta_n=\det A_n, \quad n\in N.$$

Let us consider the following system in space X

$$\hat{u}_{kn}=\sum_{i=1}^{m}a_{ik}^{(n)}u_{in}, \quad k=\overline{1,m}; n\in N. \tag{3.1}$$

Theorem 3.1 *If the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ forms a basis for X and*

$$\Delta_n\neq 0,\ \forall n\in N, \tag{3.2}$$

then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms a basis with parentheses for X. If the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ is p-basis and in addition the conditions

$$\sup_n\{\ \|A_n\|,\left\|A_n^{-1}\right\|\ \}<\infty, \quad \sup_n\{\|u_{kn}\|,\|\vartheta_{kn}\|\}<\infty, \tag{3.3}$$

holds, where $\{\vartheta_{kn}\}_{k=\overline{1,m};n\in N}\subset X^*$ is a biorthogonal system to $\{u_{kn}\}_{k=\overline{1,m};n\in N}$, then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ also forms p-basis for X.

Proof. From the representation (3.1) and from the minimality of the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$, follows the minimality of the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ and the biorthogonal system has the form

$$\hat{\vartheta}_{in}=\sum_{l=1}^{m}b_{li}^{(n)}\vartheta_{\ln}\ ,\ i=\overline{1,m}:\ n\in N, \tag{3.4}$$

where the numbers $b_{li}^{(n)}$ are elements of the inverse matrix $\left(A_n^{-1}\right)^*$. Taking these expressions into account, for $x\in X$ we have

$$\sum_{i=1}^m\left\langle x,\hat{\vartheta}_{in}\right\rangle\hat{u}_{in}=\sum_{i=1}^m\sum_{j=1}^m\sum_{l=1}^m a_{ij}^{(n)}b_{li}^{(n)}\left\langle x,\vartheta_{\ln}\right\rangle u_{jn}$$

$$=\sum_{j=1}^m\sum_{l=1}^m\left(\sum_{i=1}^m b_{li}^{(n)}a_{ij}^{(n)}\right)\left\langle x,\vartheta_{\ln}\right\rangle u_{jn}$$

$$=\sum_{j=1}^m\sum_{l=1}^m\delta_{lj}\left\langle x,\vartheta_{\ln}\right\rangle u_{jn}=\sum_{j=1}^m\left\langle x,\vartheta_{jn}\right\rangle u_{jn}.$$

Consequently

$$S_N(x)=\sum_{n=1}^{N}\sum_{i=1}^{m}\left\langle x,\hat{\vartheta}_{in}\right\rangle\hat{u}_{in}=\sum_{n=1}^{N}\sum_{j=1}^{m}\left\langle x,\vartheta_{jn}\right\rangle u_{jn}=\sum_{j=1}^{m}\sum_{n=1}^{N}\left\langle x,\vartheta_{jn}\right\rangle u_{jn}\to x, \quad \text{as } N\to\infty.$$

Thus, the system $\{\hat{u}_{in}\}_{i=\overline{1,m};\,n\in N}$ forms a basis with parentheses for X.

Now let the conditions (3.3) be satisfied. Then from the representations (3.1) and (3.4) we obtain

$$\sup_{i,n}\left\{\|\hat{u}_{in}\|\ ;\ \left\|\hat{\vartheta}_{in}\right\|\right\}<+\infty.$$

Consequently, the system $\{\hat{u}_{in}\}_{i=\overline{1,m};\, n\in N}$ is uniformly minimal, and by Statement 1 it forms the ordinary basis for X. On the other hand, if the system $\{u_{kn}\}_{k=\overline{1,m}; n\in N}$ forms a p-basis for X, i.e.

$$\left(\sum_{i=1}^{m}\sum_{n=1}^{\infty}|\langle x,\vartheta_{in}\rangle|^{p}\right)^{\frac{1}{p}}\leq M\,\|x\|\,,$$

then from (3.3) and (3.4) follows that the following inequality is valid

$$\left(\sum_{i=1}^{m}\sum_{n=1}^{\infty}\left|\left\langle x,\hat{\vartheta}_{in}\right\rangle\right|^{p}\right)^{\frac{1}{p}}\leq M_{1}\,\|x\|\,.$$

Consequently, the system $\{\hat{u}_{in}\}_{i=\overline{1,m};\, n\in N}$ also forms a p-basis for X.
The theorem is proved.

Theorem 3.2 *Let X be a Hilbert space and the system $\{u_{kn}\}_{k=\overline{1,m}; n\in N}$ form a Riesz basis for X. Then, under condition* (3.2)*, the system $\{\hat{u}_{kn}\}_{k\in\overline{1,m}; n\in N}$ forms a Riesz basis with parentheses for X. If, in addition, the conditions* (3.3) *hold, then the system $\{\hat{u}_{kn}\}_{k\in\overline{1,m}; n\in N}$ forms a usual Riesz basis for X.*

Proof. If the system $\{u_{kn}\}_{k=\overline{1,m}; n\in N}$ forms a Riesz basis for X, then for any $x\in X$ we have the following the expansion

$$x=\sum_{k=1}^{m}\sum_{n=1}^{\infty}\langle x,\vartheta_{kn}\rangle\,u_{kn}\ ,$$

where the series are unconditionally convergent. Then the series

$$x=\sum_{n=1}^{\infty}\sum_{k=1}^{m}\langle x,\vartheta_{kn}\rangle\,u_{kn},$$

is also unconditionally convergent. Now, taking into account the equality

$$\sum_{k=1}^{m}\langle x,\vartheta_{kn}\rangle\,u_{kn}\ =\sum_{k=1}^{m}\left\langle x,\hat{\vartheta}_{kn}\right\rangle\hat{u}_{kn},$$

we obtain that the series

$$\sum_{n=1}^{\infty}\sum_{k=1}^{m}\left\langle x,\hat{\vartheta}_{kn}\right\rangle\hat{u}_{kn}$$

also converges unconditionally to an element x. Consequently, the system $\{\hat{u}_{kn}\}_{k\in\overline{1,m}; n\in N}$ forms a Riesz basis with parentheses for X. On the other hand, if the conditions (3.3) hold, then it follows from (3.1) that

$$\sup_{i,n}\left\{\|\hat{u}_{in}\|\ ;\ \left\|\hat{\vartheta}_{in}\right\|\right\}<+\infty.$$

So, applying Statement 2, we get the validity of the theorem.

Using Theorems 3.1 and 3.2, we obtain the following main results relating to the basis properties of the eigenfunctions of problem (1.1), (1.2) in spaces $L_p\,(0,1)\oplus C$ and $L_p\,(0,1)$.

Theorem 3.3 *The system* $\{\hat{y}_{i,n}\}_{i=1,2;n\in N}$ *of eigen and associated vectors of the operator* L *forms a bases for space* $L_p(0,1)\oplus C$, $1<p<\infty$. *For* $p=2$ *this basis is a Riesz basis.*

Theorem 3.4 *If from the system of eigen and associated functions of problem* (1.1), (1.2) $\{y_0\}\cup\{y_{i,n}\}_{i=1,2;\,n\in N}^{\infty}$ *we eliminate any function* $y_{2,n_0}(x)$, *corresponding to a simple eigenvalue* $\lambda_{2,n}$, *then the obtaining system forms a basis for* $L_p(0,1)$, $1<p<\infty$. *And if we eliminate any function* $y_{1,n_0}(x)$ *from this system, then the obtaining system does not form a basis in* $L_p(0,1)$*; moreover, in this case the obtained system is not complete and is not minimal in this space.*

Remark 3.1 For $m>0$, the linearizing operator L of the problem (1.1),(1.2) is a self-adjoint operator in $L_2\oplus C$, and in this case all eigenvalues are real and simple, and to each eigenvalue there corresponds only one eigenvector. If $m<0$, then the operator L is a J-self-adjoint operator in $L_2\oplus C$ and in this case, applying the results of [2], we obtain that all eigenvalues are real and simple, with the exception of, may be either one pair of complex conjugate simple eigenvalues or one non-simple real value. In the case of a complex value m the operator L has an infinite number of complex eigenvalues that are asymptotically simple and, consequently, the operator L can have a finite number of associated vectors. If there are associated vectors, they are determined up to a linear combination with the corresponding eigenvector, and in this case there always exists an associated vector for which $z_{2,n}\left(\frac{1}{3}\right)=0$, as well as an associated vector for which $z_{2,n}\left(\frac{1}{3}\right)\neq 0$.

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