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Two weighted inequalities for fractional integrals on commutative hypergroups

Mubariz G. Hajibayov · **Elmira A. Gadjieva**

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Abstract. *In the present paper we prove a two weighted inequality for the fractional integrals on commutative hypergroups with quasi-metric and doubling Haar measure upper Ahlfors N-regular on an identity. This result is an generalization of the Heinig theorem obtained in [22], [6], [7] for the classical Riesz potential, for for the fractional integrals (B fractional integrals) associated with the Laplace-Bessel differential operator and for the fractional integrals on Laguerre hypergroups respectively. Also the Stein-Weiss inequality for the fractional integrals on commutative hypergroups is proved as an application of the main result.*

Keywords. hypergroup · Riesz potential · Hardy-Littlevood maximal function.

Mathematics Subject Classification (2010): 47G40 · 20N20 · 43A62 · 26A33

1 Introduction

Let $0 < \alpha < n$ and $R_\alpha f = |\cdot|^{\alpha-n} * f$ be the classical Riesz potential. By the classical Hardy-Littlewood-Sobolev theorem ([26]) the operator R_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, with $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$, and is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ with $1 < q < \infty$, $1 - 1/q = \alpha/n$.

The Hardy-Littlewood-Sobolev theorem is an important result in the potential theory and harmonic analysis. There are a lot of analogues and generalizations of that theorem. In [13], [14], [15], [16], [11], [10] Riesz potentials on different partial hypergroups were defined and analogues of the Hardy-Littlewood-Sobolev theorem were given for these operators. The generalization of that theorem was proved in [17], [18], [19], [20] for the general hypergroups.

Heinig ([22]) proved the boundedness of the classical Riesz potential $R_\alpha f$ in weighted Lebesgue spaces.

M.G. Hajibayov
National Aviation Academy, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9, B.Vahabzade str., AZ1141, Baku, Azerbaijan,
E-mail: hajibayovm@yahoo.com

E.A. Gadjieva
Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9, B.Vahabzade str., AZ1141, Baku, Azerbaijan,
E-mail: egadjieva@sgmail.com

Theorem A. ([22]) *Suppose u and v are defined on $\mathbb{R}^n$ and $U = u^*$, $\frac{1}{V} = \left(v^{-1}\right)^*$. If $1 \le p \le q \le \infty$, $p < \infty$ and, for some r, $1 < r < \frac{n}{\alpha}$*

$$\sup_{s>0} \left(\int_s^{\infty} \left[U(t) t^{\alpha - n} \right]^q t^{n-1} dt \right)^{\frac{1}{q}} \left(\int_0^s V(t)^{-p'} t^{n-1} dt \right)^{\frac{1}{p'}} < \infty$$

and

$$\sup_{s>0} \left(\int_0^s \left[U(t) t^{\alpha - \frac{n}{r}} \right]^q t^{n-1} dt \right)^{\frac{1}{q}} \left(\int_s^{\infty} \left[V(t) t^{\frac{n}{r'}} \right]^{-p'} t^{n-1} dt \right)^{\frac{1}{p'}} < \infty,$$

then $R_\alpha : L_{p,v}(\mathbb{R}^n) \to L_{q,u}(\mathbb{R}^n)$ is bounded.

In [6] and [7] the analogues of the Heinig Theorem are given for the Riesz potentials (B fractional integrals) associated with the Laplace-Bessel differential operator and for the fractional integrals on Laguerre hypergroups correspondingly. The purpose of the present paper is to extend these results to the commutative hypergroups with quasi-metric and doubling Haar measure upper Ahlfors N-regular on an identity. Also we give an application of our main result and prove the Stein-Weiss inequality for these potentials.

2 Preliminaries

Let K be a set. A function $\varrho : K \times K \to [0, \infty)$ is called quasi-metric if:

1 $\varrho(x, y) = 0 \Leftrightarrow x = y$;
2 $\varrho(x, y) = \varrho(y, x)$;
3 there exists a constant $c \ge 1$ such that for every $x, y, z \in K$

$$\varrho(x, y) \le c\left(\varrho(x, z) + \varrho(z, y)\right).$$

Let all balls $B(x, r) = \{y \in K : \varrho(x, y) < r\}$ be λ-measurable and assume that the measure λ fulfils the doubling condition

$$0 < \lambda B(x, 2r) \le D \lambda B(x, r) < \infty.$$

A space (K, ϱ, λ) which satisfies all conditions mentioned above is called a space of homogeneous type (see [5]).

In the theory of locally compact groups there arise certain spaces which, though not groups, have some of the structure of groups. Often, the structure can be expressed in terms of an abstract convolution of measures on the space.

A hypergroup $(K, *)$ consists of a locally compact Hausdorff space K together with a bilinear, associative, weakly continuous convolution on the Banach space of all bounded regular Borel measures on K with the following properties:

1. For all $x, y \in K$, the convolution of the point measures $\delta_x * \delta_y$ is a probability measure with compact support.
2. The mapping: $K \times K \to \mathcal{C}(K)$, $(x, y) \mapsto supp(\delta_x * \delta_y)$ is continuous with respect to the Michael topology on the space $\mathcal{C}(K)$ of all nonvoid compact subsets of K, where this topology is generated by the sets

   $$U_{V,W} = \{L \in \mathcal{C}(K) : L \cap V \ne \varnothing, L \subset W\}$$

   with V, W open in K.
3. There is an identity $e \in K$ with $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in K$.

4. There is a continuous involution $\sim$ on K such that

$$(\delta_x * \delta_y)^{\sim} = \delta_{y^{\sim}} * \delta_{x^{\sim}}$$

and $e \in supp(\delta_x * \delta_y) \Leftrightarrow x = y^{\sim}$ for $x, y \in K$ (see [23], [25], [2], [24], [12]).

A hypergroup K is called commutative if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$. It is well known that every commutative hypergroup K possesses a Haar measure which will be denoted by λ (see [25]). That is, for every Borel measurable function f on K,

$$\int_K f(\delta_x * \delta_y) d\lambda(y) = \int_K f(y) d\lambda(y), \quad x \in K.$$

Define the generalized translation operators T^x, $x \in K$, by

$$T^x f(y) = \int_K f(z) d(\delta_x * \delta_y)(z)$$

for all $y \in K$. If K is a commutative hypergroup, then $T^x f(y) = T^y f(x)$ and the convolution of two functions is defined by

$$(f * g)(x) = \int_K T^x f(y) g(y^{\sim}) d\lambda(y).$$

Let $(K, *)$ be a commutative hypergroup, with quasi-metric ϱ, Haar measure λ and all balls $B(x, r) = \{y \in K : \varrho(x, y) < r\}$ be λ-measurable and $N \in (0, \infty)$. We will say Haar measure λ is upper Ahlfors N-regular on an identity, if there exists a constant $A > 0$, not depending $r > 0$, such that

$$\lambda B(e, r) \leq A r^N.$$

Define the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{\lambda B(e, r)} \left(|f| * \chi_{B(e,r)}\right)(x) \tag{2.1}$$

and the fractional integral

$$I_\alpha f(x) = \left(\varrho(e, \cdot)^{\alpha - N} * f\right)(x), \; 0 < \alpha < N \tag{2.2}$$

on the commutative hypergroup $(K, *)$.

Let w be a weight function on the measure space (X, μ) , i.e., w is a non-negative and measurable function on X. The weighted Lebesgue space $L_{p,w}(X, \mu) \equiv L_{p,w}(X)$, $1 \leq p < \infty$, is the set of all classes of measurable functions f with finite norm

$$\|f\|_{L_{p,w}(X)} = \Big(\int_X |f(x)|^p w(x) d\mu(x) \Big)^{\frac{1}{p}}.$$

If $p = \infty$, we assume

$$L_{\infty,w}(X, \mu)) = \Big\{ f : \|f\|_{L_{\infty,w}(X)} = \operatorname*{ess\,sup}_{x \in X} |w(x) f(x)| < \infty \Big\}.$$

If $w = 1$, then space $L_{p,w}(X, \mu)$ and norm $\|f\|_{L_{p,w}(X)}$ are denoted by $L_p(X, \mu)$ and $\|f\|_{L_p(X)}$ respectively.

Let X and T be a linear operator from $L_p(X, \mu)$ to $L_q(X, \mu)$, where $p, q \in (0, \infty)$. T is said to be an operator of strong type (p, q) on X, if there exists a positive constant C such that

$$\big\|Tf\big\|_{L_q(X)} \leq C \big\|f\big\|_{L_p(X)} \;\text{ for }\; f \in L_p(X, \mu).$$

If for the arbitrary $\beta > 0$ and for the function $f \in L^p(X,\mu)$

$$\mu\{x : |Tf(x)| > \beta\} \leq \left(\frac{C\|f\|_{X,p}}{\beta}\right)^q,$$

then T is called an operator of weak type (p,q) on X.

Let f be a λ-measurable function defined on the hypergroup K. The distribution function $f_{*,K}$ of the function f is given by

$$f_{*,K}(s) = \lambda\{x \in K : |f(x)| > s\}, \text{ for } s \geq 0.$$

The distribution function $f_{*,K}$ is non-negative, non-increasing and continuous from the right. With the distribution function we associate the non-increasing rearrangement of f on $[0,\infty)$ defined by

$$f_K^*(t) = \inf\{s > 0 : f_{*,K}f(s) \leq t\}.$$

Lemma 2.1 *([3,21,28]) (Hardy inequalities) Suppose ξ and θ are non-negative locally integrable functions defined on $(0,\infty)$ and $1 < p \leq q < \infty$. Then there exists a constant $C > 0$ such that for all non-negative Lebesgue measurable function ψ on $(0,\infty)$, the inequality the inequality*

$$\left(\int_0^{+\infty}\left(\int_0^t \psi(s)ds\right)^q \xi(t)dt\right)^{\frac{1}{q}} \leq C\left(\int_0^{+\infty}\psi(t)^p\theta(t)dt\right)^{\frac{1}{p}}$$

is satisfied if and only if

$$\sup_{s>0}\left(\int_s^{+\infty}\xi(t)dt\right)^{\frac{1}{q}}\left(\int_0^s \theta(t)^{1-p'}dt\right)^{\frac{1}{p'}} < \infty. \tag{2.3}$$

Similarly for the dual operator,

$$\left(\int_0^{+\infty}\left(\int_t^{+\infty} \psi(s)ds\right)^q \xi(t)dt\right)^{\frac{1}{q}} \leq C\left(\int_0^{+\infty}\psi(t)^p\theta(t)dt\right)^{\frac{1}{p}}$$

is satisfied if and only if

$$\sup_{s>0}\left(\int_0^s \xi(t)dt\right)^{\frac{1}{q}}\left(\int_s^{+\infty} \theta(t)^{1-p'}dt\right)^{\frac{1}{p'}} < \infty. \tag{2.4}$$

Lemma 2.2 *([1], [21], [22]) Let f and g be non-negative measurable functions on K. Then*

$$\int_K f(x)g(x)\lambda(x) \leq \int_0^{+\infty} f_K^*(t)g_K^*(t)dt \tag{2.5}$$

and

$$\int_0^{+\infty} f_K^*(t)\frac{dt}{(g^{-1})_K^*(t)} \leq \int_K f(x)g(x)d\lambda(x). \tag{2.6}$$

Lemma 2.3 *([4]) Let $1 \leq p_1 < p_2 < +\infty$ and $1 \leq q_1 < q_2 < +\infty$. A linear operator T is a weak-type (p_1,q_1) and (p_2,q_2) on K if and only if*

$$(Tf)_K^*(t) \leq C\Big(t^{-\frac{1}{q_1}}\int_0^{t^{\frac{\sigma_1}{\sigma_2}}} s^{\frac{1}{p_1}-1}f_K^*(s)ds + t^{-\frac{1}{q_2}}\int_{t^{\frac{\sigma_1}{\sigma_2}}}^{+\infty} s^{\frac{1}{p_2}-1}f_K^*(s)ds\Big),$$

where $\sigma_1 = \frac{1}{q_1} - \frac{1}{q_2}$ and $\sigma_2 = \frac{1}{p_1} - \frac{1}{p_2}$.

The following two theorems have been proved in [20].

Theorem 2.1 *([20]) Let $(K,*)$ be a commutative hypergroup, with quasi-metric ϱ and doubling Haar measure λ, upper Ahlfors N-regular on an identity. Assume $0<\alpha<N$, $1<p<\frac{N}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{N}$ and the maximal function* (2.1) *is an operator of strong type (p,p) on $(K,*)$. Then the fractional integral* (2.2) *is an operator of strong type (p,q) on $(K,*)$.*

Theorem 2.2 *([20]) Let $(K,*)$ be a commutative hypergroup, with quasi-metric ϱ and doubling Haar measure λ, upper Ahlfors N-regular by an identity and let $0<\alpha<N$, $\frac{1}{q}=1-\frac{\alpha}{N}$ and assume that the maximal operator M is an operator of weak type $(1,1)$ on $(K,*)$. Then the fractional integral* (2.2) *is an operator of weak type $(1,q)$ on $(K,*)$.*

3 Main result

In this section the Heinig theorem for the fractional integrals on commutative hypergroups.

Theorem 3.1 *Let $(K,*)$ be a commutative hypergroup, with quasi-metric ϱ and doubling Haar measure λ, upper Ahlfors N-regular on an identity. Assume $0<\alpha<N$, $1<r<\frac{N}{\alpha}$, $1<p\le q<+\infty$, the maximal function* (2.1) *is an operator of strong type (p,p) on $(K,*)$ and u, v are non-negative λ-locally integrable functions on K with conditions*

$$\sup_{s>0}\left(\int_s^{+\infty}u_K^*(t)t^{-q(1-\frac{\alpha}{N})}dt\right)^{\frac{1}{q}}\left(\int_0^s\left((\frac{1}{v})_K^*(t)\right)^{p'-1}dt\right)^{\frac{1}{p'}}<+\infty, \tag{3.1}$$

and

$$\sup_{s>0}\left(\int_0^s u_K^*(t)t^{-q(\frac{1}{r}-\frac{\alpha}{N})}dt\right)^{\frac{1}{q}}\left(\int_s^{+\infty}\left((\frac{1}{v})_K^*(t)\right)^{p'-1}t^{p'(\frac{1}{r}-1)}dt\right)^{\frac{1}{p'}}<+\infty. \tag{3.2}$$

Then $I_K^\alpha f$ is bounded operator from $L_{p,v}(K)$ to $L_{q,u}(K)$, that is, there exists a constant $C>0$ such that for any $f\in L_{p,v}(K)$,

$$\|I_K^\alpha f\|_{K,q,u}\le C\|f\|_{K,p,v}.$$

Proof. By Theorem 2.1 and Theorem 2.2 we have that $I_K^\alpha f$ is an operator of weak-type $\left(1,\dfrac{1}{1-\frac{\alpha}{N}}\right)$ and is an operator of strong-type $\left(r,\dfrac{1}{\frac{1}{r}-\frac{\alpha}{N}}\right)$, where $1<r<\infty$. Apply to $I_K^\alpha f$ Lemma 2.3 taking

$$p_1=1,\ q_1=\frac{1}{1-\frac{\alpha}{N}},\ p_2=r,\ q_2=\frac{1}{\frac{1}{r}-\frac{\alpha}{N}}.$$

Then

$$\sigma_1=\frac{1}{q_1}-\frac{1}{q_2}=1-\frac{1}{r},\ \sigma_2=\frac{1}{p_1}-\frac{1}{p_2}=1-\frac{1}{r},\ \frac{\sigma_1}{\sigma_2}=1$$

and

$$\left[\int_0^{+\infty}\left(\left(I_K^\alpha f\right)_K^*(t)\right)^q u_K^*(t)dt\right]^{\frac{1}{q}}$$

$$\le C\left[\int_0^{+\infty}\left(t^{\frac{\alpha}{N}-1}\int_0^t f_K^*(s)ds+t^{\frac{\alpha}{N}-\frac{1}{r}}\int_t^{+\infty}s^{\frac{1}{r}-1}f_K^*(s)ds\right)^q u_K^*(t)dt\right]^{\frac{1}{q}}.$$

Applying the Minkowski inequalty we obtain

$$\left[\int_0^{+\infty}\left(\left(I_K^\alpha f\right)_K^*(t)\right)^q u_K^*(t)dt\right]^{\frac{1}{q}}$$

$$\leq C\left[\int_0^{+\infty} u_K^*(t)t^{\left(\frac{\alpha}{N}-1\right)q}\left(\int_0^t f_K^*(s)ds\right)^q dt\right]^{\frac{1}{q}}$$

$$+C\left[\int_0^{+\infty} u_K^*(t)t^{\left(\frac{\alpha}{N}-\frac{1}{r}\right)q}\left(\int_t^{+\infty} s^{\frac{1}{r}-1}f_K^*(s)ds\right)^q dt\right]^{\frac{1}{q}}. \tag{3.3}$$

If we take the denotations

$$\xi(t)=u_K^*(t)t^{\left(\frac{\alpha}{N}-1\right)q},\ \psi(t)=f_K^*(t),\ \theta(t)=\frac{1}{\left(v^{-1}\right)_K^*(t)},$$

then we have (2.3) from (3.1) and applying Lemma 2.1 we can assert that

$$\left[\int_0^{+\infty} u_K^*(t)t^{\left(\frac{\alpha}{N}-1\right)q}\left(\int_0^t f_K^*(s)ds\right)^q dt\right]^{\frac{1}{q}}$$

$$\leq C\left(\int_0^{+\infty}\frac{1}{\left(v^{-1}\right)_K^*(t)}\left(f_K^*(t)\right)^p dt\right)^{\frac{1}{p}}. \tag{3.4}$$

Now if we take

$$\xi(t)=u_K^*(t)t^{\left(\frac{\alpha}{N}-\frac{1}{r}\right)q},\ \psi(t)=t^{\frac{1}{r}-1}f_K^*(t),\ \theta(t)=\frac{1}{\left(v^{-1}\right)_K^*(t)}t^{p\left(\frac{1}{r}-1\right)},$$

then we have (2.4) from (3.2) and applying Lemma 2.1 we can assert that

$$\left[\int_0^{+\infty} u_K^*(t)t^{\left(\frac{\alpha}{N}-\frac{1}{r}\right)q}\left(\int_t^{+\infty} s^{\frac{1}{r}-1}f_K^*(s)ds\right)^q dt\right]^{\frac{1}{q}}$$

$$\leq\left(\int_0^{+\infty}\left(t^{\frac{1}{r}-1}f_K^*(t)\right)^p\frac{1}{\left(v^{-1}\right)_K^*(t)}t^{p\left(\frac{1}{r}-1\right)}dt\right)^{\frac{1}{p}}$$

$$=\left(\int_0^{+\infty}\frac{1}{\left(v^{-1}\right)_K^*(t)}\left(f_K^*(t)\right)^p dt\right)^{\frac{1}{p}}. \tag{3.5}$$

Combining (3.3), (3.4), (3.5), it yields

$$\left[\int_0^{+\infty}\left(\left(I_K^\alpha f\right)_K^*(t)\right)^q u_K^*(t)dt\right]^{\frac{1}{q}}\leq C\left(\int_0^{+\infty}\frac{1}{\left(v^{-1}\right)_K^*(t)}\left(f_K^*(t)\right)^p dt\right)^{\frac{1}{p}}. \tag{3.6}$$

Applying (2.5), (3.6) and (2.6) we have

$$\left[\int_K\left(\left(I_K^\alpha f(x)\right)\right)^q u(x)d\lambda(x)\right]^{\frac{1}{q}}\leq\left[\int_0^{+\infty}\left(\left(I_K^\alpha f\right)_K^*(t)\right)^q u_K^*(t)dt\right]^{\frac{1}{q}}$$

$$\leq C\Big(\int_0^{+\infty}\frac{1}{\left(v^{-1}\right)_K^*(t)}\left(f_K^*(t)\right)^p dt\Big)^{\frac{1}{p}}\leq C\left[\int_K\left(f(x)\right)^p v(x)d\lambda(x)\right]^{\frac{1}{p}}.$$

The theorem is proved.

4 The Stein-Weiss inequality for fractional integrals on hypergroups

In this section the Stein-Weiss inequality for the fractional integrals on commutative hypergroup is proved as an application of Theorem 3.1. Note that the Stein-Weiss inequality for the classical Riesz potentials was given in [27]. This inequality in different partial hypergroups was proved in [8], [9], [10].

Theorem 4.1 *Let* $(K,*)$ *be a commutative hypergroup, with quasi-metric* ϱ *and doubling Haar measure* λ, *and there exists a constant* $A>0$, *not depending* $r>0$, *such that* $\lambda B(e,r)=Ar^N$. *Suppose that* $0<\alpha<N$, $1<p<\dfrac{N}{\alpha}$, $\beta<0$, $0<\beta+\alpha p<N(p-1)$. *If the maximal function* (2.1) *is an operator of strong type* (p,p) *on* $(K,*)$, *then* I_K^α *is bounded operator from* $L_{p,\varrho(e,x)^{\beta+\alpha p}}(K)$ *to* $L_{q,\varrho(e,x)^{\beta}}(K)$, *that is, there exists a constant* $C>0$ *such that for any* $f\in L_{p,\varrho(e,x)^{\beta+\alpha p}}(K)$,

$$\left(\int_K |I_K^\alpha f(x)|^p \varrho(e,x)^\beta d\lambda(x)\right)^{\frac{1}{p}} \le C\left(\int_K |f(x)|^p\varrho(e,x)^{\beta+\alpha p}d\lambda(x)\right)^{\frac{1}{p}}.$$

Proof. Take

$$u(x)=\varrho(e,x)^\beta, v(x)=\varrho(e,x)^{\beta+\alpha p},\ q=r=p.$$

Since $\beta<0$ we have

$$u_{*,K}(s)=\lambda\{x:x\in K,\varrho(e,x)^\beta>s\}$$

$$=\lambda\{x:x\in K,\varrho(e,x)<s^{\frac{1}{\beta}}\}=\lambda B(0,s^{\frac{1}{\beta}})=Ar^{\frac{N}{\beta}}$$

and

$$u_K^*(t)=\inf\{s>0:u_{*,K}(s)\le t\}=A^{-\frac{\beta}{N}}t^{\frac{\beta}{N}}$$

Since $\beta+\alpha p>0$ we can write

$$\left(v^{-1}\right)_{*,K}(s)=\lambda\Big\{x:x\in K,\varrho(e,x)^{-(\beta+\alpha p)}>s\Big\}.$$

$$=\lambda\{x:x\in K,\varrho(e,x)<s^{-\frac{1}{\beta+\alpha p}}\}=\lambda B(0,s^{-\frac{1}{\beta+\alpha p}})=Ar^{-\frac{N}{\beta+\alpha p}}.$$

and

$$\left(v^{-1}\right)_K^*(t)=\inf\{s>0:\left(v^{-1}\right)_{*,K}(s)\le t\}=A^{\frac{\beta+\alpha p}{N}}t^{-\frac{\beta+\alpha p}{N}}.$$

Examine (3.1) and (3.2). Since $\beta+\alpha p<N(p-1)$ we have $\frac{\beta+\alpha p}{N}-p<-1$ and $-\frac{\beta+\alpha p}{N}(p'-1)>-1$. Then

$$\sup_{s>0}\left(\int_s^{+\infty}u_K^*(t)t^{-q(1-\frac{\alpha}{N})}dt\right)^{\frac{1}{q}}\left(\int_0^s\left(\left(v^{-1}\right)_K^*(t)\right)^{p'-1}dt\right)^{\frac{1}{p'}}$$

$$\le\sup_{s>0}\left(\int_s^{+\infty}A^{-\frac{\beta}{N}}t^{\frac{\beta}{N}}t^{-p(1-\frac{\alpha}{N})}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^s\left(A^{\frac{\beta+\alpha p}{N}}t^{-\frac{\beta+\alpha p}{N}}\right)^{p'-1}dt\right)^{\frac{1}{p'}}$$

$$=A^{\frac{\alpha}{N}}\sup_{s>0}\left(\frac{1}{\frac{\beta+\alpha p}{N}-p+1}t^{\frac{\beta+\alpha p}{N}-p+1}\Big|_s^{+\infty}\right)^{\frac{1}{p}}$$

$$\times \left(\frac{1}{-\frac{\beta+\alpha p}{N}(p'-1)+1} t^{-\frac{\beta+\alpha p}{N}(p'-1)+1} \Big|_0^s \right)^{\frac{1}{p'}}$$

$$= A^{\frac{\alpha}{N}} \left(\frac{-1}{\frac{\beta+\alpha p}{N} - p + 1} \right)^{\frac{1}{p}} \left(\frac{1}{-\frac{\beta+\alpha p}{N}(p'-1)+1} \right)^{\frac{1}{p'}}$$

$$\times \sup_{s>0} s^{\left(\frac{\beta+\alpha p}{N} - p+1\right)\frac{1}{p} + \left(-\frac{\beta+\alpha p}{N}(p'-1)+1\right)\frac{1}{p'}}$$

$$= A^{\frac{\alpha}{N}} \left(\frac{-1}{\frac{\beta+\alpha p}{N} - p + 1} \right)^{\frac{1}{p}} \left(\frac{1}{-\frac{\beta+\alpha p}{N}(p'-1)+1} \right)^{\frac{1}{p'}} < +\infty.$$

Now examine (3.2). Since $\beta+\alpha p > 0$ we have $\frac{\beta+\alpha p}{N} - 1 > -1$ and $-\frac{\beta+\alpha p}{N}(p'-1) - 1 < -1$. Then

$$\sup_{s>0} \left(\int_0^s u_K^*(t) t^{-q(\frac{1}{r} - \frac{\alpha}{N})} dt \right)^{\frac{1}{q}} \left(\int_s^{+\infty} \left((v^{-1})_K^*(t) \right)^{p'-1} t^{p'(\frac{1}{r}-1)} dt \right)^{\frac{1}{p'}}$$

$$= \sup_{s>0} \left(\int_0^s A^{-\frac{\beta}{N}} t^{\frac{\beta}{N}} t^{-p(\frac{1}{p} - \frac{\alpha}{N})} dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_s^{+\infty} \left(A^{\frac{\beta+\alpha p}{N}} t^{-\frac{\beta+\alpha p}{N}} \right)^{p'-1} t^{p'(\frac{1}{p}-1)} dt \right)^{\frac{1}{p'}}$$

$$= A^{\frac{\alpha}{N}} \sup_{s>0} \left(\frac{N}{\beta+\alpha p} t^{\frac{\beta+\alpha p}{N}} \,\big|_0^s \right)^{\frac{1}{p}}$$

$$\times \left(\frac{N}{(\beta+\alpha p)(1-p')} t^{-\frac{\beta+\alpha p}{N}(p'-1)} \,\big|_s^{+\infty} \right)^{\frac{1}{p'}}$$

$$= A^{\frac{\alpha}{N}} \frac{N}{\beta+\alpha p} \left(\frac{1}{p'-1} \right)^{\frac{1}{p'}}$$

$$\times \sup_{s>0} s^{\frac{\beta+\alpha p}{N}\frac{1}{p} - \frac{\beta+\alpha p}{N}\frac{p'-1}{p'}}$$

$$= A^{\frac{\alpha}{N}} \frac{N}{\beta+\alpha p} \left(\frac{1}{p'-1} \right)^{\frac{1}{p'}} < +\infty.$$

Therefore (3.1) and (3.2) are satisfying and from Theorem 3.1, we have the result of the theorem.

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