Transactions of NAS of Azerbaijan, Issue Mathematics, **38** (1), 79–94 (2018)
Series of Physical-Technical and Mathematical Sciences

Sublinear operators with rough kernel generated by Calderon-Zygmund operators and their commutators on generalized weighted Morrey spaces

Vugar H. Hamzayev

Received: 12.04.2017 / Revised: 12.01.2018 / Accepted: 01.03.2018

Abstract. *In this paper, we study the boundedness of a large class of sublinear operators with rough kernel T_Ω on the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ for with $q' \le p < \infty$, $p \ne 1$ and $w \in A_{p/q'}$ or $1 < p < q$ and $w^{1-p'} \in A_{p'/q'}$, where $\Omega \in L_q(S^{n-1})$ with $q > 1$ be homogeneous of degree zero. In the case when $b \in BMO$, $1 < p < \infty$ and $T_{\Omega,b}$ be is a sublinear commutator operator, we find the sufficient conditions on the pair (φ_1, φ_2) and $q' \le p < \infty$, $p \ne 1$, $w \in A_{p/q'}$ or $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the operators $T_{\Omega,b}$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1 < p < \infty$.*

Keywords. Sublinear operator · Calderón-Zygmund operator · rough kernel · generalized weighted Morrey spaces · commutator · A_p weights.

Mathematics Subject Classification (2010): 42B25 · 42B35

1 Introduction

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [3,4] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [8] states that the commutator operator $[b, K]f = K(bf) - b\,Kf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [5–7, 10, 11]).

The classical Morrey spaces were originally introduced by Morrey in [26] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [10, 11, 13, 26]. Mizuhara [25] introduced generalized Morrey spaces. Later, Guliyev [13] defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm. Recently, Komori and Shirai [23] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [14] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and

V.H. Hamzayev
Nakhchivan Teacher-Training Institute, Nakhchivan, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9, B.Vahabzade str., AZ1141, Baku, Azerbaijan,
E-mail: vugarhamzayev@yahoo.com

weighted Morrey space $L^{p,\kappa}(w)$. In [14] Guliyev also studied the boundedness of the classical operators and its commutators in these spaces $M_{p,\varphi}(w)$, see also Guliyev et al. [18, 19, 22].

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r, ${}^{\complement}B(x,r)$ denote its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$. Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$.

Let $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous of degree zero. Suppose that T_Ω represents a linear or a sublinear operator, such that that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_\Omega f(x)| \leq c_0 \int\limits_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} \, |f(y)| dy, \tag{1.1}$$

where c_0 is independent of f and x.

For a function b, suppose that the commutator operator $T_{\Omega,b}$ represents a linear or a sublinear operator, such that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_{\Omega,b} f(x)| \leq c_0 \int_{\mathbb{R}^n} |b(x)-b(y)| \, \frac{|\Omega(x-y)|}{|x-y|^n} \, |f(y)| dy, \tag{1.2}$$

where c_0 is independent of f and x.

We point out that the condition (1.1) in the case $\Omega \equiv 1$ was first introduced by Soria and Weiss in [28]. The condition (1.1) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operators, Carleson's maximal operator, Hardy–Littlewood maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, the Bochner–Riesz means and so on (see [24], [28] for details).

The following statement, was proved in [22], see also [14, 18].

Theorem 1.1 *Let* $1 \leq p < \infty$, $w \in A_p$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leq C \, \varphi_2(x,r), \tag{1.3}$$

where C *does not depend on* x *and* r. *Let* $T \equiv T_1$ *be a sublinear operator satisfying condition* (1.1) *with* $\Omega \equiv 1$ *bounded on* $L_{p,w}(\mathbb{R}^n)$ *for* $p > 1$, *and bounded from* $L_{1,w}(\mathbb{R}^n)$ *to* $WL_{1,w}(\mathbb{R}^n)$. *Then the operator* T *is bounded from* $M_{p,\varphi_1}(w)$ *to* $M_{p,\varphi_2}(w)$ *for* $p > 1$ *and from* $M_{1,\varphi_1}(w)$ *to* $WM_{1,\varphi_2}(w)$.

The following statement, was proved in [18], see also [14].

Theorem 1.2 *Let* $1 < p < \infty$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leq C \, \varphi_2(x,r), \tag{1.4}$$

where C *does not depend on* x *and* r. *Let* $T_b \equiv T_{1,b}$ *be a sublinear commutator operator satisfying condition* (1.2) *with* $\Omega \equiv 1$ *bounded on* $L_{p,w}(\mathbb{R}^n)$. *Then the operator* T_b *is bounded from* $M_{p,\varphi_1}(w)$ *to* $M_{p,\varphi_2}(w)$.

Note that, in the case $w = 1$ Theorem 1.1 was proved in [15] and for the operators M and K in [1].

Watson [29] and independently by Duoandikoetxea [9] established weighted L_p boundedness for the singular integral operators with rough kernels and their commutators.

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$.

Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{1.5}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.6}$$

where $x' = x/|x|$ for any $x \neq 0$.

The singular integral operator with homogeneous kernel $\overline{T}_\Omega$ is defined by

$$\overline{T}_\Omega(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy, \tag{1.7}$$

where Ω is homogeneous of degree zero.

Suppose that $\overline{T}_\Omega$ is a singular integral operator defined by (1.7). Let Ω be a homogeneous of degree zero on $\mathbb{R}^n$. Let $\overline{T}_{\Omega,\varepsilon}$ is the truncated operator of T_Ω defined by

$$\overline{T}_{\Omega,\varepsilon}(f)(x) = \int_{\{y\in\mathbb{R}^n:|x-y|\geq\varepsilon\}} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy, \quad \varepsilon > 0. \tag{1.8}$$

Then the operator of T^*_Ω defined by

$$\overline{T}^*_\Omega(f)(x) = \sup_{\varepsilon>0} \left|\overline{T}_{\Omega,\varepsilon}(f)(x)\right| \tag{1.9}$$

is called the maximal singular integral operator. Therefore, it will be an interesting thing to study the property of $\overline{T}^*_\Omega$. The main purpose of this paper is to show that singular integral operators with rough kernels $\overline{T}_\Omega$ are bounded from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$, $1 < p < \infty$.

The commutator of the singular integral operators with rough kernels $\overline{T}_\Omega$ is defined by

$$[b, \overline{T}_\Omega](f)(x) = p.v. \int_{\mathbb{R}^n} [b(x) - b(y)]\, \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy, \tag{1.10}$$

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The maximal operator with rough kernel M_Ω is defined by

$$M_\Omega f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |\Omega(x-y)|\, |f(y)| dy.$$

It is obvious that when $\Omega \equiv 1$, M_Ω is the Hardy-Littlewood maximal operator M. For $b \in L_1^{\rm loc}(\mathbb{R}^n)$ the commutator of the maximal operator $M_{\Omega,b}$ is defined by

$$M_{\Omega,b} f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |b(x) - b(y)|\, |\Omega(x-y)|\, |f(y)| dy. \tag{1.11}$$

Theorem 1.3 *([9]) Suppose that Ω satisfies the conditions* (1.5) *and* $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$. *Then for every* $q' \le p < \infty$, $p \ne 1$ *and* $w \in A_{p/q'}$ *or* $1 < p \le q$, $p \ne \infty$ *and* $w^{1-p'} \in A_{p'/q'}$, *there is a constant* C *independent of* f *such that*

$$\|M_\Omega(f)\|_{L_{p,w}} \le C\|f\|_{L_{p,w}}.$$

Theorem 1.4 *([2]) Suppose that Ω satisfies the conditions* (1.5) *and* $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$. *Let also* $b \in BMO(\mathbb{R}^n)$. *Then for every* $q' \le p < \infty$, $p \ne 1$ *and* $w \in A_{p/q'}$ *or* $1 < p \le q$, $p \ne \infty$ *and* $w^{1-p'} \in A_{p'/q'}$, *there is a constant* C *independent of* f *such that*

$$\|M_{\Omega,b}(f)\|_{L_{p,w}} \le C\|f\|_{L_{p,w}}.$$

Theorem 1.5 *([9, 29]) Suppose that Ω satisfies the conditions* (1.5), (1.6) *and* $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$. *Then for every* $q' \le p < \infty$, $p \ne 1$ *and* $w \in A_{p/q'}$ *or* $1 < p \le q$, $p \ne \infty$ *and* $w^{1-p'} \in A_{p'/q'}$, *there is a constant* C *independent of* f *such that*

$$\|T_\Omega(f)\|_{L_{p,w}} \le C\|f\|_{L_{p,w}}.$$

Theorem 1.6 *([9, 29]) Suppose that Ω satisfies the conditions* (1.5), (1.6) *and* $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$. *Let also* $b \in BMO(\mathbb{R}^n)$. *Then for every* $q' \le p < \infty$, $p \ne 1$ *and* $w \in A_{p/q'}$ *or* $1 < p \le q$, $p \ne \infty$ *and* $w^{1-p'} \in A_{p'/q'}$, *there is a constant* C *independent of* f *such that*

$$\big\|[b, T_\Omega](f)\big\|_{L_{p,w}} \le C\|f\|_{L_{p,w}}.$$

In [20] the authors was studied the boundedness of the singular integral operators with rough kernels $\overline{T}_\Omega$ and its commutators $[b, \overline{T}_\Omega]$ on generalized weighted Morrey spaces $M_{p,\varphi}(w)$. In this work, we prove the boundedness of the sublinear operators T_Ω satisfies the condition (1.1) generated by the Calderon-Zygmund operators from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ with $q' \le p < \infty$, $p \ne 1$, $w \in A_{p/q'}$ or $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$. We find the sufficient conditions on the pair (φ_1, φ_2) with $b \in BMO(\mathbb{R}^n)$ and $q' \le p < \infty$, $p \ne 1$, $w \in A_{p/q'}$ or $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the commutator operators $T_{\Omega,b}$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1 < p < \infty$. Note that, in [16] was studied the boundedness of the operators $\overline{T}_\Omega$ and $[b, \overline{T}_\Omega]$ on generalized Morrey spaces $M_{p,\varphi}$.

By $A \lesssim D$ we mean that $A \le CD$ with some positive constant C independent of appropriate quantities. If $A \lesssim D$ and $D \lesssim A$, we write $A \approx D$ and say that A and D are equivalent.

2 Generalized weighted Morrey spaces

We recall that a weight function w is in the Muckenhoupt class A_p [27], $1 < p < \infty$, if

$$\begin{aligned} [w]_{A_p} :&= \sup_B [w]_{A_p(B)} \\ &= \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1} \end{aligned} \tag{2.1}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls B using Hölder's inequality, we have that

$$[w]_{A_p(B)}^{1/p} = |B|^{-1}\|w\|_{L_1(B)}^{1/p}\,\|w^{-1/p}\|_{L_{p'}(B)} \ge 1. \tag{2.2}$$

For $p = 1$, the class A_1 is defined by the condition $Mw(x) \le Cw(x)$ with $[w]_{A_1} = \sup\limits_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$ $A_\infty = \bigcup_{1\le p<\infty} A_p$ and $[w]_{A_\infty} = \inf\limits_{1\le p<\infty} [w]_{A_p}$.

Remark 2.1 It is known that

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}(B)}^{q'/p'} = |B|^{-1} \|w^{1-p'}\|_{L_1(B)}^{q'/p'} \, \|w^{q'/p}\|_{L_{(p'/q')'}(B)}.$$

Moreover, we can write $w^{1-p'} \in A_{p'/q'} \Rightarrow w^{1-p'} \in A_{p'}$ because of $w^{1-p'} \in A_{p'/q'} \subset A_{p'}$. Therefore, we get

$$\begin{aligned} w^{1-p'} \in A_{p'/q'} &\Rightarrow w^{1-p'} \in A_{p'} \\ &\Rightarrow [w^{1-p'}]_{A_{p'}(B)}^{1/p'} = |B|^{-1} \|w^{1-p'}\|_{L_1(B)}^{1/p'} \, \|w^{1/p}\|_{L_p(B)}. \end{aligned} \tag{2.3}$$

But the opposite is not true.

Remark 2.2 Let's write $w^{1-p'} \in A_{p'/q'}$ and used the definitions A_p classes we get the following

$$\begin{aligned} w^{1-p'} \in A_{p'/q'} &\Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{\frac{q(p-1)}{p(q-1)}} = |B|^{-1} \|w^{1-p'}\|_{L_1(B)}^{\frac{q(p-1)}{p(q-1)}} \, \|w^{q'/p}\|_{L_{(p'/q')'}(B)} \\ &\Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{1/p'} = |B|^{-\frac{q-1}{q}} \|w^{1-p'}\|_{L_1(B)}^{1/p'} \, \|w\|_{L_{\frac{q}{q-p}}(B)}^{1/p}, \end{aligned} \tag{2.4}$$

where the following equalities are provided.

$$1-p' = -\frac{p'}{p}, \;\; \frac{q'}{p} = \frac{q}{p(q-1)}, \;\; \frac{q'}{p'} = \frac{q(p-1)}{p(q-1)}, \;\; \left(\frac{q}{p}\right)' = \frac{q}{q-p}, \;\; \left(\frac{p'}{q'}\right)' = \frac{p(q-1)}{q-p}.$$

Then from eq.(2.3) and eq.(2.4) we have

$$\begin{aligned} w^{1-p'} \in A_{p'/q'} &\Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{1/p'} \\ &= |B|^{\frac{1}{q}} [w^{1-p'}]_{A_{p'}(B)}^{1/p'} \, \|w^{1/p}\|_{L_p(B)}^{-1} \, \|w\|_{L_{\frac{q}{q-p}}(B)}^{1/p}. \end{aligned} \tag{2.5}$$

We define the generalized weighed Morrey spaces as follows.

Definition 2.1 *[14] Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L_{p,w}^{\mathrm{loc}}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, w(B(x,r))^{-\frac{1}{p}} \, \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL_{p,w}^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, w(B(x,r))^{-\frac{1}{p}} \, \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\, |f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

Remark 2.3 (1) If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(v,w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(5) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

3 Sublinear operators with rough kernel generated by Calderón-Zygmund operators in the spaces $M_{p,\varphi}(w)$

We will use the following statements on the boundedness of the weighted Hardy operators

$$H_w g(r) := \int_r^\infty g(t)w(t)dt,\ 0 < t < \infty$$

and

$$H_w^* g(r) := \int_r^\infty \big(1 + \ln \frac{t}{r}\big)\, g(t)w(t)dt,\ 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.
The following theorem was proved in [16, 17].

Theorem 3.1 *[16, 17] Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{3.1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (3.1).

The following theorem was proved in [14].

Theorem 3.2 *[14] Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{r>0} v_2(r) H_w^* g(r) \le C \operatorname*{ess\,sup}_{r>0} v_1(r) g(r) \tag{3.2}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int_r^\infty \big(1 + \ln \frac{t}{r}\big) \frac{w(t)dt}{\sup_{t<s<\infty} v_1(s)} < \infty. \tag{3.3}$$

Moreover, the value $C = B$ is the best constant for (3.1).

Remark 3.1 In (3.1) – (3.3) it is assumed that $0 \cdot \infty = 0$.

In the following lemma we get local estimate (see, for example, [12, 13] in the case $w = 1$ and [14] in the case $w \in A_p$) for the operator T_Ω.

Lemma 3.1 *Let* $1 \le p < \infty$, T_Ω *be a sublinear operator satisfying condition* (1.1) *with* $\Omega \in L_q(S^{n-1})$, $q > 1$, *be a homogeneous of degree zero, and bounded on* $L_{p,w}(\mathbb{R}^n)$ *for* $p > 1$, *and bounded from* $L_{1,w}(\mathbb{R}^n)$ *to* $WL_{1,w}(\mathbb{R}^n)$.

If $q' \le p < \infty$, $p \ne 1$ *and* $w \in A_{p/q'}$, *then the inequality*

$$\|T_\Omega(f)\|_{L_{p,w}(B(x_0,r))} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

If $1 < p \le q$, $p \ne \infty$ *and* $w^{1-p'} \in A_{p'/q'}$, *then the inequality*

$$\|T_\Omega(f)\|_{L_{p,w}(B(x_0,r))} \lesssim \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x_0,r))} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, \|w\|^{-1/p}_{L_{\frac{q}{q-p}}(B(x_0,t))} \, \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

Proof. Let Ω be satisfies the conditions (1.5), (1.6) and $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$.

Note that

$$\|\Omega(x-\cdot)\|_{L_q(B(x_0,t))} \le c_0 \, \|\Omega\|_{L_q(S^{n-1})} \, |B(0, t+|x-x_0|)|^{\frac{1}{q}},$$

where $c_0 = (n v_n)^{-1/q}$ and $v_n = |B(0,1)|$ (see, [20]).

For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \;\; f_1(y) = f(y)\chi_{2B}(y), \;\; f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \;\; r > 0 \qquad (3.4)$$

and have

$$\|T_\Omega(f)\|_{L_{p,w}(B)} \le \|T_\Omega(f_1)\|_{L_{p,w}(B)} + \|T_\Omega(f_2)\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $T_\Omega\,(f_1) \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of T_Ω in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_{p/q'}$ and $q' \le p < \infty$, $p \ne 1$ (see Theorem 1.5) it follows that

$$\begin{aligned}\|T_\Omega\,(f_1)\,\|_{L_{p,w}(B)} &\le \|T_\Omega\,(f_1)\,\|_{L_{p,w}(\mathbb{R}^n)} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,\|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\approx \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,\|f\|_{L_{p,w}(2B)}.\end{aligned}$$

It's clear that $x \in B$, $y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0-y| \le |x-y| \le \frac{3}{2}|x_0-y|$. Then by the Minkowski inequality and conditions on Ω, we get

$$T_\Omega(f_2(x)) \lesssim \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n} dy.$$

By Fubini's theorem we have

$$\int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}dy \approx \int_{\complement_{(2B)}} |\Omega(x-y)|\,|f(y)| \int_{|x-y|}^{\infty} \frac{dt}{t^{n+1}}dy$$
$$= \int_{2r}^{\infty} \int_{2r\le|x_0-y|<t} |\Omega(x-y)|\,|f(y)|dy\frac{dt}{t^{n+1}}$$
$$\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)|\,|f(y)|dy\frac{dt}{t^{n+1}}\ .$$

By applying Hölder's inequality for $q' \le p < \infty$, $p \neq 1$ and $w \in A_{p/q'}$, we get

$$\int_{\complement_{(2B)}} \frac{|\Omega(x-y)|\,|f(y)|}{|x_0-y|^n}dy \lesssim \int_{2r}^{\infty} \|\Omega(x-\cdot)\|_{L_q(B(x_0,t))}\,\|f\|_{L_{q'}(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,\|w^{-q'/p}\|^{\frac{1}{q'}}_{L_{(p/q')'}(B(x_0,t))}\,|B(0,t+|x-x_0|)|^{\frac{1}{q}}\,\frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,|B(x_0,t)|^{\frac{1}{q'}}\,|B(0,t)|^{\frac{1}{q}}\,\frac{dt}{t^{n+1}}$$
$$\approx \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}. \qquad (3.5)$$

Moreover, for all $q' \le p < \infty$, $p \neq 1$ the inequality

$$\|T_\Omega\,(f_2)\,\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}.$$

is valid. Thus

$$\|T_\Omega(f)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \Big(\|f\|_{L_{p,w}(2B)}$$
$$+\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}\Big).$$

On the other hand,

$$\|f\|_{L_{p,w}(2B)} \approx |B|\|f\|_{L_{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}}$$
$$\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$\lesssim w(B)^{\frac{1}{p}}\|w^{-1/p}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,\|w^{-1/p}\|_{L_{p'}(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$\lesssim [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}. \qquad (3.6)$$

Thus

$$\|T_\Omega(f)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}$$
$$\lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}.$$

Let also $1 < p \le q$, $p \neq \infty$ and $w^{1-p'} \in A_{p'/q'}$. Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $T_\Omega(f_1) \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of T_Ω in $L_{p,w}(\mathbb{R}^n)$ for $w^{1-p'} \in A_{p'/q'}$ and $1 < p < q$ (see Theorem 1.5) it follows that

$$\|T_\Omega(f_1)\|_{L_{p,w}(B)} \le \|T_\Omega(f_1)\|_{L_{p,w}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L_q(S^{n-1})} [w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}} \|f_1\|_{L_{p,w}(\mathbb{R}^n)}$$
$$\approx \|\Omega\|_{L_q(S^{n-1})} [w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}} \|f\|_{L_{p,w}(2B)}.$$

If $1 < p \le q$, $p \neq \infty$ and $w^{1-p'} \in A_{p'/q'}$, then Minkowski theorem and Hölder inequality,

$$\|T_\Omega(f_2)\|_{L_{p,w}(B)} \le \left(\int_B \left(\int_{2r}^\infty \int_{B(x_0,t)} |\Omega(x-y)||f(y)|dy \frac{dt}{t^{n+1}}\right)^p w(x)dx\right)^{\frac{1}{p}}$$
$$\le \int_{2r}^\infty \int_{B(x_0,t)} \|\Omega(\cdot - y)\|_{L_{p,w}(B)} |f(y)|\, dy \frac{dt}{t^{n+1}}$$
$$\lesssim \int_{2r}^\infty \int_{B(x_0,t)} \|\Omega(\cdot - y)\|_{L_q(B)} \|w\|_{L_{(q/p)'}(B)}^{\frac{1}{p}} |f(y)|\, dy \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} \|w\|_{L_{(q/p)'}(B)}^{\frac{1}{p}} \int_{2r}^\infty \int_{B(x_0,t)} |B(0, r + |x_0 - y|)|^{\frac{1}{q}} |f(y)|\, dy \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} \|w\|_{L_{(q/p)'}(B)}^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_1(B(x_0,t))} |B(0, r+t)|^{\frac{1}{q}} \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-p'/p}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}} |B(x_0,t)|^{\frac{1}{q}} \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} |B|^{\frac{1}{q}} \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \|w^{1-p'}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}} |B(x_0,t)|^{\frac{1}{q}} \frac{dt}{t^{n+1}}$$

is obtained. By applying (2.3) for $\|w^{1-p'}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}}$ and (2.5) for $\|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}}$ we have the following inequality

$$\|T_\Omega(f_2)\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} [w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}} \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}$$

is valid. Thus

$$\|T_\Omega(f)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})} [w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}} \Big(\|f\|_{L_{p,w}(2B)}$$
$$+ \|w\|_{L^{\frac{1}{p}}_{\frac{q}{q-p}}(B)} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}\Big).$$

On the other hand,

$$\|f\|_{L_{p,w}(2B)} \approx |B| \|f\|_{L_{p,w}(2B)} \int_{2r}^\infty \frac{dt}{t^{n+1}}$$

$$\begin{aligned}
&\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&= [w^{1-p'}]_{A_{p'}(B)}^{-\frac{1}{p'}} |B|^{\frac{1}{q}} \|w^{1-p'}\|_{L_1(B)}^{\frac{1}{p'}} \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,\omega}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim [w^{1-p'}]_{A_{p'}(B)}^{-\frac{1}{p'}} \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,\omega}(B(x_0,t))} |B(x_0,t)|^{\frac{1}{q}} \|w^{1-p'}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}} \frac{dt}{t^{n+1}} \\
&\lesssim \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,\omega}(B(x_0,t))} \|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|T_\Omega(f)\|_{L_{p,w}(B)} \\
&\lesssim \|\Omega\|_{L_q(S^{n-1})} [w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}} \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned}$$

Thus we complete the proof of Lemma 3.1.

Theorem 3.3 *Let $1 \le p < \infty$, T_Ω be a sublinear operator satisfying condition* (1.1) *with $\Omega \in L_q(S^{n-1})$, $q > 1$, be a homogeneous of degree zero, and bounded on $L_{p,w}(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_{1,w}(\mathbb{R}^n)$ to $WL_{1,w}(\mathbb{R}^n)$. Let also, for $q' < p < \infty$, $w \in A_{p/q'}$ the pair (φ_1, φ_2) satisfies the condition* (1.3) *and for $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$ the pair (φ_1, φ_2) satisfies the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) \|w\|_{L_{\frac{q}{q-p}}(B(x,\tau))}^{1/p}}{\|w\|_{L_{\frac{q}{q-p}}(B(x,t))}^{1/p}} \frac{dt}{t} \le C\, \varphi_2(x,r) \frac{w(B(x,r))^{\frac{1}{p}}}{\|w\|_{L_{\frac{q}{q-p}}(B(x,r))}^{\frac{1}{p}}}, \tag{3.7}$$

where C does not depend on x and r.

Then the operator T_Ω is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover

$$\|T_\Omega(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. When $q' < p < \infty$, $w \in A_{p/q'}$, by Lemma 3.1 and Theorem 3.1 with $\nu_2(r) = \varphi_2(x,r)^{-1}$, $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = w(B(x,r))^{-\frac{1}{p}} r^{-1}$ we have

$$\begin{aligned}
\|T_\Omega(f)\|_{M_{p,\varphi_2}(w)} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|\mu_\Omega(f)\|_{L_{p,w}(B(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))} \\
&= \|f\|_{M_{p,\varphi_1}(w)}.
\end{aligned}$$

For the case of $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$, by Lemma 3.1 and Theorem 3.1 with $\nu_2(r) = \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|w\|_{L_{\frac{q}{q-p}}(B(x,r))}^{\frac{1}{p}}$, $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$,

$g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = \|w\|_{L_{\frac{q}{q-p}}(B(x,r))}^{-\frac{1}{p}} r^{-1}$ we have

$$\|T_\Omega(f)\|_{M_{p,\varphi_2}(w)} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|\mu_\Omega(f)\|_{L_{p,w}(B(x,r))}$$
$$\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \int_r^\infty \|f\|_{L_{p,w}(B(x_0,t))}\, \|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}$$
$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))}$$
$$= \|f\|_{M_{p,\varphi_1}(w)}.$$

4 Commutators of linear operators with rough kernel generated by Calderón-Zygmund operators in the spaces $M_{p,\varphi}(w)$

Remark 4.1 ([21])

(1) The John-Nirenberg inequality : There are constants C_1, $C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B \,:\, |b(x) - b_B| > \beta\}| \le C_1|B|e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy\right)^{\frac{1}{p}} \tag{4.1}$$

for $1 < p < \infty$.

(3) Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_* \ln\frac{t}{r} \;\text{ for }\; 0 < 2r < t, \tag{4.2}$$

where C is independent of b, x, r and t.

In the following lemma we get local estimate (see, for example, [14]) for the commutator operator $T_{\Omega,b}$.

Lemma 4.1 *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$, T_Ω be a sublinear operator satisfying condition* (1.1) *with $\Omega \in L_q(S^{n-1})$, $q > 1$, be a homogeneous of degree zero, and bounded on $L_{p,w}(\mathbb{R}^n)$.*

If $q' < p < \infty$ and $w \in A_{p/q'}$, then the inequality

$$\|T_{\Omega,b}(f)\|_{L_{p,w}(B(x_0,r))}$$
$$\lesssim \|b\|_*\, w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^\infty \left(1 + \ln\frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, and for all $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$.

If $1 < p < q$ and $w^{1-p'} \in A_{p'/q'}$, then the inequality

$$\|T_{\Omega,b}(f)\|_{L_{p,w}(B(x_0,r))}$$
$$\lesssim \|w\|_{L_{\frac{q}{q-p}}(B(x_0,r))}^{1/p} \int_{2r}^\infty \left(1 + \ln\frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))}\, \|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-1/p} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, and for all $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $p \in (1, \infty)$ and $b \in BMO(\mathbb{R}^n)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0, 2r)$. We represent f as (3.4) and have

$$\|T_{\Omega,b}(f)\|_{L_{p,w}(B)} \leq \|T_{\Omega,b}(f_1)\|_{L_{p,w}(B)} + \|T_{\Omega,b}(f_2)\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $T_{\Omega,b}(f_1) \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of $T_{\Omega,b}$ in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_{p/q'}$ and $q' < p < \infty$ (see Theorem 1.6) it follows that

$$\begin{aligned}\|T_{\Omega,b}(f_1)\|_{L_{p,w}(B)} &\leq \|T_{\Omega,b}(f_1)\|_{L_{p,w}(\mathbb{R}^n)} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})} [w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}} \|b\|_* \|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\approx \|\Omega\|_{L_q(S^{n-1})} [w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}} \|b\|_* \|f\|_{L_{p,w}(2B)}.\end{aligned}$$

For $x \in B$ we have

$$T_{\Omega,b}(f_2(x)) \lesssim \int_{\complement_{(2B)}} |b(y) - b(x)||\Omega(x-y)| \frac{|f(y)|}{|x_0 - y|^n} dy.$$

Then

$$\begin{aligned}&\|T_{\Omega,b}(f_2)\|_{L_{p,w}(B)} \\ &\lesssim \left(\int_B \left(\int_{\complement_{(2B)}} |b(y) - b(x)||\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^n}\, dy\right)^p w(x)dx\right)^{\frac{1}{p}} \\ &\lesssim \left(\int_B \left(\int_{\complement_{(2B)}} |b(y) - b_{B,w}||\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^n}\, dy\right)^p w(x)dx\right)^{\frac{1}{p}} \\ &\quad + \left(\int_B \left(\int_{\complement_{(2B)}} |b(x) - b_{B,w}||\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^n}\, dy\right)^p w(x)dx\right)^{\frac{1}{p}} \\ &= I_1 + I_2.\end{aligned}$$

Let us estimate I_1.

$$\begin{aligned}I_1 &= w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} |b(y) - b_{B,w}||\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^n}\, dy \\ &\approx w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} |b(y) - b_{B,w}||\Omega(x-y)||f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}}\, dy \\ &\approx w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \int_{2r \leq |x_0-y| \leq t} |b(y) - b_{B,w}||\Omega(x-y)||f(y)|\, dy \frac{dt}{t^{n+1}} \\ &\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B,w}||\Omega(x-y)||f(y)| dy \frac{dt}{t^{n+1}}.\end{aligned}$$

Set $m = p/q' > 1$. Since $w \in A_m$, from (2.3), we know $w^{1-m'} \in A_{m'}$. Applying Hölder's inequality and by (4.2), we get

$$I_1 \lesssim \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|\Omega(x-\cdot)\|_{L_q(B(x_0,t))} \, \||b(y)-b_{B,w}|f\|_{L_{q'}(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|b-b_{B,w}\|_{L_{m'q',w^{1-m'}}(B(x_0,t))}$$
$$\times \|f\|_{L_{p,w}(B(x_0,t))} \, |B(x_0, t+|x-x_0|)|^{\frac{1}{q}} \, \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \, (w^{1-m'}(B(x_0,t)))^{\frac{1}{m'q'}}$$
$$\times \|f\|_{L_{p,w}(B(x_0,t))} \, |B(x_0,t)| \, \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} [w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}} \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t}.$$

In order to estimate I_2 note that

$$I_2 = \Big(\int_B |b(x)-b_{B,w}|^p w(x) dx\Big)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n} \, dy.$$

By (3.5) and (4.2), we get

$$I_2 \lesssim \|b\|_* \, w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n} \, dy$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} [w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}} \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))} \, w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t}.$$

Summing up I_1 and I_2, for all $p \in (1,\infty)$ we get

$$\|T_{\Omega,b}(f_2)\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} [w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}} \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}.$$

Thus

$$\|T_{\Omega,b}(f)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})} \, [w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}} \, \|b\|_* \Big(\|f\|_{L_{p,w}(2B)}$$
$$+ \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}\Big).$$

On the other hand, by (3.6) we get

$$\|T_{\Omega,b}(f)\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} \, [w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}} \, \|b\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t}$$
$$\lesssim \|b\|_* \, w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-\frac{1}{p}} \, \frac{dt}{t}.$$

With similar techniques for $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$ can be achieved and the proof is finished.

Theorem 4.1 *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$, T_Ω be a sublinear operator satisfying condition* (1.1) *with $\Omega \in L_q(S^{n-1})$, $q > 1$, be a homogeneous of degree zero, and bounded on $L_{p,w}(\mathbb{R}^n)$. Let also, for $q' < p < \infty$, $w \in A_{p/q'}$ the pair (φ_1, φ_2) satisfies the condition* (1.4) *and for $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$ the pair (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,\tau))}}{\|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,t))}} \frac{dt}{t} \le C\, \varphi_2(x,r)\, \frac{w(B(x,r))^{\frac{1}{p}}}{\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}}, \quad (4.3)$$

where C does not depend on x and r.

Then the operator $T_{\Omega,b}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.

$$\|T_{\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. When $q' < p < \infty$, $w \in A_{p/q'}$, by Lemma 4.1 and Theorem 3.2 with $\nu_2(r) = \varphi_2(x,r)^{-1}$, $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = w(B(x,r))^{-\frac{1}{p}} r^{-1}$ we have

$$\begin{aligned}
\|T_{\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|\mu_{\Omega,b}(f)\|_{L_{p,w}(B(x,r))} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x,t))}\, w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))} \\
&= \|b\|_* \|f\|_{M_{p,\varphi_1}(w)}.
\end{aligned}$$

For the case of $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$, by Lemma 3.1 and Theorem 3.2 with $\nu_2(r) = \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}$, $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))} r^{-1}$ we have

$$\begin{aligned}
\|T_{\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|\mu_\Omega(f)\|_{L_{p,w}(B(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \\
&\times \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))}\, \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x_0,t))} \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))} \\
&= \|f\|_{M_{p,\varphi_1}(w)}.
\end{aligned}$$

References

1. Akbulut, A., Guliyev V.S., Mustafayev, R.: *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem. **137** (1), 27–43 (2012).

2. Alvarez, J., Bagby, R.J., Kurtz, D.S., Pérez, C.: *Weighted estimates for commutators of linear operators*, Studia Math. **104**, 195–209 (1993).
3. Calderon, A.P.: *Commutators of singular integral operators*, Proc. Natl. Acad. Sci. USA, **53**, 1092–1099 (1965).
4. Calderon, A.P.: *Cauchy integrals on Lipschitz curves and related operators*, Proc. Natl. Acad. Sci. USA **74**(4), 1324–1327 (1977).
5. Chiarenza, F., Frasca, M.: *Morrey spaces and Hardy-Littlewood maximal function*, Rend Mat. **7**, 273–279 (1987).
6. Chiarenza, F., Frasca, M., Longo, P.: *$W^{2,p}$-solvability of Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336**, 841–853 (1993).
7. Chen, Y.: *Regularity of solutions to elliptic equations with VMO coefficients*, Acta Math. Sin. (Engl. Ser.) **20**, 1103–1118 (2004).
8. Coifman, R., Rochberg, R., Weiss, G.: *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103**(2), 611–635 (1976).
9. Duoandikoetxea, J.: *Weighted norm inequalities for homogeneous singular integrals*, Trans. Amer. Math. Soc. **336**, 869–880 (1993).
10. Di Fazio G., Ragusa M.A.: *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112**, 241–256 (1993).
11. Fan, D., Lu, S., Yang, D.: *Boundedness of operators in Morrey spaces on homogeneous spaces and its applications*, Acta Math. Sinica (N. S.) **14**, 625–634 (1998).
12. Guliyev, V.S.: Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$, *Doctoral dissertation, Moscow, Mat. Inst. Steklov*, 329 pp. (1994) (in Russian).
13. Guliyev, V.S.: *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. *Art. ID 503948*, 20p. (2009).
14. Guliyev, V.S.: *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J. **3**(3), 33–61 (2012).
15. Guliyev, V.S., Aliyev S.S., Karaman, T.: *Boundedness of a class of sublinear operators and their commutators on generalized Morrey spaces*, Abstr. Appl. Anal. *Art. ID 356041*, 18 p. (2011).
16. Guliyev, V.S.: *Local generalized Morrey spaces and singular integrals with rough kernel*, Azerb. J. Math. **3** (2), 79–94 (2013).
17. Guliyev, V.S.: *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N. Y.) **193**(2), 211–227 (2013).
18. Guliyev, V.S., Karaman, T., Mustafayev, R.Ch., Serbetci, A.: *Commutators of sublinear operators generated by Calderón-Zygmund operator on generalized weighted Morrey spaces*, Czechoslovak Math. J. **64**, 139 (2), 365–386 (2014).
19. Guliyev, V.S., Omarova, M.N.: *Higher order commutators of vector-valued intrinsic square functions on vector-valued generalized weighted Morrey spaces*, Azerb. J. Math. **4**(2), 64–85 (2014).
20. Guliyev, V.S., Hamzayev, V.H.: *Rough singular integral operators and its commutators on generalized weighted Morrey spaces*, Math. Ineq. Appl. **19**(3), 863–881 (2016).
21. Janson, S.: *On functions with conditions on the mean oscillation*, Ark. Mat. **14** (2), 189–196 (1976).
22. Karaman, T., Guliyev, V.S., Serbetci, A.: *Boundedness of sublinear operators generated by Calderón-Zygmund operators on generalized weighted Morrey spaces*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.), **60**(1), 227–244 (2014).
23. Komori, Y., Shirai, S.: *Weighted Morrey spaces and a singular integral operator*, Math. Nachr. **282**(2), 219–231 (2009).
24. Lu, G., Lu, S., Yang, D.: *Singular integrals and commutators on homogeneous groups*,

Anal. Math. **28**, 103–134 (2002).
25. Mizuhara, T.: *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo, 183–189 (1991).
26. MORREY, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43**, 126–166 (1938).
27. Muckenhoupt, B.: *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165**, 207–226 (1972).
28. Soria, F., Weiss, G.: *A remark on singular integrals and power weights*, Indiana Univ. Math. J. **43**, 187–204 (1994).
29. Watson, D.: *Weighted estimates for singular integrals via Fourier transform estimates*, Duke Math. J. **60**, 389–399 (1990).