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On the theory of necessary optimality conditions in discrete systems with a delay in control

Samin T. Malik

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Abstract. *In the present paper, a discrete optimization problem with a delay in control is considered. Taking into account the specific character of the discrete system, a necessary optimality condition which is not formulated in terms of the Hamilton-Pontryagin function is obtained with rather general input data (without assumptions of convexity and smoothness). It is shown by special example that the discrete maximum principle, generally speaking, does not hold even for the linear (with respect to a phase variable) discrete problem with delay in a control.*

Keywords. optimality conditions · discrete systems · maximum principle · delay in control

Mathematics Subject Classification (2010): 49J21 · 49J40 · 49L20 · 49M25 · 49M05

1 Introduction

Continuous problems of optimization with a delay in control were studied in sufficient detail and many important results were obtained (see: e.i [1, 3, 5, 6, 11, 12, 13]). Analysis of these results shows that such problems may be referred to the class of optimization problems with specifical features and principal difficulties in developing qualitative theory. Such a situation arises in the study of discrete problems with a delay in control. Taking into account this fact and the results of the papers [15, 16] direct transfer of analogs of some known necessary optimality conditions [7] for discrete problems with a delay in control, is impossible. Because, for studying discrete problems with a delay in control, such a feature requires a more subtle methodology that takes into account the specifies of the considered system. The present paper was devoted just to such a statement.

The paper is the generalization of [8] on a discrete problem with a delay in control. Here, not imposing assumptions on convexity and smoothness we obtain necessary condition for optimality not formulated by the Hamilton-Pontryagin function. This optimality condition allows to replace multi-dimensional minimization problem by the sequence of problems of less dimension. Furthermore, it is useful in implementing to narrow the set of controls suspicious for optimality, that were obtained by means of the results of the papers [15, 16 and so on]. In the paper we consider some particular optimization problems and obtain a new type necessary optimality condition. It is shown that for a linear (with respect to phase variable) problem with a delay in control, generally speaking, the discrete maximum principle does not hold.

Samin T. Malik
Institute of Mathematics and Mechanics, Baku, Azerbaijan
E-mail: saminmelik@gmail.com

The paper is structured as follows. In Section 2, we introduce the optimization problem. In Section 3, we obtain an increment formula of the objective functional to prove the necessary optimality conditions in the next section. Section 4 proves the main results of the present paper concern necessary optimality condition and gives some corollaries and an example. In section 5, we give some conclusions.

2 Problem statement

Let it be required to minimize the objective functional

$$S(u(\cdot)) = \Phi(x(t_1)) \to \min_u \tag{2.1}$$

with constraints

$$x(t+1) = f(x(t), u(t), u(t-h), t),\ t \in I := \{t_0,\ t_0+1, ..., t_1-1\}, x(t_0) = x_0, \tag{2.2}$$

$$u(t) \in U(t) \subseteq R^r,\ t \in \{t_0 - h, ..., t_0, ..., t_1 - 1\} =: I_h, \tag{2.3}$$

where $U(t),\ \ t \in I_h$, are the given subsets of $r-$ dimensional Euclidean space $R^r,\ R^1 =: R =: (-\infty, +\infty)$ and $u \in R^r$ is a control vector; $x \in R^n$ is a state vector, $x_0 \in R^n$ is a given vector; t is time (discrete); $t_0,\ t_1 \in R$ and $h \in N$ are given points such that $t_1 - t_0 > h$; $\Phi(x) : R^n \to R$ and $f(x, u, \hat{u}, t) : R^n \times R^r \times R^r \times I \to R^n$ are the given functions.

We present next the definition of an admissible control strategy and the associated notion of the controlled process. The controls satisfying condition (2.3) are called admissible. The pair $(u(\cdot), x(\cdot))$ is called an admissible process if $x(\cdot)$ is the solution of system (2.2) corresponding to the admissible control $u(\cdot)$. An admissible control $u(t), t \in I_h$, minimizing the functional (2.1) with constraints (2.2), (2.3) is called an optimal control, and the corresponding trajectory $x(t), t \in I_h$, of system (2.2) is called an optimal trajectory. In this case, the pair $(u(\cdot), x(\cdot))$ is called an optimal process.

In practice, there is a problem of the form (2.1)-(2.3). For example, we can assume that $u(t)$ is the rate of withdrawal of money for the acquisition and repair of equipment from a bank account, $x(t)$ is the amount of the size of withdrawal and company's operating capital at time t , h is a delay in the input of production capital, and the functional can describe the state of the bank account or the profit of the firm (in this case, the functional should be maximized).

3 Increment formula for quality functional

Let $(u^0(\cdot),\ x^0(\cdot))-$be some admissible process. We will determine the variation of the control $u^0(t), t \in I_h$ is the following way:

$$\tilde{u}(t) = \begin{cases} v,\ t = \theta, \\ u^0(t),\ t \in I_h \backslash \{\theta\}, \end{cases} \tag{3.1}$$

where $\theta \in I_h, v \in U(\theta)-$are arbitrary fixed points.

Obviously, $\tilde{u}(t), t \in I_h$ are admissible controls.

Also, with the process $(u^0(\cdot),\ x^0(\cdot))$ we consider another admissible process $(\tilde{u}(\cdot),\ \tilde{x}(\cdot))$. It is clear that the increment $\tilde{x}(t) - x^0(t) =: \Delta x(t), t \in \hat{I} := I \cup \{t_1\}-$ is the solution of the system.

$$\begin{cases} \Delta x(t+1) = f(x^0(t) + \Delta x(t), \tilde{u}(t), \tilde{u}(t-h), t) - f(t), \\ \Delta x(t_0) = 0, \end{cases} \tag{3.2}$$

where

$$f(t) := f(x^0(t), u^0(t), u^0(t-h), t),\ t \in I. \tag{3.3}$$

Let us consider the following cases.

Case 1 Let $\theta \in \{t_0 - h, t_0 - h + 1, ..., t_0 - 1\} =: I_0$. Then from (3.2) allowing for (3.1), (3.3), we have

$$\begin{cases} \Delta x(t+1) = f(x^0(t) + \Delta x(t),\ u^0(t), u^0(t-h), t) - f(t), \\ \qquad t \in \{\theta + h + 1, ..., t_1 - 1\}, \\ \Delta x(\theta + h + 1) = \Delta_{(u^0(\theta+h), v)} f(\theta + h), \\ \Delta x(t) = 0,\ t \in \{t_0, t_0 + 1, ..., \theta + h\}, \end{cases} \tag{3.4}$$

where

$$\Delta_{(u^0(\theta+h), v)} f(\theta + h) := f(x^0(\theta + h), u^0(\theta + h), v, \theta + h) - f(\theta + h). \tag{3.5}$$

Case 2 Let $\theta \in \{t_0, t_0 + 1, ..., t_1 - h - 1\} =: I_1$. Then the system (3.2), allowing for (3.1), (3.3), takes the form

$$\begin{cases} \Delta x(t+1) = f(x^0(t) + \Delta x(t),\ u^0(t), u^0(t-h), t) - f(t), \\ \qquad t \in \{\theta + h + 1, ..., t_1 - 1\}, \\ \Delta x(\theta + h + 1) = f(x^0(\theta + h) + \Delta x(\theta + h), u^0(\theta + h), v, \theta + h) - \\ \qquad - f(\theta + h), \\ \Delta x(t+1) = f(x^0(t) + \Delta x(t),\ u^0(t), u^0(t-h), t) - f(t), \\ \qquad t \in \{\theta + 1, ..., \theta + h - 1\}, \\ \Delta x(\theta + 1) = \Delta_{(v, u^0(\theta - h))} f(\theta), \\ \Delta x(t) = 0,\ t \in \{t_0, ..., \theta\}, \end{cases} \tag{3.6}$$

where

$$\Delta_{(v, u^0(\theta - h))} f(\theta) = f(x^0(\theta), v, u^0(\theta - h), \theta) - f(\theta). \tag{3.7}$$

Case 3 Let $\theta \in \{t_1 - h, ..., t_1 - 1\} =: I_2$. Then from (3.2) allowing for (3.1), (3.3), (3.7) we get

$$\begin{cases} \Delta x(t+1) = f(x^0(t) + \Delta x(t),\ u^0(t), u^0(t-h), t) \\ \qquad - f(t),\ t \in \{\theta + 1, ..., t_1 - 1\}, \\ \Delta x(\theta + 1) = \Delta_{(v, u^0(\theta - h))} f(\theta), \\ \Delta x(t) = 0,\ t \in \{t_0, ..., \theta\}. \end{cases} \tag{3.8}$$

Taking into accounts (3.1)-(3.8), it is easy to prove the following lemma.

Lemma 3.1 *If $\Delta x(t)$, $t \in I$, is the solution of the system (3.2), corresponding to the variation (3.1), then for every $i \in \{0, 1, 2\}$ the following equality is valid:*

$$\Delta x(t_1) = z^{(i)}(t_1; \theta, v), \forall (\theta, v) \in I_i \times U(\theta). \tag{3.9}$$

Here $z^{(i)}(\cdot; \theta, v), i \in \{0, 1, 2\}$ is the solution of the system:

$$\begin{cases} z^{(0)}(t+1; \theta, v) = f(x^0(t) + z^{(0)}(t; \theta, v), u^0(t), u^0(t-h), t) - f(t), \\ \qquad t \in \{\theta + h + 1, ..., t_1 - 1\}, \\ z^{(0)}(\theta + h + 1; \theta, v) = \Delta_{(u^0(\theta+h), v)} f(\theta + h),\ \theta \in I_0, \\ z^{(0)}(t; \theta, v) = 0,\ t \le \theta + h, \end{cases} \tag{3.10}$$

$$\begin{cases} z^{(1)}(t+1;\theta,v) = f(x^0(t)+z^{(1)}(t;\theta,v),u^0(t),u^0(t-h),t) - f(t), \\ \qquad\qquad t \in \{\theta+h+1,...,t_1-1\}, \\ z^{(1)}(\theta+h+1;\theta,v) \\ \qquad = f(x^0(\theta+h)) + z^{(1)}(\theta+h;\theta,v),u^0(\theta+h),v,\theta+h) - f(\theta+h), \\ z^{(1)}(t+1;\theta,v) = f(x^0(t)+z^{(1)}(t;\theta,v),u^0(t),u^0(t-h),t) - f(t), \\ \qquad\qquad t \in \{\theta+1,...,\theta+h-1\}, \\ z^{(1)}(\theta+1;\theta,v) = \Delta_{(v,u^0(\theta-h))} f(\theta),\ \theta \in I_1, \\ z^{(1)}(t;\theta,v) = 0,\ t \le \theta, \end{cases} \tag{3.11}$$

$$\begin{cases} z^{(2)}(t+1;\theta,v) = f(x^0(t)+z^{(2)}(t;\theta,v),u^0(t),u^0(t-h),t) - f(t), \\ \qquad\qquad t \in \{\theta+1,...,t_1-1\}, \\ z^{(2)}(\theta+1;\theta,v) = \Delta_{(v,u^0(\theta-h))} f(\theta),\ \theta \in I_2, \\ z^{(2)}(t;\theta,v) = 0,\ t \le \theta. \end{cases} \tag{3.12}$$

Thus, by Lemma 3.1, the increments $\Phi(x^0(t_1)+\Delta x(t_1)) - \Phi(x^0(t_1)) =: \Delta S(u^0(\cdot))$ of the functional (2.1), may be written in the following form:

$$\Delta S(u^0(\cdot)) = \Phi(x^0(t_1) + z^{(i)}(t_1;\theta,v)) - \Phi(x^0(t_1)),$$
$$(\theta,v) \in I_i \times U(\theta),\ i \in \{0,1,2\}. \tag{3.13}$$

4 Necessary optimality conditions

Theorem 4.1 *Let $(u^0(\cdot),x^0(\cdot))$ be an optimal process. Then for every $i \in \{0,1,2\}$ and for all $(\theta,v) \in I_i \times U(\theta)$ the following inequality is fulfilled:*

$$\Phi(x^0(t_1) + z^{(i)}(t_1;\theta,v)) - \Phi(x^0(t_1)) \ge 0, \tag{4.1}$$

where $z^{(i)}(\cdot\,;\theta,v),\ i \in \{0,1,2\}$ are determined from (3.10)-(3.12).

Proof. As the inequality $\Delta S(u^0(\cdot)) \ge 0$ holds along the optimal process $(u^0(\cdot),x^0(\cdot))$, then taking into account (3.13), we get the validity of inequality (4.1). The theorem is proved.

Remark 4.1 Since $t_1 - t_0 > h \ge 1$, then $\theta = t_1 - 1 \in I_2$. Therefore, at the point $\theta = t_1 - 1$ the optimality condition (4.1) accepts a simpler form:

$$\Phi(f(x^0(t_1-1),v,u^0(t_1-1-h),t_1-1)) - \Phi(x^0(t_1)) \ge 0,\ \forall v \in U(t_1-1). \tag{4.2}$$

We give effective (both in the checking and computational aspects) corollaries of the theorem 4.1.

Corollary 4.1 *In the problem (2.1)-(2.3) suppose that $\Phi(x) = c^T x$ and $f(x,u,\hat{u},t) = A(t,u,\hat{u})x + b(t,u,\hat{u})$, where $A(\cdot)-$ is a continuous matrix function of order $n \times n$, $b(\cdot)$ is an $n-$dimensional continuous vector function and $c-$ is an $n-$dimensional vector. Then for the optimality of the process $(u^0(\cdot),x^0(\cdot))$ it is necessary that the inequality*

$$\chi_I(\theta)\Delta_{(v,u^0(\theta-h))}H(\theta) + \chi_I(\theta+h)\Delta_{(u^0(\theta+h),v)}H(\theta+h)$$
$$+\chi_I(\theta)\chi_I(\theta+h)\Delta_{(u^0(\theta+h),v)}H_x^T(\theta+h)q(\theta+h;\theta,v) \le 0, \forall(\theta,v) \in I_h \times U(\theta) \tag{4.3}$$

be fulfilled.

Here $\chi_I(t),\ t \in R-$ is the characteristic function of the set I, i.e. $\chi_I(t) = 1,\ t \in I$ and $\chi_I(t) = 0, t \notin I$, $H(\mathring{\psi}(t), x^0(t), u^0(t), u^0(t-h), t) = \mathring{\psi}^T(t) f(x^0(t), u^0(t), u^0(t-h), t)-$ is the Hamilton-Pontryagin function, $\mathring{\psi}(\cdot)-$ is the solution of the system

$$\begin{cases} \mathring{\psi}(t-1) = \mathring{\psi}^T(t-1) A(t, u^0(t), u^0(t-h)),\ t \in \{t_1 - 1, ..., t_0 + 1\}, \\ \mathring{\psi}(t_1 - 1) = -c; \end{cases} \tag{4.4}$$

$$\Delta_{(v,u^0(\theta-h))} H(\theta) = H(\mathring{\psi}(\theta), x^0(\theta), v, u^0(\theta-h), \theta) - H(\theta), \tag{4.5}$$

$$\Delta_{(u^0(\theta+h),v)} H(\theta+h) = H(\mathring{\psi}(\theta+h), x^0(\theta+h), u^0(\theta+h), v, (\theta+h)) - H(\theta+h), \tag{4.6}$$

$$\begin{aligned} &\Delta_{(u^0(\theta+h),v)} H_x(\theta+h) \\ &= \mathring{\psi}^T(\theta+h)[A(\theta+h, u^0(\theta+h), v) - A(\theta+h, u^0(\theta+h)), u^0(\theta)], \end{aligned} \tag{4.7}$$

$q(\cdot;\theta, v)-$ solution of the system

$$\begin{cases} q(t+1;\theta,v) \\ = A(t, u^0(t), u^0(t-h)) q(t;\theta,v), t \in \{\theta+1, \theta+2, ...\} \cap I, \\ q(\theta+1;\theta,v) \\ = \Delta_{(v,u^0(\theta-h))}[A(\theta, u^0(\theta), x^0(\theta-h)) x^0(\theta) + b(\theta, u^0(\theta) u^0(\theta-h))]. \end{cases} \tag{4.8}$$

Proof. Let $(u^0(\cdot), x^0(\cdot))-$ be an optimal control. Let us consider the following cases.
Case 1. Let $(\theta, v) \in I_0 \times U(\theta)$. Then by virtue of Theorem 4.1, the inequality

$$c^T z^{(0)}(t_1;\theta,v) \geq 0$$

is valid.

Hence, taking into account (3.10), (4.4), (4.6) and the condition of the Corollary 4.1, we have

$$\begin{aligned} &-\mathring{\psi}^T(t_1-1) A(t_1-1, u^0(t_1-1), u^0(t_1-1-h)) z^{(0)}(t_1-1;\theta,v) \\ &= -\mathring{\psi}^T(t_1-2) z^{(0)}(t_1-1;\theta,v) = ... = \mathring{\psi}^T(\theta+h) z^{(0)}(\theta+h+1;\theta,v) \\ &= -\mathring{\psi}^T(\theta+h)\Delta_{(u^0(\theta+h),v)}[A(\theta+h, u^0(\theta+h), u^0(\theta)) x^0(\theta+h) \\ &\quad + b(\theta+h, u^0(\theta+h), u^0(\theta))] = -\Delta_{(u^0(\theta+h),v)} H(\theta+h) \geq 0. \end{aligned} \tag{4.9}$$

Case 2. Let $(\theta, v) \in I_1 \times U(\theta)$ be an arbitrary fixed point. Then by virtue of Theorem 4.1, the inequality

$$c^T z^{(1)}(t_1;\theta,v) \geq 0 \tag{4.10}$$

is valid.

Hence, since $f(t) = A(t, u^0(t), u^0(t-h)) x^0(t) + b(t, u^0(t), u^0(t-h))$, then by (3.11), (4.4), (4.6), (4.7) we get

$$\begin{aligned} &-\mathring{\psi}^T(t_1-1) A(t_1-1, u^0(t_1-1), u^0(t_1-1-h)) z^{(1)}(t_1-1;\theta,v) \\ &= -\mathring{\psi}^T(t_1-2) z^{(1)}(t_1-1;\theta,v) = ... = -\mathring{\psi}^T(\theta+h) z^{(1)}(\theta+h+1;\theta,v) \\ &= -\Delta_{(u^0(\theta+h),v)} H(\theta+h) - \Delta_{(u^0(\theta+h),v)} H_x(\theta+h) z^{(1)}(\theta+h;\theta,v) \end{aligned}$$

$$-\mathring{\psi}^T(\theta+h)A(\theta+h,u^0(\theta+h),u^0(\theta))z^{(1)}(\theta+h;\theta,v) \tag{4.11}$$

In the sequel, for the last term of (4.11), in the same way, taking into account (3.11), (4.5) we have

$$\mathring{\psi}^T(\theta+h)A(\theta+h,u^0(\theta+h),u^0(\theta))z^{(1)}(\theta+h;\theta,v)$$

$$=\mathring{\psi}^T(\theta+h-1)z^{(1)}(\theta+h;\theta,v)=...=\mathring{\psi}^T(\theta)z^{(1)}(\theta+1;\theta,v)$$

$$=\varDelta_{(v,u^0(\theta-h)}H(\theta). \tag{4.12}$$

and according to

$$\varDelta_{(v,u^0(\theta-h)}f(\theta)=\varDelta_{(v,u^0(\theta-h)}[A(\theta,u^0(\theta),u^0(\theta-h))x^0(\theta)+b(\theta,u^0(\theta),u^0(\theta-h))]$$

(3.11), (4.8), the equality

$$z^{(1)}(\theta+h;\theta,v)=q(\theta+h;\theta,v). \tag{4.13}$$

is valid

Thus, taking into account (4.11)-(4.13) from (4.10) for every $(\theta,v)\in I_1\times U(\theta)$ we get the validity of the inequality

$$\varDelta_{(v,u^0(\theta-h))}H(\theta)+\varDelta_{(u^0(\theta+h),v)}H(\theta+h)$$

$$+\varDelta_{(u^0(\theta+h),v)}H_x^T(\theta+h)\;q(\theta+h;\theta,v)\leq 0, \tag{4.14}$$

Case 3. Let $(\theta,v)\in I_2\times U(\theta)-$ be an arbitrary fixed point. Then, by theorem 4.1 we have

$$c^Tz^{(2)}(t_1;\theta,v)\geq 0.$$

From this inequality, similar to case 1, taking into account (3.12), (4.4), (4.5) and $f(\cdot)=A(t,u,\bar{u})x+b(t,u,\bar{u})$ we get

$$\mathring{\psi}^T(t_1-1)z^{(2)}(t_1;\theta,v)$$

$$=\mathring{\psi}^T(\theta)\varDelta_{(v,u^0(\theta-h))}[A(\theta,u^0(\theta),u^0(\theta-h))x^0(\theta)+b(\theta,u^0(\theta),u^0(\theta-h))]$$

$$=\varDelta_{(v,u^0(\theta-h))}H(\theta)\leq 0,\forall(\theta,v)\in I_2\times U(\theta). \tag{4.15}$$

Thus, based on inequalities (4.9), (4.14), (4.15) and using the characteristic function $\chi_I(\cdot)$ we get the validity of (4.3).

Corollary 4.1 is completely proved.

Corollary 4.2 *Let the conditions of Corollary 4.1 be fulfilled and suppose that the matrix $A(t,u,\hat{u})$ does not depend on the variable $\hat{u}$. Then for the optimality of the process $(u^0(\cdot),x^0(\cdot))$ it is necessary the inequality*

$$\chi_I(\theta)\varDelta_{(v,u^0(\theta-h))}H(\theta)+\chi_I(\theta+h)\varDelta_{(u^0(\theta+h),v)}H(\theta+h)\leq 0,$$

$$\forall(\theta,\tilde{u})\in I_h\times U(\theta), \tag{4.16}$$

be fulfilled, where $\varDelta_{(v,u^0(\theta-h))}H(\theta)$, $\varDelta_{(u^0(\theta+h),v)}H(\theta+h)$ are determined by (4.5), (4.6) respectively.

The proof of this Corollary directly follows from optimality condition (4.3), since we have

$$\varDelta_{(u^0(\theta+h),v)}H_x(\theta+h)=0, \quad \forall(\theta,h)\in I_h\times U(\theta).$$

It should be specially noticed that the optimality conditions (4.16) in essence is the discrete analog of the Pontryagin maximum principle for the problem (2.1)-(2.3) under the conditions of Corollary 4.2.

Example 4.1 Let us consider the problem

$$S(u(\cdot))=-x_2(3)\to min,$$

$$\begin{cases} x_1(t+1)=x_1(t)+2u(t),\\ x_2(t+1)=-u(t-1)x_1(t)+x_2(t)+u^2(t),\ t\in\{0,1,2\}=:I, \end{cases}$$

$$x_1(0)=3,\ \ x_2(0)=0,\ \ h=1,$$

$$U(t)=[-1,1]\subset R,\ \ t\in\{-1,0,1,2\}=:I_h.$$

Show that the discrete maximum principle (4.16) is not valid for this linear problem.

Calculate $S(u(\cdot))$. Then we have

$$S(u(\cdot))=3u(-1)+3(u(0)+u(1))+(u(0)+u(1))^2-u^2(2)\to min,$$

$$u(t)\in[-1,1],\ \ t\in\{-1,0,1,2\}$$

hence we get $u^0(t), t\in I_h$, where $u^0(-1)=-1, u^0(0)=-\frac{1}{2}, u^0(1)=u^0(2)=-1$ is an optimal control, while $x^0(t)=(x_1^0(t),\ x_2^0(t))^T,\ t\in\{0,1,2,3\}$, where $x_1^0(0)=3,\ x_1^0(1)=2,\ x_1^0(2)=0,\ x_1^0(3)=-2,\ x_2^0(0)=0,\ x_2^0(1)=3,25,\ x_2^0(2)=5,25,\ x_2^0(3)=6,25$.

Now check fulfillment of the maximum principle (4.16) along the optimal process $(u^0(\cdot),x^0(\cdot))$.

We carry out the following calculations by (4.4)-(4.7):

$$H(\psi,x,u,v,t)=\psi_1(x_1+2u)+\psi_2(x_2+u^2-vx_1),$$

$$\begin{cases} \overset{\circ}{\psi}_1(t-1)=\overset{\circ}{\psi}_1(t)-u^0(t-1)\overset{\circ}{\psi}_2(t),\\ \overset{\circ}{\psi}_2(t-1)=\overset{\circ}{\psi}_2(t),\ t\in\{2,1\}, \end{cases}$$

$$\begin{cases} \overset{\circ}{\psi}_1(2)=0,\\ \overset{\circ}{\psi}_2(2)=1, \end{cases}\quad \begin{cases} \overset{\circ}{\psi}_1(2)=0,\\ \overset{\circ}{\psi}_2(2)=1, \end{cases}\quad \begin{cases} \overset{\circ}{\psi}_1(1)=1,\\ \overset{\circ}{\psi}_2(1)=1, \end{cases}$$

$$\triangle_{(v,u^0(-1))}\ H(0)=v^2+3v+1,25,\ v\in[-1,1],$$

$$\triangle_{(u^0(1),v)}\ H(1)=-2v-1,\ v\in[-1,1].$$

Taking into account that these calculations are not fulfilled, we get the optimality condition (4.16) at the point $t=0$

$$\triangle_{(v,u^0(-1))}\ H(0)+\ \triangle_{(u^0(1),v)}\ H(1)=v^2+v+0,25\le 0,\ \forall v\in[-1,1].$$

Consequently, this example shows that the discrete maximum principle generally speaking, does not hold if even the problem (2.1)-(2.3) is linear (with respect to phase variable and with linear quality criterion), i.e. in the problem (2.1)-(2.3) $f(\cdot)=A(t,u(t),u(t-h))x(t)+b(t,u(t),u(t-h)),\varPhi(\cdot)=c^Tx$.

As a significant difference, we note that for a linear discrete optimal control problem without a delay, the discrete maximum principle holds [2]. We also note that by Corollary 4.1 for studying example 1, we can apply the necessary condition for optimality (4.3). Continuing the calculation, in example 1 we easily get that condition (4.3) is fulfilled along the optimal control $u^0(\cdot)$: $-(v+0,5)^2\le 0,\ \forall v\in[-1,1]$, since we have $\varDelta_{(u^0(1),v)}H_x^T(1)q(1,0,v)=-2v^2-2v-0,5$.

5 Conclusion

In conclusion, it should be noted that the necessary optimality condition (4.1) is rather general. The merit of the condition (4.1) compared with discrete maximum principle [2, 4, 9, 10, 14] is that it has no convexity or smoothness conditions with respect to the input data of problem (2.1)-(2.3). Therefore the optimality condition (4.1) has a wider range of applications than the earlier known necessary optimality conditions (for example, [6, 15, 16]). Note that the optimality condition (4.1) allows one to replace the many-dimensional minimization problem by a sequence of problems of less dimensions, and its application is convenient when its test is performed sequentially with respect to $\theta = t_1 - 1, t_1 - 2, ..., t_0$.

By Corollaries 4.1, 4.2, it is shown that the optimality condition (4.1) contains as a particular case both new (see (4.3)) and known (see (4.16) [15, 16] optimality conditions. Also, by these corollaries and Example 4.1, how important it is to take into account the specific character of the concrete optimization problem (2.1)-(2.3).

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