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Some spectral properties of an eigenvalue problem with spectral parameter contained in the boundary conditions

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Abstract. *In this paper we consider a spectral problem with describes bending vibrations of a homogeneous rod, in cross-sections of which the longitudinal force acts, the left end of which is fixed, and the right end resiliently fastened and on this end an inertial mass is concentrated. We give the location of the eigenvalues on the real axis and the structure of root subspaces of this problem.*

Keywords. fourth order eigenvalue problem · flexural vibrations of a homogeneous rod · eigenfunction · root subspace · Pontryagin space

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1 Introduction

We consider the following eigenvalue problem

$$y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ 0 < x < 1, \tag{1.1}$$

$$y(0) = y'(0) = 0, \tag{1.2}$$

$$y''(1) - (a\lambda + b)y'(1) = 0, \tag{1.3}$$

$$Ty(1) - c\lambda y(1) = 0, \tag{1.4}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty(x) \equiv y'''(x) - q(x)y'(x)$, $x \in [0,1]$, $q(x)$ is a positive absolutely continuous function on the interval $[0,1]$, $a,\ b$ and c are real constants such that $a > 0,\ b < 0$ and $c > 0$.

The spectral problems for second and fourth order ordinary differential operators with spectral parameter in the boundary conditions were studied by many authors (see, e.g., [1-5, 9-17]). A number of problems in mathematical physics can be reduced to such problems (e.g., see [5, 9, 10, 11, 16, 17]).

The eigenvalue problem (1.1)-(1.4) describes the flexural vibrations of a homogeneous rod, in the cross sections of which the longitudinal force acts, the left end is fixed rigidly, and the right end resiliently fastened and on this end an inertial mass is concentrated (see [10]).

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The location of eigenvalues in the complex plane (on the real axis), the structure of the root subspaces, oscillation properties of eigenfunctions (and their derivatives), the asymptotic formulas for eigenvalues and eigenfunctions, the basis property of subsystems of eigenfunctions of problem (1.1)-(1.4) for $a > 0$, $c < 0$ and $b = 0$ were investigated in the recent paper [5]. In this paper is established as a necessary and sufficient condition, as well as sufficient conditions for the system of eigenfunctions of this problem after removing two functions to form a basis in $L_p(0,1)$, $1 < p < \infty$. The establishment of sufficient conditions is based on the rough asymptotic behavior of eigenvalues and oscillation properties of eigenfunctions and their derivatives.

Note that the signs of the parameters a and c play an important role. If $a > 0$ and $c < 0$, then problem (1.1)-(1.4), can be treated as a spectral problem for a self-adjoint operator in the Hilbert space $H = L_2(0,1)\oplus\mathbb{C}^2$. If $a > 0$ and $c > 0$, then this problem is equivalent to a spectral problem for the J-self-adjoint operator in the Pontryagin space $\Pi_1 = L_2(0,1)\oplus\mathbb{C}^2$ with the corresponding inner product (e.g., see [2-4, 6-8, 16]). In the case $a > 0$ and $c < 0$ all eigenvalues of problem (1.1)-(1.4) are positive, simple, and form an infinitely increasing sequence. In the case $a > 0$, $b < 0$ and $c > 0$ we show that problem (1.1)-(1.4) has one negative simple eigenvalue and a sequence of positive and simple eigenvalues tending to infinity.

The purpose of the present paper is to investigated the location of the eigenvalues on the real axis and the structure of root subspaces of problem (1.1)-(1.4).

2 Operator interpretation of the spectral problem (1.1)-(1.4)

The considered problem (1.1)-(1.4) can be reduced to the eigenvalue problem for the linear operator L in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^2$ with the inner product

$$(\hat{u},\hat{v}) = (\{y,m,n\},\{v,s,t\}) = (y,v)_{L_2} + |a|^{-1}m\bar{s} + |c|^{-1}n\bar{t}, \tag{2.1}$$

where

$$L\hat{y} = L\{y,m,n\} = \{(Ty(x))',\, y''(1) - by'(1),\, Ty(1)\}$$

is an operator with the domain

$$D(L) = \{\{y\,(x),\, m,\, n\}\ :\ y \in W_2^4(0,1),\ (Ty(x))' \in L_2(0,1),$$

$$y(0) = y'(0) = 0,\ m = ay'(1),\ n = cy(1)\}$$

dense everywhere in H (see [16]). L is a closed operator in H with a compact resolvent. The eigenvalue problem of operator L is equivalent to problem (1.1)-(1.4): the spectra of operator L and problem (1.1)-(1.4) coincide, as do their multiplicities; between the eigenvectors of operator L and the eigenfunctions of problem (1.1)-(1.4) corresponding to one and the same eigenvalue, it is possible to establish a one-to-one correspondence

$$y_k(x) \leftrightarrow\ \{y_k(x),\, m_k,\, n_k\},\ \ m_k = ay_k'(1),\ n_k = cy_k\,(1).$$

The eigenvalue problem (1.1)-(1.4) in the case $a \neq 0$ and $c \neq 0$ is strongly regular in the sense of [16]; in particular, this problem has discrete spectrum.

In the case $a > 0$ and $c < 0$ the operator L is a self-adjoint discrete lower-semibounded in H and hence has a system of eigenvectors $\{y_k(x), m_k, n_k\}$, $k \in \mathbb{N}$, that forms an orthogonal basis in H [5].

In the case $a > 0$ and $c > 0$ the operator L is a nonself-adjoint in H. In this case we define an operator $J : H \to H$ as follows:

$$J\{y,m;n\} = \{y,m,-n\}.$$

J is a unitary, symmetric operator on H. Its spectrum consists of two eigenvalues: -1 with multiplicity 1, and $+1$ with infinite multiplicity. This operator generates the Pontryagin space $\Pi_1 = L_2(0,1) \oplus \mathbb{C}^2$ with inner product [6]

$$[\hat{u}, \hat{v}] = (\hat{u}, \hat{v})_{\Pi_1} = (\{y, m, n\}, \{u, s, t\})_{\Pi_1} = (u, v)_{L_2} + a^{-1} m\bar{s} - c^{-1} n\bar{t}. \tag{2.2}$$

Theorem 2.1 *L is $J-$self-adjoint operator in Π_1.*

Proof. J-self-adjointness of L on Π_1 follows from [7, Section 3, Proposition 3^0].

Theorem 2.2 *If L^* is the adjoint operator of L in H, then $L^* = JLJ$. The system of eigenvectors $\{\hat{y}_k\}_{k=1}^{\infty}$, $\hat{y}_k = \{y_k, m_k, n_k\}$, of L forms an unconditional basis in H.*

Proof. The proof of the first part of this theorem follows from [7, Section 3, Proposition 5^0] and the second part from [8].

3 Some auxiliary statements and main properties of the solution of problem (1.1)-(1.3)

We introduce the boundary conditions (see [1, 2, 13])

$$y(1)\cos\delta - Ty(1)\sin\delta = 0, \tag{3.1}$$

where $\delta \in [\frac{\pi}{2}, \pi]$.

Alongside the problem (1.1)-(1.4) we shall consider the problem (1.1)-(1.3), (3.1). The problem (1.1)-(1.3), (3.1) have been considered in [1]. In [1] study the oscillation properties of eigenfunctions and their derivatives, and investigate the basis property in the space $L_p, 1 < p < \infty$, of the system of eigenfunctions of this problem.

The following theorem is a special case of the general result of [1].

Theorem 3.1 (see [1, Theorem 2.2]) *There exists an unboundedly increasing sequence of eigenvalues $\lambda_1(\delta), \lambda_2(\delta), \ldots, \lambda_n(\delta), \ldots$ of boundary value problem* (1.1)-(1.3), (3.1), *moreover, $\lambda_n > 0$ for $n \in \mathbb{N}$. The corresponding eigenfunctions and their derivatives have the following oscillation properties:*

(i) the eigenfunction $y_n^{(\delta)}(x)$, $n \in \mathbb{N}$, corresponding to the eigenvalue $\lambda_n(\delta)$, has $n-1$ simple zeros in $(0,1)$ in the case $a\lambda_n(\delta) + b \leq 0$, has either $n-2$ or $n-1$ simple zeros in $(0,1)$ in the case $a\lambda_n(\delta) + b > 0$;

(ii) the function $(y_n^{(\delta)}(x))'$, $n \in \mathbb{N}$, has exactly $n-1$ simple zeros in $(0,1)$.

It follows from the proof of [1, Theorem 2.2] that

$$\lambda_1\left(\frac{\pi}{2}\right) < \lambda_1(\pi) < \lambda_2\left(\frac{\pi}{2}\right) < \lambda_1(\pi) < \ \ldots\ . \tag{3.2}$$

Theorem 3.2 *For each fixed $\lambda \in \mathbb{C}$ there exists a nontrivial solution $y(x, \lambda)$ of the problem* (1.1)-(1.3) *which is unique up to a constant coefficient.*

Proof. We denote by $\varphi_k(x, \lambda), k = \overline{1,4}$, be solutions of equation (1.1), normalized for $x = 0$ by the Cauchy conditions

$$\varphi_k^{(s-1)}(0, \lambda) = \delta_{ks}, \ s = \overline{1,3}, \ T\varphi_k(0, \lambda) = \delta_{k4}, \tag{3.3}$$

where δ_{ks} is the Kronecker delta.

We will seek the function $y(x, \lambda)$ in the following form

$$y(x, \lambda) = \sum_{k=1}^{4} C_k \varphi_k(x, \lambda), \tag{3.4}$$

where $C_k, k = \overline{1,4}$, are constants.

It follows from (3.3), (3.4) and boundary conditions (1.2), (1.3) that $C_1 = C_2 = 0$ and

$$C_3 \left(\varphi_3''(1,\lambda) - (a\lambda + b)\,\varphi_3'(1,\lambda)\right) + C_4 \left(\varphi_4''(1,\lambda) - (a\lambda + b)\,\varphi_4'(1,\lambda)\right) = 0.$$

For the completion of the proof of this theorem it is sufficient to show that

$$|\varphi_3''(1,\lambda) - (a\lambda + b)\,\varphi_3'(1,\lambda)| + |\varphi_4''(1,\lambda) - (a\lambda + b)\,\varphi_4'(1,\lambda)| > 0. \tag{3.5}$$

Let $\lambda \in \mathbb{C}\backslash\mathbb{R}$. If for such λ, (3.5) is not satisfied, then the functions $\varphi_3(x,\lambda)$ and $\varphi_4(x,\lambda)$ are solutions of the problem (1.1)-(1.3). We define the function $w(x,\lambda)$ in the following way:

$$w(x,\lambda) = T\varphi_4(1,\lambda)\,\varphi_3(x,\lambda) - T\varphi_3(1,\lambda)\,\varphi_4(x,\lambda).$$

Since $Tw(1,\lambda) = 0$, the function $w(x,\lambda)$ is an eigenfunction of the spectral problem (1.1)-(1.3), (3.1) for $\delta = \frac{\pi}{2}$ corresponding to the eigenvalue $\lambda \in \mathbb{C}\backslash\mathbb{R}$. By virtue of Theorem 3.1 we have $\lambda \in \mathbb{R}$ which contradicts the relation $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

Now let $\lambda \in \mathbb{R}$ and (3.5) be not satisfied. In this case alongside the function $w(x,\lambda)$ we define the function:

$$\psi(x,\lambda) = \varphi_4(1,\lambda)\,\varphi_3(x,\lambda) - \varphi_3(1,\lambda)\,\varphi_4(x,\lambda).$$

Since $\psi(1,\lambda) = 0$, the function $\psi(x,\lambda)$ is an eigenfunction of the spectral problem (1.1)-(1.3), (3.1) for $\delta = \pi$ corresponding to the eigenvalue $\lambda \in \mathbb{R}$. On the other hand, by the above arguments λ is also an eigenvalue of this problem for $\delta = \frac{\pi}{2}$, which contradicts the relation (3.2). The proof of this theorem is complete.

Remark 3.1 From the proof of Theorem 4.1 it seen that without loss of generality we can regard solution $y(x,\lambda)$ of the problem (1.1)-(1.3) for each fixed $x \in [0,1]$ as an entire function of λ of the following form

$$\begin{aligned} y(x,\lambda) = &(\varphi_4''(1,\lambda) - (a\lambda + b)\,\varphi_4'(1,\lambda))\,\varphi_3(x,\lambda) \\ &- (\varphi_3''(1,\lambda) - (a\lambda + b)\,\varphi_3'(1,\lambda))\,\varphi_4(x,\lambda). \end{aligned} \tag{3.6}$$

Let $\mathcal{B}_k = (\lambda_{k-1}(\pi), \lambda_k(\pi))$, $n = 1, 2, ...$, where $\lambda_0(\pi) = -\infty$.

Obviously, the eigenvalues $\lambda_n(\pi)$ and $\lambda_n(\pi/2)$ of the spectral problem (1.1)-(1.3), (2.1) for $\delta = \pi$ and $\delta = \pi/2$ are zeros of the entire functions $y(1,\lambda)$ and $Ty(1,\lambda)$, respectively.

Note that the function

$$F(\lambda) = \frac{Ty(1,\lambda)}{y(1,\lambda)}$$

is defined for

$$\lambda \in \mathcal{B} \equiv (\mathbb{C}\backslash\mathbb{R}) \cup \left(\bigcup_{k=1}^{\infty} \mathcal{B}_k\right),$$

and $\lambda_k(\pi/2)$ and $\lambda_k(\pi)$, $n \in \mathbb{N}$, are the zeros and poles of this function, respectively.

Lemma 3.1 *The following formula holds:*

$$\frac{dF(\lambda)}{d\lambda} = \frac{1}{y^2(1,\lambda)} \left\{ \int_0^l y^2(x,\lambda)\,dx + ay'^2(1,\lambda) \right\},\ \lambda \in \mathcal{B}. \tag{3.7}$$

Proof. By (1.1) we have

$$(Ty(x,\mu))'\, y(x,\lambda) - (Ty(x,\lambda))'\, y(x,\mu) = (\mu-\lambda) y(x,\mu) y(x,\lambda).$$

Integrating this relation from 0 to 1 (using the formula for the integration by parts) and taking into account boundary conditions (1.1) and (1.3) we obtain

$$\begin{gathered} Ty(1,\mu)\, y(1,\lambda) - Ty(1,\lambda)\, y(1,\mu) \\ = (\mu-\lambda)\left\{\int_0^1 y(x,\mu)\, y(x,\lambda) dx + a y'(1,\mu)\, y'(1,\lambda)\right\}. \end{gathered} \tag{3.8}$$

By (3.8) for $\mu, \lambda \in \mathcal{B},\ \mu \neq \lambda$, we have

$$\frac{Ty(1,\mu)}{y(1,\mu)} - \frac{Ty(1,\lambda)}{y(1,\lambda)} = (\mu-\lambda)\ \frac{\int_0^l y(x,\mu)\, y(x,\lambda)\, dx + a\, y'(1,\mu)\, y'(1,\lambda)}{y(1,\mu)\, y(1,\lambda)}\ . \tag{3.9}$$

Dividing both sides of relation (3.9) by $\mu - \lambda$ ($\mu \neq \lambda$) and by passing to the limit as $\mu \to \lambda$ we obtain (3.7). The proof of this lemma is complete.

Corollary 3.1 *The function $F(\lambda)$ of λ strictly increases on each interval $\mathcal{B}_k,\ k \in \mathbb{N}$.*

Lemma 3.2 *The following relation holds:*

$$\lim_{\lambda \to -\infty} F(\lambda) = -\infty. \tag{3.10}$$

The proof is similar to that of [5, Lemma 3.4].

Remark 3.2 By Theorem 3.1 it follows from Lemmas 3.1 and 3.2 that $F(0) < 0$.

Following the corresponding arguments of the proof of [5, Lemma 3.17], we can show that the following assertion holds.

Lemma 3.3 *The following representation holds:*

$$F(\lambda) = F(0) + \sum_{k=1}^{\infty} \frac{\lambda\, c_k}{\lambda_k(\pi)(\lambda - \lambda_k(\pi))}\,, \tag{3.11}$$

where $c_k = \operatorname*{res}_{\lambda=\lambda_k(\pi)} F(\lambda)$, *and* $c_k < 0,\ k \in \mathbb{N}$.

The proof is similar to that of [13, Lemma 3.17].

Corollary 3.2 *The function $F(\lambda)$ is convex in the interval $(-\infty,\ \lambda_1(\pi))$.*

4 The location of the eigenvalues and structure of root subspaces of problem (1.1)-(1.4)

Lemma 4.1 *The eigenvalues of the boundary value problem* (1.1)-(1.4) *are real, simple and form an at most countable set without finite limit point.*

Proof. It is easy to see that the eigenvalues of problem (1.1)-(1.4) are the roots of the equation

$$Ty(1,\lambda) - c\lambda\, y(1,\lambda) = 0. \tag{4.1}$$

If λ is a nonreal eigenvalue of problem (1.1)-(1.4), then $\bar{\lambda}$ is also an eigenvalue of this problem, because the coefficients $q(x),\ a,\ b,\ c$ are real. In this case $y(x,\bar{\lambda}) = \overline{y(x,\lambda)}$, so that if equality (4.1) holds for λ, then it also holds for $\bar{\lambda}$.

Setting $\mu = \bar{\lambda}$ in (3.8), we obtain

$$\overline{Ty(1,\lambda)}\, y(1,\lambda) - Ty(1,\lambda)\, \overline{y(1,\lambda)} = (\bar{\lambda} - \lambda) \left\{ \int_0^1 |\, y(x,\lambda)|^2 dx + a|\, y'(1,\lambda)|^2 \right\}. \tag{4.2}$$

By virtue of (1.4) from (4.2) we get

$$c\,(\bar{\lambda} - \lambda)\, |y'(1,\lambda)|^2 = (\bar{\lambda} - \lambda) \left\{ \int_0^1 |\, y(x,\lambda)|^2 dx + a|\, y'(1,\lambda)|^2 \right\}.$$

Since $\bar{\lambda} \neq \lambda$, it follows that

$$\int_0^1 |\, y(x,\lambda)|^2 dx + \ a\,|\, y'(1,\lambda)|^2 - c\,|\, y(1,\lambda)|^2 = 0. \tag{4.3}$$

On the other hand multiplying both sides of equation (1.1) by $\overline{y(x,\lambda)}$, and integrating resulting equality from 0 to 1, using the formula of integration by parts and taking into account (1.2)-(1.4), we have

$$\begin{aligned} &\int_0^1 |\, y''(x,\lambda)|^2 dx + \int_0^1 q(x)|\, y'(x,\lambda)|^2 dx \\ &= \lambda \left\{ \int_0^1 |\, y(x,\lambda)|^2 dx + a\,|\, y'(1,\lambda)|^2 - c\,|\, y(1,\lambda)|^2 \right\}. \end{aligned} \tag{4.4}$$

By (4.3) from (4.4) we obtain

$$\int_0^1 |y''(x,\lambda)|^2 dx + \int_0^1 q(x)|y'(x,\lambda)|^2 dx = 0.$$

which implies (by (1.2)) that $y(x,\lambda) \equiv 0$. This contradiction shows that the eigenvalues of problem (1.1)-(1.4) are real. The proof of this lemma is complete.

The entire function occurring on the left-hand side in equation (4.1) does not vanish for non-real λ. Consequently, it does not vanish identically. Therefore, its zeros form an at most countable set without finite limit points.

If λ is an eigenvalue of problem (1.1)-(1.4), then it follows from (3.2) that $y(1,\lambda) \neq 0$. Hence each root (with regard of multiplicities) of equation (4.1) is a root of the equation

$$F(\lambda) = c\lambda. \tag{4.5}$$

Let us show that equation (4.5) has only simple roots. Indeed, if $\lambda = \tilde{\lambda}$ is a multiple root of (4.5), then

$$F(\tilde{\lambda}) = c\lambda, \;\; F'(\tilde{\lambda}) = c. \tag{4.6}$$

Hence by (3.7) from (4.6) we obtain

$$\int_0^1 y(x,\tilde{\lambda})^2 dx + a\, y'(1,\tilde{\lambda})^2 - c\, y(1,\tilde{\lambda})^2 = 0. \tag{4.7}$$

Since the eigenvalues of problem (1.1)-(1.4) are real it follows from (4.4) that

$$\begin{aligned} &\int_0^1 y''(x,\tilde{\lambda})^2 dx + \int_0^1 q(x)\, y'(x,\tilde{\lambda})^2 dx \\ &= \lambda \left\{ \int_0^1 y(x,\tilde{\lambda})^2 dx + a\, y'(1,\tilde{\lambda})^2 - c\, y(1,\tilde{\lambda})^2 \right\}. \end{aligned} \tag{4.8}$$

which implies that

$$\int_0^1 y''(x,\tilde{\lambda})^2 dx + \int_0^1 q(x)\, y'(x,\tilde{\lambda})^2 dx = 0. \tag{4.9}$$

By virtue of (4.9) we have $y(x,\tilde{\lambda}) \equiv 0$ which contradicts condition $y(x,\tilde{\lambda}) \not\equiv 0$. The resulting contradiction completes the proof of Lemma 5.1.

Remark 4.1 By Remark 3.2 it follows from (4.5) that $\lambda = 0$ is not an eigenvalue of the problem (1.1)-(1.4).

Lemma 4.2 *The eigenvalue problem* (1.1)-(1.4) *can have only one eigenvalue in each interval* $\mathcal{B}_1 \cap (-\infty, 0)$ *and* $\mathcal{B}_k \cap (0, +\infty)$, $k = 1, 2, \ldots$.

Proof. Let $\tilde{\lambda} \in \mathcal{B}_k \cap (0, +\infty)$ be an eigenvalue of problem (1.1)-(1.4) for some $k \in \mathbb{N}$. Then from equality (4.8) we obtain

$$\int_0^1 y^2(x,\tilde{\lambda})\, dx - \; a_2 y^2(1,\tilde{\lambda}) + a_1 y'^2(1,\tilde{\lambda}) > 0,$$

which implies by (3.7) that

$$(F(\lambda) - c\lambda)'\big|_{\lambda=\tilde{\lambda}} > 0.$$

Hence it follows from this inequality that the function $F(\lambda) - c\lambda$ takes zero value only strictly increasing in the interval $\mathcal{B}_k \cap (0, +\infty)$. Since $F(\tilde{\lambda}) - c\tilde{\lambda} = 0$ equation (4.5) has a unique solution $\tilde{\lambda}$ in the interval $\mathcal{B}_k \cap (0, +\infty)$. In a similar way, one can show that problem (1.1)-(1.4) can have only one eigenvalue in the interval $\mathcal{B}_1 \cap (-\infty, 0)$. The proof of Lemma 4.2 is complete.

Theorem 4.1 *There exists an infinitely increasing sequence of eigenvalues* $\lambda_1, \lambda_2, ..., \lambda_k, ...$ *of the boundary value problem* (1.1)-(1.4) *such that*

$$\lambda_1 \in (-\infty, 0)) \text{ and } \lambda_k \in (\lambda_{k-1}(\pi/2), \lambda_{k-1}(\pi),) \;\text{ for } k > 1. \tag{4.10}$$

Proof. We recall that the eigenvalues of problem (1.1)-(1.4) are the roots of the equation $F(\lambda) = c\,\lambda$ where $c > 0$. By virtue of Corollary 3.2 $F(\lambda)$ is convex function in the interval $\mathcal{B}_1$. By virtue of the relations (3.10) and (3.11) we have

$$\lim_{\lambda\to-\infty} F(\lambda) = -\infty \ \text{ and } \lim_{\lambda\to\lambda_1(\pi)-0} F(\lambda) = +\infty.$$

Therefore, the line $c\,\lambda$ intersect the graph of the function $F(\lambda)$ at two points $\lambda_1 < \lambda_2$ such that $\lambda_1 \in (-\infty, 0)$ and $\lambda_2 \in (\lambda_1(\pi/2), \lambda_1(\pi) \subset (0, \lambda_1(\pi)$. Consequently, in the interval $\mathcal{B}_1$ problem (1.1)-(1.4) has two simple eigenvalues λ_1 and λ_2 such that $\lambda_1 \in (-\infty, 0)$ and $\lambda_2 \in (\lambda_1(\pi/2), \lambda_1(\pi)$.

It follows from (3.2), (3.7) and (3.11) that

$$\lim_{\lambda\to\lambda_k(\pi)-0} F(\lambda) = +\infty, \quad \lim_{\lambda\to\lambda_{k-1}(\pi)+0} F(\lambda) = -\infty, \; k \geq 2. \tag{4.11}$$

Hence equation (4.5) has at least one solution in each interval $\mathcal{B}_k$, $k = 2, 3, \ldots$. Then by virtue of Lemma 4.2, in the interval $\mathcal{B}_k$, $k = 2, 3, \ldots$, problem (1.1)-(1.4) has the unique (simple) eigenvalue λ_{k+1}. The proof of this theorem is complete.

References

1. Aliyev, Z.S.: *Basis properties of a fourth order differential operator with spectral parameter in the boundary condition*, Cent. Eur. J. Math., **8**(2), 378–388 (2010).
2. Aliev, Z.S.: *Basis properties in L_p of systems of root functions of a spectral problem with spectral parameter in a boundary condition*, Diff. Equ., **47**(6), 766–777 (2011).
3. Aliev, Z.S., Dunyamalieva, A.A.: *Defect basis property of a system of root functions of a Sturm-Liouville problem with spectral parameter in the boundary conditions*, Diff. Equ., **51**(10), 1249–1266 (2015).
4. Aliyev, Z.S., Dunyamaliyeva, A.A., Mehraliyev, Y.T.: *Basis properties in L_p of root functions of Sturm-Liouville problem with spectral parameter-dependent boundary conditions*, Mediterr. J. Math, **14**(3), 1–23 (2017).
5. Aliyev, Z.S., Guliyeva, S.B.: *Properties of natural frequencies and harmonic bending vibrations of a rod at one end of which is concentrated inertial load*, J. Differential Equations, **263**(9), 5830–5845 (2017).
6. Azizov, T.Ya., Iokhvidov, I.S.: Linear operators in spaces with an indefinite metric, *John Vielcy, Chichester, UK* (1989).
7. Azizov, T.Ya., Iokhvidov, I.S.: *Linear operators in Hilbert spaces with G-metric*, Uspekhi Mat. Nauk, **26**(4), 43–92 (1971).
8. Azizov, T.Ya., Iokhvidov, I.S.: *A test of whether the root vectors of a completely continuous J-self-adjoint operator in a Pontrjagin space $\Pi_\varkappa$ complete and form a basis*, Mat. Issled. **6**(1), 158–161 (1971).
9. Amara, J. Ben, Vladimirov, A.A: *On a fourth-order problem with spectral and physical parameters in the boundary condition*, Izvestiya: Mathematics, **68**(4), 645–658 (2004).
10. Bolotin, B.B.: Vibrations in technique: Handbook in 6 volumes, The vibrations of linear systems, I, *Engineering Industry, Moscow* (1978).
11. Fulton, C.T.: *Two-point boundary value problems with eigenvalue parameter in the boundary conditions*, Proc. Roy. Soc. Edinburgh, Sect.**A**, **77**(3–4), 293-308 (1977).
12. Kerimov, N.B., Aliev, Z.S.: *Basis properties of a spectral problem with spectral parameter in the boundary condition*, Sbornik: Mathematics, **197**(10), 1467–1487 (2006).
13. Kerimov, N.B., Aliev, Z.S.: *Spectral properties of the differential operators of the fourth-order with eigenvalue parameter dependent boundary condition*, Int. J. Math. Math. Sci., **2012** (2012), Article ID 456517, 28 pages.

14. Kerimov, N.B., Poladov, R.G.: *Basis properties of the system of eigenfunctions in the Sturm-Liouville problem with a spectral parameter in the boundary conditions*, Doklady Math., **85**(1), 8–13 (2012).
15. Möller, M., Zinsou, B.: *Self-adjoint fourth order differential operators with eigenvalue parameter dependent boundary conditions*, Quaestiones Mathematicae, **34**(3), 393–406 (2011).
16. Shkalikov, A.A.: *Boundary value problems for ordinary differential equations with a parameter in the boundary conditions*, J. Sov. Math., **33**, 1311–1342 (1986).
17. Walter, J.: *Regular eigenvalue problems with eigenvalue parameter in the boundary condition*, Math. Z., **133**(4), 301–312 (1973).