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On a new metric in the cotangent bundle

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Abstract. *In this paper we construct a new metric* $\tilde{G} = {}^{R}\nabla + \sum\limits_{i,j=1}^{n} g^{ji}\delta p_j \delta p_i$ *in the cotangent bundle, where* ${}^{R}\nabla$ *is the Riemannian extension. Some curvature properties and geodesics are investigated for the metric* $\tilde{G}$*.*

Keywords. Riemannian extension · cotangent bundle · Levi-Civita connection · geodesic

1 Introduction

Let M^n be an n-dimensional differentiable manifold equipped with a torsion free connection ∇ and π the bundle projection $T^*M^n \to M^n$ where T^*M^n being the cotangent bundle of M^n. The local coordinates $\left(U, x^i\right), i = 1, ..., n$ on M^n induces on T^*M^n a system of local coordinates $\left(\pi^{-1}\left(U\right), x^i, x^{\bar{i}} = p_i\right), \bar{i} = n+1, ..., 2n$, where $x^{\bar{i}} = p_i$ are the components of the covector p in each cotangent space $T_x^*M^n, x \in U$ with respect to the natural coframe $\left\{dx^i\right\}, i = 1, ..., n$. Let $F\left(M^n\right)\left(F(T^*M^n)\right)$ be the ring of real-valued C^∞ functions on $M^n(T^*M^n)$ and $\Im_s^r\left(M^n\right)\left(\Im_s^r\left(T^*M^n\right)\right)$ be the module over $F\left(M^n\right)\left(F(T^*M^n)\right)$ of C^∞ tensor fields of type (r, s).

Suppose that the vector and a covector (1-form) field $X \in \Im_0^1(M^n)$ and $\omega \in \Im_1^0(M^n)$ have the local expression $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions in $U \subset M^n$, respectively. The complete and horizontal lifts ${}^{C}X, {}^{H}X \in \Im_0^1(T^*M^n)$ of $X \in \Im_0^1(M^n)$ and the vertical lift ${}^{V}\omega \in \Im_0^1(T^*M^n)$ of $\omega \in \Im_1^0(M^n)$ are given, respectively, by

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$$^{C}X = X^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.1}$$

$$^{H}X = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.2}$$

$$^{V}\omega = \sum_i \omega_i \frac{\partial}{\partial x^{\bar{i}}} \tag{1.3}$$

with respect to the natural frame $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}} \right\}$, where Γ^h_{ij} are the coefficients of the torsion-free connection ∇_g on M^n (see [14] for more details).

We put

$$X_{(i)} = \frac{\partial}{\partial x^i}, \quad \theta^{(i)} = dx^i, \quad i = 1, ..., n$$

Using (1.2) and (1.3) we obtain

$$\tilde{e}_{(i)} = {}^{H}X_{(i)} = \frac{\partial}{\partial x^i} + \sum_h p_a \Gamma^a_{hi} \frac{\partial}{\partial x^{\bar{h}}}, \tag{1.4}$$

$$\tilde{e}_{(\bar{i})} = {}^{V}\theta^{(i)} = \frac{\partial}{\partial x^{\bar{i}}}. \tag{1.5}$$

The set $\{\tilde{e}_{(\alpha)}\} = \{\tilde{e}_{(i)}, \tilde{e}_{(\bar{i})}\} = \{{}^{H}X_{(i)}, {}^{V}\theta^{(i)}\}$ is called the frame adapted to the affine connection ∇_g. The indices $\alpha, \beta, ... = 1, ..., 2n$ indicate the indices with respect to the adapted frame.

Then from equations (1.2), (1.3), (1.4) and (1.5), we see that ${}^{H}X$ and ${}^{V}\omega$ have respectively of the form

$$^{H}X = X^i \tilde{e}_{(i)}, \quad {}^{H}X = ({}^{H}X^{\alpha}) = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, \tag{1.6}$$

$$^{V}\omega = \sum_i \omega_i \tilde{e}_{(\bar{i})}, \quad {}^{V}\omega = ({}^{V}\omega^{\alpha}) = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix} \tag{1.7}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$ [14].

Suppose now that local 1-forms $\tilde{\omega}^{\alpha}$ in $\pi^{-1}(U)$ defined by

$$\tilde{\omega}^{\alpha} = \bar{A}^{\alpha}{}_{B} dx^B,$$

where

$$A^{-1} = (\bar{A}^{\alpha}{}_{B}) = \begin{pmatrix} \bar{A}^i{}_j & \bar{A}^i{}_{\bar{j}} \\ \bar{A}^{\bar{i}}{}_j & \bar{A}^{\bar{i}}{}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0 \\ -p_a \Gamma^a_{ij} & \delta^j_i \end{pmatrix} \tag{1.8}$$

The matrix (1.8) is the inverse of the matrix

$$A = (A_{\beta}{}^{A}) = \begin{pmatrix} A_j{}^i & A_{\bar{j}}{}^i \\ A_j{}^{\bar{i}} & A_{\bar{j}}{}^{\bar{i}} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0 \\ p_a \Gamma^a_{ij} & \delta^j_i \end{pmatrix} \tag{1.9}$$

of the transformation $\tilde{e}_\beta = A_\beta{}^A \partial_A$ (see (1.4) and (1.5)). Then we see that the set $\{\tilde{\omega}^\alpha\}$ is the coframe dual to the adapted frame $\{\tilde{e}_{(\beta)}\}$, i.e. $\tilde{\omega}^\alpha(\tilde{e}_{(\beta)}) = \bar{A}^\alpha{}_B A_\beta{}^B = \delta^\alpha_\beta$.

For the Lie bracket of the adapted frame $\{\tilde{e}_{(\beta)}\}$, we have

$$[\tilde{e}_\gamma, \tilde{e}_\beta] = \Omega_{\gamma\beta}{}^\alpha \tilde{e}_\alpha$$

where

$$\Omega_{\gamma\beta}{}^\alpha = (\tilde{e}_\gamma A_\beta{}^A - \tilde{e}_\beta A_\gamma{}^A)\bar{A}^\alpha{}_A.$$

Using (1.4), (1.5), (1.8) and (1.9), the non-holonomic object $\Omega_{\gamma\beta}{}^\alpha$ has components of the form

$$\begin{cases} \Omega_{l\bar{j}}{}^{\bar{i}} = -\Omega_{\bar{j}l}{}^{\bar{i}} = \Gamma^j_{li}, \\ \Omega_{lj}{}^{\bar{i}} = p_a R_{lji}{}^a, \end{cases} \tag{1.10}$$

all the others being zero, where $R_{lji}{}^a$ being local components of the curvature tensor R of ∇_g.

The geometry of tangent bundles of M^n goes back to the fundamental paper [12] of Sasaki. On the tangent bundle of M^n by using g, one can construct several Riemannian metrics (Sasakian, Cheeger-Gromoll, g-natural metric, complete lift etc.). Cotangent bundle dual space of the tangent bundle but constraction of its lifts differ from the tangent bundle. Using different metrics are very useful tool for the geometry of the cotangent bundle. The Sasaki metric is well known and intensively studied for the cotangent bundle (see for example [2, 9, 11]). Also, the Cheeger-Gromoll metric is very important to the cotangent bundle ([1, 10]). The Riemannian extension is a metric on the cotangent bundle (see [13, 14] for more detail). The applications of this metric are contained in [3, 4, 5, 6, 7, 8]. The main aim of this paper is to study the metric $\tilde{G}$ which defined with Riemannian extension in the cotangent bundle and also investigate the metric connection and geodesics with respect to this metric.

The Riemannian extension ${}^R\nabla \in \Im^0_2(T^*M^n)$ defines a pseudo-Riemannian metric in T^*M^n. The line element of the Riemannian extension ${}^R\nabla$ is given by

$$ds^2 = 2dx^i \delta p_i$$

where $\delta p_i = dp_i - p_h \Gamma^h_{ji} dx^i$ (see [13, 14] for more detail).

On the cotangent space $\pi^{-1}(x) = T^*_x(M^n)$, the scalar product $g^{-1} = (g^{ji})$ for each $x \in M^n$ is defined by

$$g^{-1}(\omega, \theta) = g^{ji}\omega_j \theta_i$$

for all $\omega, \theta \in \Im^0_1(M^n)$.

Using Riemannian extension and quadratic differential form $\sum\limits_{i,j=1}^{n} g^{ji}\delta p_j \delta p_i$, where $\delta p_i = dp_i - p_h \Gamma^h_{ji} dx^i$, we have a new metric

$$\tilde{G} = 2dx^j \delta p_i + \sum_{i,j=1}^{n} g^{ji}\delta p_j \delta p_i \tag{1.11}$$

on the cotangent bundle T^*M^n. The main aim of this paper is to study the metric $\tilde{G}$ in the cotangent bundle and also the metric connection and geodesics with respect to this metric.

From (1.9) and (1.11) we prove that $\tilde{G}$ has components of the form

$$\tilde{G} = \begin{pmatrix} \tilde{G}_{ji} & \tilde{G}_{j\bar{i}} \\ \tilde{G}_{\bar{j}i} & \tilde{G}_{\bar{j}\bar{i}} \end{pmatrix} = \begin{pmatrix} 0 & \delta^i_j \\ \delta^j_i & g^{ji} \end{pmatrix} \tag{1.12}$$

with respect to the adapted frame $\left\{\tilde{e}_{(\alpha)}\right\}$ in T^*M^n and components

$$\tilde{G} = \begin{pmatrix} \tilde{G}_{ji} & \tilde{G}_{j\bar{i}} \\ \tilde{G}_{\bar{j}i} & \tilde{G}_{\overline{ji}} \end{pmatrix} = \begin{pmatrix} g^{hs}p_m p_a \Gamma^m_{js}\Gamma^a_{ih} - 2p_m\Gamma^m_{ji} & \delta^i_j - g^{hi}p_m\Gamma^m_{jh} \\ \delta^j_i - g^{jh}p_m\Gamma^m_{ih} & g^{ji} \end{pmatrix} \tag{1.13}$$

with respect to the to the natural frame $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\right\}$, where g^{ji} denote contravariant components of g.

From (1.2), (1.3) and (1.13) we have

$$\begin{aligned} &\tilde{G}\left({}^HX, {}^HY\right) = 0, \\ &\tilde{G}\left({}^HX, {}^V\omega\right) = {}^V(\omega(X)) = \omega(X)\circ\pi, \\ &\tilde{G}({}^V\omega, {}^V\theta) = {}^V\left(g^{-1}(\omega,\theta)\right) = g^{-1}(\omega,\theta)\circ\pi \end{aligned} \tag{1.14}$$

for any $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^0_1(M^n)$. The equation (1.14) is the definition of the metric $\tilde{G}$ because the tensor field of type (0,2) on T^*M^n is completely determined with the vector fields of type ${}^HX$ and ${}^V\omega$ (see [14, p.280]).

By means of (1.1) and (1.2), the complete lift ${}^CX$ of $X \in \Im^1_0(M^n)$ is given by

$${}^CX = {}^HX - {}^V(p(\nabla X)), \tag{1.15}$$

where $p(\nabla X) = p_k\left(\nabla_i X^k\right)dx^i$.

Taking account of (1.14) and (1.15), we obtain

$$\begin{aligned} \tilde{G}\left({}^CX, {}^CY\right) = &-{}^V\left[(p(\nabla X))(Y) + (p(\nabla Y))(X)\right] \\ &-{}^V\left(g^{-1}(p(\nabla X), p(\nabla Y))\right), \end{aligned} \tag{1.16}$$

where $g^{-1}(p(\nabla X), p(\nabla Y)) = g^{kl}\left(p_t\nabla_k X^t\right)(p_m\nabla_l Y^m)$.

Since the tensor field of $\tilde{G} \in \Im^0_2(T^*M^n)$ is completely determined with the complete lift of vector fields (see [14, p.237]), we say that the equation (1.16) is an alternative characterization of $\tilde{G}$.

From (1.16), we also have

Theorem 1.1 *If X, Y are parallel on M^n.The complete lifts of vector fields X, Y on T^*M^n with metric $\tilde{G}$ are orthogonal.*

2 Levi-Civita connection of $\tilde{G}$

The Lie bracket operation for horizontal and vertical vector fields on T^*M^n of M^n satisfies the following

$$\begin{aligned} &i)[{}^HX, {}^HY] = {}^H[X,Y] + \gamma R(X,Y) = {}^H[X,Y] + {}^V(pR(X,Y)), \\ &ii)[{}^V\omega, {}^V\theta] = 0, \\ &iii)[{}^HX, {}^V\omega] = {}^V(\nabla_X\omega) \end{aligned} \tag{2.1}$$

for any $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^0_1(M^n)$, where R denotes the curvature tensor of ∇ (See [14, p.238 and p.277] for more details).

Theorem 2.1 *Let (M^n, g) be a n-dimensional manifold and $\tilde{\nabla}$ be the Levi-Civita connection of the cotangent bundle $\left(T^*M^n, \tilde{G}\right)$.The Levi-Civita connection $\tilde{\nabla}$ holds the following*

$$
\begin{aligned}
& i)\tilde{\nabla}_{^H X}{}^H Y = {}^H(\nabla_X Y) + \tfrac{1}{2}{}^H\left(g^{-1} \circ pR(X,Y)\right) + {}^V\left(\tilde{Y}R(X,\tilde{p})\right), \\
& ii)\tilde{\nabla}_{^H X}{}^V \omega = {}^V(\nabla_X \omega) + {}^H\left(g^{-1} \circ \nabla_X \omega\right) + \tfrac{1}{2}{}^V\left(\tilde{X}R(\tilde{\omega},\tilde{p})\right), \\
& iii)\tilde{\nabla}_{^V \omega}{}^H Y = \tfrac{1}{2}{}^V\left(\tilde{Y}R(\tilde{\omega},\tilde{p})\right), \\
& iv)\tilde{\nabla}_{^V \omega}{}^V \theta = 0
\end{aligned}
$$

for all $X, Y \in \Im_0^1(M^n)$, $\omega, \theta \in \Im_1^0(M^n)$, where $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M^n)$, $\tilde{X} = g \circ X \in \Im_1^0(M^n)$, $\tilde{Y}R(X,\tilde{p}) \in \Im_1^0(M^n)$, R denotes the curvature tensor of ∇.

Proof. It is known that the Koszul formula for the Levi-Civita connection $\tilde{\nabla}$ is given by

$$
\begin{aligned}
2\bar{g}\left(\tilde{\nabla}_{^i U}{}^j V, {}^k W\right) &= {}^i U\left(\tilde{G}\left({}^j V, {}^k W\right)\right) + {}^j V\left(\tilde{G}\left({}^k W, {}^i U\right)\right) - {}^k W\left(\tilde{G}\left({}^i U, {}^j V\right)\right) \\
&- \tilde{G}\left({}^i U, \left[{}^j V, {}^k W\right]\right) + \tilde{G}\left({}^j V, \left[{}^k W, {}^i U\right]\right) + \tilde{G}\left({}^k W, \left[{}^i V, {}^j W\right]\right)
\end{aligned}
$$

for all $U, V, W \in \Im_0^1(M^n)$ and $i, j, k \in \{H, V\}$.
i) Taking account of Koszul formula and (2.1), we have

$$
\begin{aligned}
2\tilde{G}\left(\tilde{\nabla}_{^H X}{}^H Y, {}^V \omega\right) &= {}^H X\left(\tilde{G}\left({}^H Y, {}^V \omega\right)\right) + {}^H Y\left(\tilde{G}\left({}^V \omega, {}^H X\right)\right) \\
&- {}^V \omega\left(\tilde{G}\left({}^H X, {}^H Y\right)\right) - \tilde{G}\left({}^H X, \left[{}^H Y, {}^V \omega\right]\right) + \tilde{G}\left({}^H Y, \left[{}^V \omega, {}^H X\right]\right) \\
&+ \tilde{G}\left({}^V \omega, \left[{}^H X, {}^H Y\right]\right) \\
&= 2{}^V(\omega(\nabla_X Y)) + {}^V\left(g^{-1}(\omega, pR(X,Y))\right) \\
&= 2\tilde{G}\left({}^V \omega, {}^H(\nabla_X Y)\right) + \tilde{G}\left({}^H\left(g^{-1} \circ pR(X,Y)\right), {}^V \omega\right).
\end{aligned}
$$

where

$$
\begin{aligned}
{}^V\left(g^{-1}(\omega, pR(X,Y))\right) &= g^{ij}\omega_i (pR(X,Y))_j = \omega_i \left(g^{-1} \circ pR(X,Y)\right)^i \\
&= \tilde{G}\left({}^H\left(g^{-1} \circ pR(X,Y)\right), {}^V \omega\right)
\end{aligned}
$$

and

$$
\begin{aligned}
2\tilde{G}\left(\tilde{\nabla}_{^H X}{}^H Y, {}^H Z\right) &= -2p_a R_{ijk}{}^a Y^i Z^j X^k = -2p_a g^{at} R_{ijkt} Y^i Z^j X^k \\
&= 2p_a g^{at} R_{ktij} Y^i Z^j X^k = 2\tilde{p}^t g_{is} R_{ktj}{}^s Y^i Z^j X^k = 2\tilde{p}^t R_{ktj}{}^s \tilde{Y}_s Z^j X^k \\
&= 2{}^V\left(\tilde{Y}R(X,\tilde{p})Z\right) = 2\tilde{G}\left({}^V\left(\tilde{Y}R(X,\tilde{p})\right), {}^H Z\right).
\end{aligned}
$$

Then we have

$$
\tilde{\nabla}_{^H X}{}^H Y = {}^H(\nabla_X Y) + \frac{1}{2}{}^H\left(g^{-1} \circ pR(X,Y)\right) + {}^V\left(\tilde{Y}R(X,\tilde{p})\right).
$$

$$
\begin{aligned}
ii)2\tilde{G}\left(\tilde{\nabla}_{^H X}{}^V \omega, {}^H Z\right) &= {}^H X\left(\tilde{G}\left({}^V \omega, {}^H Z\right)\right) + {}^V \omega\left(\tilde{G}\left({}^H Z, {}^H X\right)\right) \\
&- {}^H Z\left(\tilde{G}\left({}^H X, {}^V \omega\right)\right) - \tilde{G}\left({}^H X, \left[{}^V \omega, {}^H Z\right]\right) + \tilde{G}\left({}^V \omega, \left[{}^H Z, {}^H X\right]\right) \\
&+ \tilde{G}\left({}^H Z, \left[{}^H X, {}^V \omega\right]\right) \\
&= 2{}^V((\nabla_X \omega)Z) + {}^V\left(g^{-1}(\omega, pR(Z,X))\right) \\
&= 2\tilde{G}\left({}^V(\nabla_X \omega), {}^H Z\right) + \tilde{G}\left({}^V\left(\tilde{X}R(\tilde{\omega},\tilde{p})\right), {}^H Z\right),
\end{aligned}
$$

$$2\tilde{G}\left(\tilde{\nabla}_{{}^{H}X}{}^{V}\omega, {}^{V}\theta\right) = 2^{V}\left(g^{-1}\left(\nabla_X\omega, \theta\right)\right) = 2\tilde{G}\left({}^{H}\left(g^{-1}\circ\left(\nabla_X\omega\right)\right), {}^{V}\omega\right)$$

so

$\tilde{\nabla}_{{}^{H}X}{}^{V}\omega = {}^{V}\left(\nabla_X\omega\right) + {}^{H}\left(g^{-1}\circ\nabla_X\omega\right) + \frac{1}{2}{}^{V}\left(\tilde{X}R\left(\tilde{\omega}, \tilde{p}\right)\right).$

By the calculations similar to those in (i) and (ii), the proofs of (iii) and (iv) are obtained easily.

If we compute components of the Christoffel symbols $\tilde{\Gamma}^{\delta}_{\alpha\beta}$ with respect to the adapted frame $\left\{\tilde{e}_{(\alpha)}\right\}$ of T^*M^n with the metric $\tilde{G}$ by making use of $\tilde{\nabla}_{\tilde{e}_\alpha}\tilde{e}_\beta = \tilde{\Gamma}^{\delta}_{\alpha\beta}\tilde{e}_\delta$, then we have

Corollary 2.1 *Let (M^n, g) be a n-dimensional manifold and $\tilde{\nabla}$ be the Levi-Civita connection of the cotangent bundle $\left(T^*M^n, \tilde{G}\right)$. The components of the Christoffel symbols $\tilde{\Gamma}^{\delta}_{\alpha\beta}$ are found as follows*

$$\begin{aligned}
&\tilde{\Gamma}^{k}_{ij} = \Gamma^{k}_{ij} + \tfrac{1}{2}p_a R_{ijt}{}^{a}g^{tk},\\
&\tilde{\Gamma}^{k}_{i\bar{j}} = p_a R_{kji}{}^{a},\\
&\tilde{\Gamma}^{\bar{k}}_{i\bar{j}} = \tfrac{1}{2}p_a R_{kjt}{}^{a}g^{ti},\\
&\tilde{\Gamma}^{\bar{k}}_{ij} = -\Gamma^{j}_{ik} + \tfrac{1}{2}p_a R_{kit}{}^{a}g^{jt},\\
&\tilde{\Gamma}^{\bar{k}}_{\bar{i}j} = -\Gamma^{j}_{it}g^{tk}\\
&\tilde{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}} = \tilde{\Gamma}^{k}_{\bar{i}\bar{j}} = \tilde{\Gamma}^{k}_{\bar{i}j} = 0.
\end{aligned} \tag{2.2}$$

with respect to the adapted frame $\left\{\tilde{e}_{(\alpha)}\right\}$.

From (1.4), (1.5), (1.6), (1.7) and (2.2) we obtain that the covariant derivatives $\tilde{\nabla}^{V}\omega$, $\tilde{\nabla}^{H}X$ and $\tilde{\nabla}^{C}X$ have respectively of the form

$$\tilde{\nabla}_{\kappa}{}^{V}\omega^{\alpha} = \begin{pmatrix} -\Gamma^{t}_{km}g^{im}\omega_t & 0 \\ \nabla_k\omega_i + \frac{1}{2}p_a R_{ik.}{}^{ta}\omega_t & 0 \end{pmatrix}, \tag{2.3}$$

$$\tilde{\nabla}_{\kappa}{}^{H}X^{\alpha} = \begin{pmatrix} \nabla_k X^i + \frac{1}{2}p_a R_{kt.}{}^{is}X^t & 0 \\ p_a R_{itk}{}^{a}X^t & -\frac{1}{2}p_a R_{ti.}{}^{ka}X^t \end{pmatrix}, \tag{2.4}$$

$$\tilde{\nabla}_{\kappa}{}^{C}X^{\alpha} = \begin{pmatrix} \nabla_k X^i + \frac{1}{2}p_a R_{kt.}{}^{ia}X^k - \Gamma^{t}_{km}g^{mi}p_h\nabla_t X^h & 0 \\ \beta & -\nabla_i X^k - \frac{1}{2}p_a R_{ti.}{}^{ka}X^t \end{pmatrix} \tag{2.5}$$

where $\beta = -p_h\nabla_k\nabla_i X^m + p_a R_{itk}{}^{a}X^t - \frac{1}{2}p_a p_m R_{ik.}{}^{ta}\nabla_t X^m$.

If M^n has a pseudo-Riemannian metric g, then we have

$$\begin{aligned}
p_a R_{itk}{}^{a}X^t &= p_a g^{as}R_{itks}X^t = -p_a g^{as}R_{tiks}X^t = -p_a g^{as}R_{ksti}X^t\\
&= -p_a g^{as}g_{if}R_{kst.}{}^{f}X^t = -2p_a g^{as}g_{if}\nabla_{[k}\nabla_{s]}X^f.
\end{aligned} \tag{2.6}$$

From (2.3), (2.4), (2.5) and (2.6) we have

Theorem 2.2 *The vertical lift of covector field $\omega \in \Im^0_1(M^n)$ to T^*M^n with metric $\tilde{G}$ is never parallel.*

Theorem 2.3 *The complete and horizontal lifts of a vector field $X \in \Im^1_0(M^n)$ to T^*M^n with metric $\tilde{G}$ are parallel if and only if X is parallel to ∇ on M^n.*

3 Curvature properties of $\tilde{G}$

The curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ calculate with
$\tilde{R}\left(\tilde{e}_{(\alpha)}, \tilde{e}_{(\beta)}\right)\tilde{e}_{(\gamma)} = \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{e}_{(\gamma)} - \tilde{\nabla}_\beta \tilde{\nabla}_\alpha \tilde{e}_{(\gamma)} - \Omega_{\alpha\beta}{}^{\varepsilon}\tilde{\nabla}_\varepsilon \tilde{e}_{(\gamma)}$
where $\tilde{\nabla}_\alpha = \tilde{\nabla}_{\tilde{e}_{(\alpha)}}$ and we have
$\tilde{R}^\sigma_{\alpha\beta\gamma} = \tilde{e}_\alpha \tilde{\Gamma}^\sigma_{\beta\gamma} - \tilde{e}_\beta \tilde{\Gamma}^\sigma_{\alpha\gamma} + \tilde{\Gamma}^\sigma_{\alpha\varepsilon}\tilde{\Gamma}^\varepsilon_{\beta\gamma} - \tilde{\Gamma}^\sigma_{\beta\varepsilon}\tilde{\Gamma}^\varepsilon_{\alpha\gamma} - \Omega_{\alpha\beta}{}^{\varepsilon}\tilde{\Gamma}^\sigma_{\varepsilon\gamma}$
with respect to the adapted frame $\left\{\tilde{e}_{(\alpha)}\right\}$.

By virtue of (1.10) and (2.2), we obtain

$$
\begin{aligned}
\tilde{R}_{kij}{}^{l} &= R_{kij}{}^{l} + \tfrac{1}{4}p_a p_m \left(R_{kt.}{}^{lm}R_{ij.}{}^{ta} - R_{it.}{}^{lm}R_{kj.}{}^{ta}\right) \\
&\quad + \tfrac{1}{2}p_a g^{lt}\left(\nabla_k R_{ijt}{}^{a} - \nabla_i R_{kjt}{}^{a}\right) - p_a g^{ml}\left(R_{tjk}{}^{a}\Gamma^t_{im} - R_{tji}{}^{a}\Gamma^t_{km}\right), \\
\tilde{R}_{\bar{k}ij}{}^{l} &= \tfrac{1}{2}R_{ij.}{}^{lk} + \tfrac{1}{2}p_a \Gamma^t_{if}R_{tj.}{}^{ka}g^{fl}, \\
\tilde{R}_{ki\bar{j}}{}^{l} &= R_{ik.}{}^{lj} - \Gamma^j_{kt}\Gamma^t_{if}g^{fl} + \Gamma^j_{it}\Gamma^t_{kf}g^{fl} + \tfrac{1}{2}p_a\left(R^{f}_{.\,k.}{}^{la}\Gamma^j_{if} - R^{f}_{.\,i.}{}^{la}\Gamma^j_{kf}\right) \\
&\quad + \tfrac{1}{2}p_a g^{ml}\left(R_{tk.}{}^{ja}\Gamma^t_{im} - R_{ti.}{}^{ja}\Gamma^t_{km}\right), \\
\tilde{R}_{ki\bar{j}}{}^{\bar{l}} &= R_{ikl}{}^{j} + \tfrac{1}{2}p_a\left(g^{mj}\nabla_k R_{lim}{}^{a} - g^{tj}\nabla_i R_{lkt}{}^{a}\right) \\
&\quad - p_a g^{mt}\left(R_{ltk}{}^{a}\Gamma^j_{im} - R_{lti}{}^{a}\Gamma^j_{km}\right) \\
&\quad + \tfrac{1}{4}p_a p_m\left(R_{lk.}{}^{tm}R_{ti.}{}^{ja} - R_{li.}{}^{tm}R_{tk.}{}^{ja}\right), \\
\tilde{R}_{kij}{}^{\bar{l}} &= p_m\left(\nabla_k R_{lji}{}^{m} - \nabla_i R_{ljk}{}^{m}\right) + \tfrac{1}{2}p_a p_m\left(R_{lk.}{}^{tm}R_{tji}{}^{a} - R_{li.}{}^{tm}R_{tjk}{}^{a}\right) \\
&\quad + \tfrac{1}{2}p_a p_m\left(R_{ij.}{}^{tm}R_{ltk}{}^{a} - R_{kj.}{}^{tm}R_{lti}{}^{a}\right) - \tfrac{1}{2}p_a p_m R_{kit}{}^{a}R_{lj.}{}^{tm}, \\
\tilde{R}_{\bar{k}ij}{}^{\bar{l}} &= R_{lji}{}^{k} - \tfrac{1}{2}p_a g^{tk}\nabla_i R_{ljt}{}^{s} + \tfrac{1}{4}p_a p_m\left(R_{lt.}{}^{km}R_{ij.}{}^{ta} - R_{li.}{}^{tm}R_{tj.}{}^{ka}\right), \\
\tilde{R}_{\bar{k}\bar{i}j}{}^{\bar{l}} &= R_{lj.}{}^{ik}, \\
\tilde{R}_{\bar{k}i\bar{j}}{}^{\bar{l}} &= \tfrac{1}{2}R_{li.}{}^{jk} + \tfrac{1}{2}p_a R^{f}_{.\,l.}{}^{ka}\Gamma^j_{if}, \\
\tilde{R}_{\bar{k}\bar{i}\bar{j}}{}^{\bar{l}} &= \tilde{R}_{\bar{k}\bar{i}\bar{j}}{}^{l} = \tilde{R}_{k\bar{i}\bar{j}}{}^{l} = \tilde{R}_{\bar{k}i\bar{j}}{}^{l} = \tilde{R}_{\bar{k}\bar{i}j}{}^{l} = 0.
\end{aligned}
\tag{3.1}
$$

Then we have the following theorem

Theorem 3.1 *Let (M^n, g) be a n-dimensional manifold and T^*M^n be its cotangent bundle with metric $\tilde{G}$. Then $(T^*M^n, \tilde{G})$ is flat if and only if M^n is flat.*

Proof. Firstly, we assume that $R = 0$, then from the equaions (3.1) it implies that $\tilde{R} = 0$. Conversely, we assume that $\tilde{R} = 0$ in the point $(x^i, p_i) = (x^i, 0) \in T^*M^n$. Then we have

$$
\tilde{R}_{ki\bar{j}}{}^{l}|_{(x^i,0)} = \left(R_{ik.}{}^{lj} - \Gamma^j_{kt}\Gamma^t_{if}g^{fl} + \Gamma^j_{it}\Gamma^t_{kf}g^{fl} + \frac{1}{2}p_a\left(R^{f}_{.\,k.}{}^{la}\Gamma^j_{if} - R^{f}_{.\,i.}{}^{la}\Gamma^j_{kf}\right)\right)|_{(x^i,0)} = 0
$$

for $R = 0$.

4 The metric connection with respect to the metric $\tilde{G}$

The Levi-Civita connection $\tilde{\nabla}$ of the metric $\tilde{G}$ on T^*M^n given in Theorem 2.1. This is the unique connection which satisfies $\tilde{\nabla}\tilde{G} = 0$ which has no torsion. But we shall find another connection which satisfies $\bar{\nabla}\tilde{G} = 0$ and has non-trivial torsion tensor. This connection is called the metric connection of $\tilde{G}$.

The horizontal lift ${}^H\nabla$ of any connection ∇ on the cotangent bundle T^*M^n is defined by

$$\begin{cases} {}^H\nabla_{{}^V\theta}{}^V\omega = 0, & {}^H\nabla_{{}^V\theta}{}^HY = 0, \\ {}^H\nabla_{{}^HX}{}^V\omega = {}^V(\nabla_X\omega), & {}^H\nabla_{{}^HX}{}^HY = {}^H(\nabla_XY) \end{cases} \tag{4.1}$$

for any $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$.

We put ${}^H\nabla_\alpha = {}^H\nabla_{\tilde{e}_{(\alpha)}}$. Then taking account of ${}^H\nabla_\alpha\tilde{e}_{(\beta)} = {}^H\Gamma_{\alpha\beta}^\gamma\tilde{e}_{(\gamma)}$ and writing ${}^H\Gamma_{\alpha\beta}^\gamma$ for the different indices, from (4.1) we have

$$\begin{cases} {}^H\Gamma_{ij}^k = {}^H\Gamma_{ij}^k, & {}^H\Gamma_{i\bar{j}}^{\bar{k}} = -{}^H\Gamma_{ik}^j, \\ {}^H\Gamma_{\bar{i}j}^k = {}^H\Gamma_{i\bar{j}}^k = {}^H\Gamma_{\bar{i}\bar{j}}^k = {}^H\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = {}^H\Gamma_{ij}^{\bar{k}} = {}^H\Gamma_{\bar{i}j}^{\bar{k}} = 0. \end{cases} \tag{4.2}$$

Let T be the torsion tensor of the horizontal lift ${}^H\nabla$. Then T is the skew-symmetric tensor field of type (1,2) on T^*M^n determined by [14, p.287]

$$\mathrm{T}\left({}^V\omega, {}^V\theta\right) = 0, \quad \mathrm{T}\left({}^HX, {}^V\theta\right) = 0, \quad \mathrm{T}\left({}^HX, {}^HY\right) = -\gamma R(X,Y)$$

where R is the curvature tensor of ∇ and $\gamma R(X,Y) = \sum_i p_h R_{kli}{}^h X^k Y^l \frac{\partial}{\partial x^{\bar{i}}}$. Thus the connection ${}^H\nabla$ has non-trivial torsion even for the Levi-Civita connection ∇_g determined by g, unless g is locally flat.

Using (1.14) and (4.1), we have

$$\begin{aligned} \left({}^H\nabla_{{}^V\omega}\tilde{G}\right)\left({}^V\theta, {}^V\varepsilon\right) &= {}^H\nabla_{{}^V\omega}\tilde{G}\left({}^V\theta, {}^V\varepsilon\right) - \tilde{G}\left({}^H\nabla_{{}^V\omega}{}^V\theta, {}^V\varepsilon\right) - \tilde{G}\left({}^V\theta, {}^H\nabla_{{}^V\omega}{}^V\varepsilon\right) \\ &= {}^H\nabla_{{}^V\omega}{}^V\left(g^{-1}(\theta,\varepsilon)\right) = {}^V\omega{}^V\left(g^{-1}(\theta,\varepsilon)\right) = 0, \end{aligned}$$

$$\begin{aligned} \left({}^H\nabla_{{}^HX}\tilde{G}\right)\left({}^V\theta, {}^V\varepsilon\right) &= {}^H\nabla_{{}^HX}\tilde{G}\left({}^V\theta, {}^V\varepsilon\right) - \tilde{G}\left({}^H\nabla_{{}^HX}{}^V\theta, {}^V\varepsilon\right) - \tilde{G}\left({}^V\theta, {}^H\nabla_{{}^HX}{}^V\varepsilon\right) \\ &= {}^H\nabla_{{}^HX}{}^V\left(g^{-1}(\theta,\varepsilon)\right) - \tilde{G}\left({}^V(\nabla_X\theta), {}^V\varepsilon\right) - \tilde{G}\left({}^V\theta, {}^V(\nabla_X\varepsilon)\right) \\ &= {}^HX^V\left(g^{-1}(\theta,\varepsilon)\right) - {}^V\left(g^{-1}(\nabla_X\theta,\varepsilon)\right) - {}^V\left(g^{-1}(\theta,\nabla_X\varepsilon)\right) \\ &= {}^V\left(Xg^{-1}(\theta,\varepsilon)\right) - {}^V\left(g^{-1}(\nabla_X\theta,\varepsilon)\right) - {}^V\left(g^{-1}(\theta,\nabla_X\varepsilon)\right) \\ &= {}^V\left(\left(\nabla_Xg^{-1}\right)(\theta,\varepsilon)\right) = 0, \end{aligned}$$

$$\begin{aligned} \left({}^H\nabla_{{}^V\omega}\tilde{G}\right)\left({}^V\theta, {}^HZ\right) &= {}^H\nabla_{{}^V\omega}\tilde{G}\left({}^V\theta, {}^HZ\right) - \tilde{G}\left({}^H\nabla_{{}^V\omega}{}^V\theta, {}^HZ\right) - \tilde{G}\left({}^V\theta, {}^H\nabla_{{}^V\omega}{}^HZ\right) \\ &= {}^H\nabla_{{}^V\omega}{}^V(\theta(Z)) = {}^V\omega{}^V(\theta(Z)) = 0, \end{aligned}$$

$$\begin{aligned} \left({}^H\nabla_{{}^HX}\tilde{G}\right)\left({}^V\theta, {}^HZ\right) &= {}^H\nabla_{{}^HX}\tilde{G}\left({}^V\theta, {}^HZ\right) - \tilde{G}\left({}^H\nabla_{{}^HX}{}^V\theta, {}^HZ\right) - \tilde{G}\left({}^V\theta, {}^H\nabla_{{}^HX}{}^HZ\right) \\ &= {}^V((\nabla_X\theta)Z) - {}^V((\nabla_X\theta)Z) = 0, \end{aligned}$$

$$\begin{aligned} &\left({}^H\nabla_{{}^V\omega}\tilde{G}\right)\left({}^HY, {}^V\varepsilon\right) = 0, \\ &\left({}^H\nabla_{{}^HX}\tilde{G}\right)\left({}^HY, {}^V\varepsilon\right) = 0, \\ &\left({}^H\nabla_{{}^V\omega}\tilde{G}\right)\left({}^HY, {}^HZ\right) = 0, \\ &\left({}^H\nabla_{{}^HX}\tilde{G}\right)\left({}^HY, {}^HZ\right) = 0 \end{aligned}$$

for any $X, Y, Z \in \Im_0^1(M^n)$ and $\omega, \theta, \varepsilon \in \Im_1^0(M^n)$, i.e. the horizontal lift ${}^H\nabla$ of ∇_g is a metric connection with respect to the metric $\tilde{G}$. Thus we have

Theorem 4.1 *Let ∇ be the Levi-Civita connection on (M^n, g). Then the horizontal lift ${}^H\nabla$ is a metric connection of the metric $\tilde{G}$.*

Let ${}^HR$ be the curvature tensor field of ${}^H\nabla$. The components of ${}^HR$ is given by

$$ {}^HR_{\delta\gamma\beta}{}^{\alpha} = 2\left(\tilde{e}_{[\delta}\,{}^H\Gamma^{\alpha}_{\gamma]\beta} + {}^H\Gamma^{\alpha}_{[\delta|\varepsilon|}\,{}^H\Gamma^{\varepsilon}_{\gamma]\beta}\right) - \Omega_{\delta\gamma}{}^{\varepsilon H}\Gamma^{\alpha}_{\varepsilon\beta} $$

with respect to the adapted frame $\left\{\tilde{e}_{(\alpha)}\right\}$. Using (1.4), (1.5), (1.10), (4.2) and computing the components of the Ricci tensor field ${}^HR_{\gamma\beta} = {}^HR_{\alpha\gamma\beta}{}^{\alpha}$, we have

$$ \begin{cases} {}^HR_{kj} = {}^HR_{\alpha kj}{}^{\alpha} = {}^HR_{ikj}{}^{i} + {}^HR_{\bar{i}kj}{}^{\bar{i}} = R_{ikj}{}^{i} = R_{kj}, \\ {}^HR_{\bar{k}\bar{j}} = {}^HR_{\bar{k}j} = {}^HR_{k\bar{j}} = 0, \end{cases} \tag{4.3} $$

where R_{kj} is the Ricci tensor field of ∇_g on M^n [14].

For the scalar curvature of ${}^H\nabla$ with respect to the metric $\tilde{G}$, we have

$$ \begin{aligned} {}^Hr &= \tilde{G}^{\gamma\beta H}R_{\gamma\beta} = \tilde{G}^{kjH}R_{kj} + \tilde{G}^{k\bar{j}H}R_{k\bar{j}} + \tilde{G}^{\bar{k}jH}R_{\bar{k}j} + \tilde{G}^{\bar{k}\bar{j}H}R_{\bar{k}\bar{j}} \\ &= \tilde{G}^{kjH}R_{kj} = -g^{kj}R_{kj} = -r \end{aligned} $$

by means of (4.3) and $\left(\tilde{G}\right)^{-1} = \left(\tilde{G}^{\alpha\beta}\right) = \begin{pmatrix} \tilde{G}^{ji} & \tilde{G}^{j\bar{i}} \\ \tilde{G}^{\bar{j}i} & \tilde{G}^{\bar{j}\bar{i}} \end{pmatrix} = \begin{pmatrix} -g^{ji} & \delta^i_j \\ \delta^j_i & 0 \end{pmatrix}$.

Thus we have

Theorem 4.2 *The cotangent bundle $(T^*M^n, \tilde{G})$ with the metric connection ${}^H\nabla$ has vanishing scalar curvature ${}^Hr$ with respect to the metric if and only if the scalar curvature r of ∇_g on M^n is zero.*

5 Geodesics on $(T^*M^n, \tilde{G})$

Let now we investigate the geodesics of the $(T^*M^n, \tilde{G})$. Firstly, $C : x^h = x^h(t)$ be a curve in M^n and $\vartheta_h(t)$ be a covector field along C. Then we suppose that $\tilde{C}$ be a curve on T^*M^n and locally expressed by

$$ x^h = x^h(t), \qquad x^{\bar{h}} \overset{def}{=} p_h = \vartheta_h(t). \tag{5.1} $$

If the curve C satisfies at all the points the relation

$$ \frac{\delta\vartheta_h}{dt} = \frac{d\vartheta_h}{dt} - \Gamma^i_{jh}\frac{dx^j}{dt}\vartheta_i = 0 $$

then the curve $\tilde{C}$ is said to be a horizontal lift of the curve C in M^n . Hence, the initial condition $\vartheta_h = \vartheta^0_h$ for $t = t_0$ is given, there exists a unique horizontal lift given by (5.1).

Let t be the arc length of a curve $x^A = x^A(t)$, $A = (i, \bar{i})$ in T^*M^n. Equations of the geodesic have the usual form

$$ \frac{\delta^2x^A}{dt^2} = \frac{d^2x^A}{dt^2} + \tilde{\Gamma}^A_{CB}\frac{dx^C}{dt}\frac{dx^B}{dt} = 0 \tag{5.2} $$

with respect to the induced coordinates $\left(x^i, x^{\bar{i}}\right) = \left(x^i, p_i\right)$ in T^*M^n, where $\tilde{\Gamma}^A_{CB}$ are components of $\tilde{\nabla}$ defined by (2.2).

Now we write the equations (5.2) to the adapted frame $\left\{\tilde{e}_{(\alpha)}\right\}$. Taking account of (1.8), we have

$$\theta^{\alpha} = \tilde{A}^{\alpha}{}_{A} dx^{A},$$

i.e.

$$\theta^{h} = \tilde{A}^{h}{}_{A} dx^{A} = \delta_{i}^{h} dx^{i} = dx^{h}$$

for $\alpha = h$ and

$$\theta^{\bar{h}} = \tilde{A}^{\bar{h}}{}_{A} dx^{A} = -p_a \Gamma_{hj}^{a} dx^{j} + \delta_{j}^{h} dx^{j} = \delta p_h$$

for $\alpha = \bar{h}$. Also we put

$$\frac{\theta^{h}}{dt} = \tilde{A}^{h}{}_{A} \frac{dx^{A}}{dt} = \frac{dx^{h}}{dt},$$
$$\frac{\theta^{\bar{h}}}{dt} = \tilde{A}^{\bar{h}}{}_{A} \frac{dx^{A}}{dt} = \frac{\delta p_h}{dt}$$

along a curve $x^{A} = x^{A}(t)$ in $T^{*}M^{n}$.

So we obtain

$$\frac{d}{dt}\left(\frac{\theta^{\alpha}}{dt}\right) + \tilde{\Gamma}_{\gamma\beta}^{\alpha} \frac{\theta^{\gamma}}{dt}\frac{\theta^{\beta}}{dt} = 0$$

whit respect to adapted frame $\left\{\tilde{e}_{(\alpha)}\right\}$ and from (2.2), we have

$$\begin{aligned} &(a)\tfrac{\delta^2 x^h}{dt^2} + \tfrac{1}{2} p_m R_{ijt}{}^{m} g^{th} \tfrac{dx^i}{dt}\tfrac{dx^j}{dt} - \Gamma_{it}^{j} g^{th} \tfrac{dx^i}{dt}\tfrac{\delta p_j}{dt} = 0, \\ &b)\tfrac{\delta^2 p_h}{dt^2} + p_m R_{hji}{}^{m} \tfrac{dx^i}{dt}\tfrac{dx^j}{dt} + p_m R_{hit}{}^{s} g^{tj} \tfrac{dx^i}{dt}\tfrac{\delta p_j}{dt} = 0. \end{aligned} \tag{5.3}$$

Because of $R_{(ij)t}{}^{m} = 0$, then we have $R_{ijt}{}^{m} \frac{dx^i}{dt}\frac{dx^j}{dt} = 0$. So we obtain

$$\begin{aligned} &(a)\tfrac{\delta^2 x^h}{dt^2} - \Gamma_{it}^{j} g^{th} \tfrac{dx^i}{dt}\tfrac{\delta p_j}{dt} = 0, \\ &b)\tfrac{\delta^2 p_h}{dt^2} + p_m R_{hji}{}^{m} \tfrac{dx^i}{dt}\tfrac{dx^j}{dt} + p_m R_{hit}{}^{s} g^{tj} \tfrac{dx^i}{dt}\tfrac{\delta p_j}{dt} = 0. \end{aligned} \tag{5.4}$$

Theorem 5.1 *$\tilde{C}$ be a curve in $T^{*}M^{n}$ expressed locally by $x^{h} = x^{h}(t)$, $p_h = \vartheta_h(t)$ with respect to the induced coordinates $\left(x^{i}, x^{\bar{i}}\right) = \left(x^{i}, p_i\right)$ in $T^{*}M^{n}$. The curve $\tilde{C}$ is a geodesic of $\tilde{G}$, if it satisfies the equation (5.4).*

If a curve satisfies (5.4) lies on a fibre given by $x^{h} = const.$, then (5.4, b) reduces to

$$\frac{\delta^2 p_h}{dt^2} = 0$$

so that $x^{h} = x^{h}(t)\,, p_h = p_h(t)$, a_h and b_h being constant. Thus we have

Theorem 5.2 *If geodesic $x^{h} = x^{h}(t)$, $p_h = p_h(t)$ lies on a fibre of $T^{*}M^{n}$ with respect to the metric $\tilde{G}$, then the geodesic is given by linear equations*
$x^{h} = c^{h}$,
$p_h = a_h t + b_h$
where a_h, b_h and c^{h} are constant.

Let now $\tilde{C} : x^{h} = x^{h}(t)\,, \quad x^{\bar{h}} \overset{def}{=} p_h = \vartheta_h(t)$ be a horizontal lift $\left(\frac{\delta p_h}{dt} = \frac{\delta \vartheta_h}{dt} = 0\right)$ of the geodesic $C : x^{h} = x^{h}(t)$ $\left(\frac{\delta^2 x^h}{dt^2} = 0\right)$ in M^{n} of ∇_g. Then by virtue of (5.4), we have

Theorem 5.3 *The horizontal lift of a geodesic on (M^{n}, g) need not be a geodesic on $T^{*}M^{n}$ with respect to the connection $\tilde{\nabla}$.*

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