Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
38 (2018), no. 4, Mathematics, 22-38.

A growth property assertion for positive solutions of non-uniformly degenerate elliptic equations of second order

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Received: 11.12.2017 / Revised: 31.05.2018 / Accepted: 11.09.2018

Abstract. *In this paper, we consider a class of non-divergent form elliptic equations satisfying non-uniform ellipticity condition concerning the leading coefficients. For those equations, E.M.Landis's type growth assertion of positive solutions has been proved.*

Keywords. non-uniformly elliptic, degenerate elliptic equations, qualitative properties

Mathematics Subject Classification (2010): 35B09, 35B45, 35B50, 35B65, 35J70

1 Introduction

Let E_n be n-dimensional Euchideam space of points $x = (x_1, ..., x_n)$, $n \leq 3$, D be a bounded domain contained in E_n such that $0 \in D$. The ∂D is a boundary of the domain D. We consider in D an elliptic equation of the form

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{ij} + \sum_{i=1}^{n} b_{ij}(x)u_i + c(x) = 0, \tag{1.1}$$

where for $i, j = 1, ..., n$, $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. For $x \in D$ the matrix $\|a_{ij}(x)\|$ is symmetric and has elements measurable functions satisfying non-uniformly ellipticity condition: for all $x \in D$, $\xi \in E_n$ it is fulfilled the condition

$$\gamma \sum_{i=1}^{n} \lambda_i(x)\xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \gamma^{-1} \sum_{i=1}^{n} \lambda_i(x)\xi_i^2, \tag{1.2}$$

where $\gamma \in (0, 1]$ is a constant, the functions $\lambda_i(x)$, $i = 1, ..., n$ are finite and positive a.e. in D. Here the lower term coefficients satisfy the conditions

$$|b_i| \leq b_0,\ i = 1, ...n,\ -c_0 \leq c(x) \leq 0, \tag{1.3}$$

by some positive constants C_0, $b_0 > 0$.

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Let $\omega_i(t)$ be positive continuous and monotony increasing functions on $[0, diamD]$, $\omega_i(0) = 0$. Set $\rho(x) = \sum\limits_{i=1}^{n} \omega_i(|x_i|)$, $\lambda_i(t) = g_i(\rho(x))$, $g_i(t) = \left(\omega_i^{-1}(t)/t\right)^2$, $i = 1, ..., n$. Here $\omega_i^{-1}(t)$ are the functions inverse to $\omega_i(t)$, wherein

$$\alpha\omega_i(t) \le \omega_i(\eta t) \le \beta\omega_i(t) \tag{1.4}$$

with positive constants $\alpha, \beta, \eta \in (0, \infty)$ not depending on $t \in (0, diamD)$.

Under the solution of equation (1) we will mean its classical solution, -i.e. the functions $u(x) \in C^2(D) \cap C(\bar{D})$ satisfying pointwise equation (1). The function $u(x) \in C^2(D)$ is said to be L subelliptic in D if $Lu \ge 0$. The function $u(x)$ is called L- superelliptic in D if the function $-u(x)$ is L -subelliptic in D.

Let $x^0 \in E_n$, $R > 0$, $k > 0$, $\ni_{R:K}$ (x^0) be the ellipsoid $\left\{x : x \in D \sum\limits_{i=1}^{n} \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2} < K^2\right\}$, $\Pi_{R:K}(x^0)$ be the parallelepiped $\{x : x \in D, |x_i - x_i^0| < K\omega_i^{-1}(R), i = 1, ..., n\}$, the ball $B_R(x^0) = \{x : x \in D, |x - x^0| < R\}$.

Denote $B^1 = \ni_{R:17}$ (0), $B^2 = \ni_{R:1}$ (0), $B^3 = B^1 \setminus B^2$. For $x, y \in B^3$, $x \ne y$ we introduce the function

$$G(x, y) = r^{-S},$$

where $r = \sqrt{\sum\limits_{i=1}^{n} \frac{(x_i - y_i)^2}{(\omega_i^{-1}(R))^2}}$, S-is a positive number, $R \le 1$ and $\omega_i^{-1}(1) \le 1$.

We assume that the coefficients of operator L are defined all over E_n and satisfy conditions (1.2)-(1.4).

The goal of the paper is to prove an assertion on growth of positive solutions for a class of non-divergent non-uniformly degenerating elliptic equations of second order. A basic study the uniform ellipticity cases, see e.g. in [12, 9, 7, 8, 13, 10]. On a study of qualitative behavior of solutions we refer to the papers [5, 4] and [3, 2, 11, 6], in cases uniform and nonuniform degeneration, respectively. In their study of non-uniformly degeneration we refer to [2, 6, 3], where power type weights or at most leading terms have been considered. Also, another approach has been applied in this study.

The notation $C(\cdots)$ means that a positive constant C depends only on the content of bracket.

Similarly to [11, 6], one can show that, there exist constants $C_1(n)$ and $C_2(n)$ such that

$$C_1(n)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2 \le \lambda_i(x) \le C_2(n)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2, \; x \in B^3. \tag{1.5}$$

Let E be a Borel set contained in B^3. The positive measure $\mu(y)$ defined on measurable subsets of E is admissible if

$$\int\limits_E G(x, y) d\mu(y) \le 1, \; x \bar{\in} E.$$

The number $C_S(E) = \sup \mu(E)$, where the least upper bound is taken all over the admissible measures μ is called elliptic $S-$ capacity of set E.

Lemma 1.1 *Let the coefficients of operator L satisfy conditions* (1.2)-(1.4). *Then there exists a positive number $S(\gamma, n, b_0)$ such that for any fixed point $y \in B^3$ the function $G(x, y)$ is L -subelliptic in B^3 for $R \le 1$, $\omega_i^{-1}(1) \le 1$, $i = 1, ..., n$.*

Proof. For $i, j = 1, .., n$ we have

$$\frac{\partial r}{\partial x_i} = \frac{x_i - y_i}{r\left(\omega_i^{-1}(R)\right)^2};\ \frac{\partial G}{\partial x_i} = -\frac{S}{r^{S+2}} \cdot \frac{x_i - y_i}{r\left(\omega_i^{-1}(R)\right)^2};$$

$$\frac{\partial G}{\partial x_i^2} = S(S+2)r^{-S-4} \cdot \frac{(x_i - y_i)^2}{\left(\omega_i^{-1}(R)\right)^4} - \frac{S}{r^{S+2}\left(\omega_i^{-1}(R)\right)^2};$$

$$\frac{\partial^2 G}{\partial x_i \partial x_j} = \frac{S(S+2)}{r^{S+4}} \cdot \frac{(x_i - y_i)(x_j - y_j)}{\left(\omega_i^{-1}(R)\right)^2 \left(\omega_j^{-1}(R)\right)^2}.$$

Then

$$LG = r^{-S-2}\left[S(S+2)\sum_{i,j=1}^{n} a_{ij}(x)\frac{(x_i - y_i)(x_j - y_j)}{r^2\left(\omega_i^{-1}(R)\right)^2\left(\omega_j^{-1}(R)\right)^2}\right.$$
$$\left. -S\sum_{i=1}^{n}\frac{u_{ii}(x)}{\left(\omega_i^{-1}(R)\right)^2} - S\sum_{i=1}^{n} b_i(x)\frac{x_i - y_i}{\left(\omega_i^{-1}(R)\right)^2} + C(x)r^2\right].$$

Using condition (1.3) and the Cauchy inequality, we get

$$LG \geq r^{-S-2}\left[\frac{S(S+2)\gamma}{r^2}\sum_{i=1}^{n}\frac{\lambda_i(x)(x_i - y_i)^2}{\left(\omega_i^{-1}(R)\right)^4} - \frac{S}{\gamma}\sum_{i=1}^{n}\frac{\lambda_i(x)}{(\omega_i^{-1}(R))^2}\right.$$
$$\left. -S\left(\sum_{i=1}^{n}\frac{b_i^2(x)}{(\omega_i^{-1}(R))^2}\right)^{1/2} \cdot \left(\sum_{i=1}^{n}\frac{(x_i - y_i)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} + C(x)r^2\right]$$

Taking into account (1.5) in the last inequality, we get

$$LG \geq r^{-S-2}\left[\frac{S(S+2)\gamma C_1}{R^2} - \frac{SnC_2}{\gamma R^2} - Sr\sqrt{\sum_{i=1}^{n}\frac{b_i^2(x)}{(\omega_i^{-1}(R))^2}} - C(x)r^2\right] \tag{1.6}$$

Using (1.2),

$$\left(\sum_{i=1}^{n}\frac{b_i^2(x)}{(\omega_i^{-1}(R))^2}\right)^{1/2} \leq b_0\sqrt{\sum_{i=1}^{n}\frac{1}{(\omega_i^{-1}(R))^2}}.$$

For $x, y \in B^3$ it follows that

$$r = \sqrt{\sum_{i=1}^{n}\frac{(x_i - y_i)^2}{(\omega_i^{-1}(R))^2}}$$

$$\leq \sqrt{\sum_{i=1}^{n}\frac{x_i^2}{(\omega_i^{-1}(R))^2} + \sum_{i=1}^{n}\frac{y_i^2}{(\omega_i^{-1}(R))^2}} \leq \sqrt{2(17^2 + 17^2)} = 2 \cdot 17 = 34.$$

Here, we have used inequality $(a-b)^2 \leq 2(a^2 + b^2)$. Hence, we conclude

$$r\sqrt{\sum_{i=1}^{n}\frac{b_i^2(x)}{(\omega_i^{-1}(R))^2}} \leq 34 b_0\sqrt{n}. \tag{1.7}$$

By condition (1.2)

$$c(x)r^2 \geq -C_0 r^2 \geq -34^2 C_0 = -1156 C_0. \tag{1.8}$$

Taking into account (1.7)-(1.8) in (1.6), we get

$$LG \geq r^{-s-2}\left(S\left[(S+2)\frac{\gamma C_1}{R^2} - \frac{nC_2}{\gamma R^2} - 34b_0\sqrt{n}\right] - 1156C_0\right)$$

$$\geq r^{-s-2}\left(S\left((S+2)\gamma C_1 - nC_2\gamma^{-1} - 34b_0\sqrt{n}\right) - 1156C_0\right).$$

Let S_1 be such that

$$(S_1+2)\gamma C_1 = \frac{nC_2}{\gamma}34b_0\sqrt{n} + 1,$$

i.e.

$$S_1 = \frac{nC_2}{\gamma C_1} + \frac{34b_0\sqrt{n}}{\gamma C_1} + \frac{1}{\gamma C_1} - 2.$$

Then for a fixed $S \geq S_1$ it holds

$$LG \geq r^{-S-2}(S - 1156C_0).$$

Denote $S = 1156C_0$, and fix $S = \max\{S_1, S_2\}$.

We get $LG \geq 0$, this completes the proof of lemma.

□

Lemma 1.2 *Let* $b =\in_{R;\rho} (x^0)$, $\bar{B} \subset B^3$ *with* $\rho > 0, R \leq 1$. *Then* $C_S(B) \geq \rho^S$.

Proof. Consider the function $W(x) = G(x, x^0)$ for $x \bar{\in} \bar{B}$.

If $x \bar{\in} \bar{B}$ then

$$\sum_{i=1}^{n} \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2} \geq \rho^2, \text{ therefore } W(x) = G(x, x^0) = \left[\sum_{i=1}^{n} \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2}\right]^{-S/2} \leq \rho^{-S}.$$

Hence

$$\sup_{E_n \setminus \bar{B}} W(x) \leq \rho^{-S}. \tag{1.9}$$

Now, we consider a singular measure μ concentrated at the point x^0 with volume ρ^{-S}. Allowing for (1.9) $x \bar{\in} \bar{B}$ we have

$$\int_B G(x,y)d\mu(y) = \int_{x^0} W(x)d\mu(x^0) \leq \rho^{-S}\mu(x^0) = \rho^{-S} \cdot \rho^S = 1,$$

i.e. the measure μ is S -admissible. Therefore,

$$C_S(B) \geq \mu(B) = \mu(x^0) = \rho^S.$$

This completes the proof of Lemma.

Let $B^4 =\ni_{R:9} (0)$, $x^0 \in \partial B^4$, $B^5 =\ni_{R:8} (x^0)$, $B^6 =\ni_{R:1} (x^0)$.

Lemma 1.3 *Let the domain* D *have limit points on* ∂B^5*, and has nonempty intersection with* B^5*. Let* $H = B^6 \setminus D$ *and a positive* L *-subelliptic function* $u(x)$ *be continuous in* $\bar{D}$ *and vanishes on part of boundary* ∂D *that strongly located in* B^5*. Then then there exists a constant* $\eta_1(S) > 0$ *such that it holds*

$$\sup_D u(x) \geq (1 + \eta_1(S) \cdot C_S(H)) \sup_{D \cap B^6} u(x). \tag{1.10}$$

provided the conditions (1.2)-(1.4) *and* $R \leq 1$ *are fulfilled.*

Proof. Without loss of generality, we can assume $C_S(H) > 0$, otherwise (1.10) is evidently. Fix an arbitrary $\epsilon \in (0, C_S(H))$ such that measure $\mu(H)$ be such that $\mu(H) > C_S(H) - \varepsilon$ and

$$U(x) = \int_H G(x,y)d\mu(y) \leq 1 \quad \text{for} \quad x\bar{\in}H. \tag{1.11}$$

Fix a point $y \in H$ and a point $x \in \partial B^5$. Then

$$\left(\sum_{i=1}^n \frac{(x_i - y_i)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \geq \left(\sum_{i=1}^n \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} - \left(\sum_{i=1}^n \frac{(y_i - x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \geq 8 - 1 = 7.$$

If $y \in H$ then $x \in \partial B^6$, therefore

$$\left(\sum_{i=1}^n \frac{(x_i - y_i)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \leq \left(\sum_{i=1}^n \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} + \left(\sum_{i=1}^n \frac{(x_i^0 - y_i)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \leq 1 + 1 = 2.$$

We have

$$G(x,y)|_{\partial B^5} \leq \frac{1}{7^S} \quad \text{and} \quad U(x)|_{\partial B^5} \leq \frac{1}{7^S} C_S(H). \tag{1.12}$$

On the other hand,

$$G(x,y)\Big|_{\substack{\partial B^6 \\ y\in H}} \geq \frac{1}{7^S} \quad \text{and} \quad U(x)\,|_{\partial B^6} \geq \frac{1}{7^S} C_S(H) \geq \frac{1}{2^S}\mu(H). \tag{1.13}$$

Consider the auxiliary functions

$$W(x) = M\left[1 - U(x) + \left(\frac{1}{7}\right)^S C_S(H)\right] - u(x), \tag{1.14}$$

where $M = \sup\limits_D u(x)$.

By lemma 1.1, the function L -superelliptic in D . According to inequality (1.13) $W(x) \geq 0$ for $x \in \partial B^5$. Furthermore, by inequality (1.11) $W(x) \geq 0$ for $x \in \partial D \cap B^5$. Thus, $W(x) \geq 0$ for $x \in \partial D \cap B^5$. By the maximum principle, $W(x) \geq 0$ in D.

In particular, inserting (1.14) in (1.12), we get

$$\sup_{D\cap B^6} u(x) \leq M\left[1 - \frac{1}{2^S}\mu(H) + \frac{1}{7^S}C_s(H)\right] \leq M\left[1 - \frac{1}{2^S}(C_S(H) - \varepsilon) + \frac{1}{7^S}C_s(H)\right]$$
$$= M\left[1 - \left(\frac{1}{2^S} - \frac{1}{7^S}\right)C_S(H) + \varepsilon \cdot \frac{1}{2^S}\right].$$

Now taking into account arbitrariness of $\varepsilon > 0$ and denoting $\frac{1}{2^S} - \frac{1}{7^S} = \eta_1(S)$, we get

$$\sup_{D\cap B^6} u(x) \leq M(1 - \eta_1(S)C_S(H)).$$

This completes the proof of Lemma 1.3.

Corollary 1.1 *Let the notations of Lemma 1.3 be satisfied. If H contains ellipsoid $\ni_{R:\rho}$ (x'), then*

$$\sup_D u(x) \geq (1 + \eta_2) \sup_{D\cap B^6} u(x),$$

where $\eta_2 = \eta_2(s, \rho)$.

Proof. As H contains the ellipsoid $\ni_{R:\rho}(x')$, then

$$C_S(H) \geq C_S(\ni_{R:\rho}(x')).$$

On the other hand using (1.10), we get

$$\sup_D u(x) \geq \left(1 + \eta_1(S) C_3(S) \sup_{D\cap B^6} u(x)\right).$$

Inserting $\eta_2 = \eta_1 C_3(S)\rho^S$ this completes the proof of Corollary 1.1.

Lemma 1.4 *Let the conditions of Lemma 1.3 be fulfilled. Then there exists $\delta(\gamma, n, b_0)$ such that*

$$\sup_D u(x) \geq (1 + \eta_3) \sup_{D\cap B^6} u(x). \tag{1.15}$$

provided that $mes\, D \leq \delta \cdot mes\, B^5$, with $\eta_3 = \frac{61}{64}$.

Proof. Consider the auxiliary function

$$W_1(x) = \frac{M}{64}\left[1 + \sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right]$$

where M has the same sense as in the previous lemma. We have

$$W_1(x)|_{\partial B^5} = \frac{M}{64}(1+64) = \frac{65\mu}{64}, \tag{1.16}$$

$$W_1(x)|_{\partial D\cap B^6} \leq \frac{M}{64}, \tag{1.17}$$

$$W_1(x)|_{D\cap B^6} \leq \frac{\mu}{64}[1+1] = \frac{2\mu}{64} = \frac{M}{32}, \tag{1.18}$$

$$LW_1(x) = \frac{M}{32}\sum_{i=1}^{n} a_{ii}(x)\frac{1}{\left(\omega_i^{-1}(R)\right)^2} + \frac{M}{32}\sum_{i=1}^{n} b_i(x)\frac{x_i - x_i^0}{\left(\omega_i^{-1}(R)\right)^2} + \frac{MC(x)}{64}$$

$$\left(1 + \sum_{i=1}^{n}\frac{\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right) \leq \frac{M}{32\gamma}\sum_{i=1}^{n}\lambda_i(x)\frac{1}{\left(\omega_i^{-1}(R)\right)^2} + \frac{M}{32}$$

$$\times\left(\sum_{i=1}^{n}\frac{b_i^2\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} \cdot \left(\sum_{i=1}^{n}\frac{1}{\left(\omega_i^{-1}(R)\right)^2}\right) \leq \frac{MnC_2}{32\gamma R^2} + \frac{Mb_0}{32}$$

$$\times\left(\sum_{i=1}^{n}\frac{\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} \cdot \left(\sum_{i=1}^{n}\frac{1}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} \leq \frac{MnC_2}{32\gamma R^2} + \frac{Mb_0\sqrt{n}}{32}$$

$$\leq \frac{M}{R^2}\left(\frac{nC_2}{32\gamma} + \frac{b_0\sqrt{n}}{4}\right) = C_5(n,\gamma,b_0)M \text{ where } C_5 = \frac{nC_2}{32\gamma} + \frac{b_0\sqrt{n}}{4}. \tag{1.19}$$

Consequently,

$$L(W_1(x) - u(x)) = LW_1(x) - Lu(x) \leq \frac{C_5 M}{R^2}. \tag{1.20}$$

In the parts $\partial D \cap B^5$ and ∂B^5 we have the estimates

$$(W_1(x) - u(x))|_{\partial B^5} \geq \frac{65M}{64} - M = \frac{M}{64}, \tag{1.21}$$

$$(W_1(x) - u(x))|_{\partial D \cap B^5} \geq \frac{M}{64}. \tag{1.22}$$

Let D' be a domain located in $\ni_{R:15,5}(0) \setminus \ni_{R:1/2}(0)$ and such that $\overline{D} \subset D'$, $\quad mes\, D' \leq 2\, mes\, D$. One can easily construct a function $f \in C^\infty(\bar{D}')$ with inequality $0 \geq f(x) \geq -\frac{C_5 M}{R^2}$, $x \in D' \setminus D$ and such that

$$f(x) = \begin{cases} -\frac{C_5 M}{R^2}, & x \in D \\ 0, & x \bar{\in} D' \end{cases}$$

Denote by $z(x)$ the solution of Dirichlet problem in $\ni_{R:17,5}(0) \setminus \ni_{R:0,5}(0)$

$$Lz(x) = f(x),\ z(x)|_{\partial\left(\ni_{R:17,5}(0) \setminus \ni_{R:0,5}(0)\right)} = 0. \tag{1.23}$$

Obviously,

$$\det(a_{ij}) \geq C_6(n,\gamma) \prod_{i=1}^{n} \lambda_i(x) \geq C_7(n,\gamma) \prod_{i=1}^{n} \left(\frac{\omega_i^{-1}(R)}{R}\right)^2. \tag{1.24}$$

We make change of variables

$$y_i = \frac{x_i}{17,5\omega_i^{-1}(R)},\ \ i = 1, ..., n.$$

Then the ellipsoid layer $\ni_{R:17,5}(0) \setminus \ni_{R:0,5}(0)$ will go into spherical layer $Q = Q_1^0 \setminus Q_{1/35}^0$ and problem (1.24) will get the following form

$$\widetilde{L}\widetilde{z}(y) = f(y);\ \widetilde{z}(y)|_{\partial Q} = 0,$$

where

$$\widetilde{L} = \sum_{i=1}^{n} \frac{a_{ij}(y)}{(17,5)^2\, \omega_i^{-1}(R)\omega_j^{-1}(R)} \cdot \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{n} \frac{b_i(y)}{17,5 \cdot \omega_i^{-1}(R)} \cdot \frac{\partial}{\partial y_i} + c(y). \tag{1.25}$$

Denote

$$\widetilde{a}_{ij}(y) = \frac{a_{ij}(y)}{(17,5)^2\, \omega_i^{-1}(R)\omega_j^{-1}(R)}, \widetilde{b}_i(y) = \frac{b_i(y)}{17,5 \cdot \omega_i^{-1}(R)}.$$

From (1.25) it follows that

$$\det(\widetilde{a}_{ij}) = -\frac{\det(a_{ij})}{\prod_{i=1}^{n} (17,5)^2 \left(\omega_i^{-1}(R)\right)^2} \geq \frac{1}{\prod_{i=1}^{n} (17,5)^2 \left(\omega_i^{-1}(R)\right)^2}$$

$$\times C_7 \prod_{i=1}^{n} \left(\frac{\left(\omega_i^{-1}(R)\right)}{R}\right)^2 = \frac{C_8(n,\gamma)}{R^{2n}}.$$

According to A.D. Alexandov's maximum principle [1],

$$\widetilde{z}(y) \le C_9(n,\gamma) \left\| \frac{f}{\sqrt[n]{\det(\widetilde{a}_{ij})}} \right\|_{L_n(Q)} \cdot F_n \left(\left\| \frac{\widetilde{b}}{\sqrt[n]{\det(\widetilde{a}_{ij})}} \right\|_{L_n(Q)} \right), \tag{1.26}$$

where $F_n(t)$ is a bounded function of t.

We have

$$j_1 = \left\| \frac{\widetilde{b}}{\sqrt[n]{\det(\widetilde{a}_{ij})}} \right\|_{L_n(Q)} = \frac{1}{\sqrt[n]{\det(\widetilde{a}_{ij})}} \left(\int_Q \left(\sqrt{\sum_{i=1}^{n} \widetilde{b}_i^2} \right)^n dy \right)^{1/n}$$

$$\le \frac{C_{10}(n,\gamma)}{\sqrt[n]{\det(\widetilde{a}_{ij})}} \left(\sum_{i=1}^{n} \int_Q \left| \widetilde{b}_i \right|^n dy \right)^{1/n}$$

(where ω_n is the volume of n dimensional unique ball)

$$\le \frac{C_{10}(n,\gamma)}{\sqrt[n]{\det(\widetilde{a}_{ij})}} \left(\sum_{i=1}^{n} \int_Q \left| \widetilde{b}_i \right|^n dy \right)^{1/n}$$

$$\le C_{11}(n,\gamma) R^2 \left(\sum_{i=1}^{n} \int_Q \frac{|b_i|^n}{\left(\omega_i^{-1}(R) \right)^n} \right)^{1/n} \le C_{11} b_0 \sqrt[n]{n\omega_n} \le C_{12}(n,\gamma,b_0),$$

$$\left\| \widetilde{b} \right\|_{L_n(Q)} = \left\| \sqrt{\sum_{i=1}^{n} b_i^2} \right\|_{L_n(Q)}.$$

Hence, the second multiplier in (1.26) is bounded:

$$F_n(j_1) \le C_{13}(n,\gamma,b_0). \tag{1.27}$$

We have

$$j_2 = \left\| \frac{f}{\sqrt[n]{\det(\widetilde{a}_{ij})}} \right\|_{L_n(Q)} = \frac{1}{\sqrt[n]{\det(\widetilde{a}_{ij})}} \left(\int_Q |f|^n dy \right)^{1/n} \le R^2 C_{14}(n,\gamma) \left(\int_Q |f|^n dy \right)^{1/n}.$$

We insert the last inequality and (1.27) in (1.26) and get

$$\widetilde{z}(y) \le C_{15}(n,\gamma,b_0) R^2 \left(\int_Q |f|^n dy \right)^{1/n}.$$

Coming back to the initial coordinates,we conclude

$$z(x) \le C_{15}(n,\gamma,b_0) R^2 \left(\int_{\ni_{R:17,5}(0) \setminus \ni_{R:0,5}(0)} \left(\mathop{\Pi}_{i=1}^{n} \omega_i^{-1}(R) \right)^{-1} dx \right)^{1/n}$$

$$= \frac{C_{16}R^2}{\left(\prod\limits_{i=1}^{n}\omega_i^{-1}(R)\right)^{1/n}} \cdot \left(\int\limits_{D'} |f|^n \, dx\right)^{1/n} = \frac{C_{16}R^2}{\left(\prod\limits_{i=1}^{n}\omega_i^{-1}(R)\right)^{1/n}}$$

$$\times \left(\int\limits_{D'} \left(\frac{MC_5}{R^2}\right)^n dx\right)^{1/n} = \frac{MC_{17}(n,\gamma,b_0)}{\left(\prod\limits_{i=1}^{n}\omega_i^{-1}(R)\right)^{1/n}} \left(\int\limits_{D'} dx\right)^{1/n}$$

$$= \frac{MC_{17}}{\left(\prod\limits_{i=1}^{n}\omega_i^{-1}(R)\right)^{1/n}} \left(mes\, D'\right)^{1/n} \leq \frac{MC_{17}}{\left(\prod\limits_{i=1}^{n}\omega_i^{-1}(R)\right)^{1/n}} \left(2\, mes\, D'\right)^{1/n}$$

$$\leq \frac{MC_{18}}{\left(\prod\limits_{i=1}^{n}\omega_i^{-1}(R)\right)^{1/n}} \left(\delta\, mes\, B^5\right)^{1/n}$$

$$\leq \frac{M\sqrt[n]{\delta}\left(\prod\limits_{i=1}^{n}\omega_i^{-1}(R)\right)^{1/n} \cdot C_{19}(n,\gamma,b_0,C_0)}{\left(\prod\limits_{i=1}^{n}\omega_i^{-1}(R)\right)^{1/n}} = M\sqrt[n]{\delta}C_{19}.$$

Choose $\delta = (64C_{19})^{-n}$. For $x \in \ni_{R:17,5}(0) \setminus \ni_{R:17,5}(0)$

$$0 \leq z(x) \leq \frac{M}{64}. \tag{1.28}$$

Let $W_2(x) = W_1(x) - u(x) + z(x)$. Then from (1.17)-(1.24) and (1.28) it follows that

$$LW_2 \leq 0 \quad \text{and} \quad W_2(x)|_{\partial B^5} \geq \frac{65M}{64} - M + 0 = \frac{\mu}{64};\ W_2(x)|_{\partial D \cap B^5} \geq \frac{M}{64}.$$

By the maximum principle, $W_2(x) \geq 0$ in D i.e. $u(x) \leq W_1(x) + z(x)$. Hence, if follows

$$\sup_{D\cap B^6} u(x) \leq \sup_{D\cap B^6} W_1(x) + \sup_{D\cap B^6} z(x) \leq \frac{M}{32} + \frac{M}{64} = \frac{3M}{64}$$

$$= M\left(1 - \frac{61}{64}\right) \leq M - \frac{61}{64}\sup_{D\cap B^6} u(x),$$

therefore,

$$\sup_{D} u(x) \geq \left(1 - \frac{61}{64}\right)\sup_{D\cap B^6} u(x). \tag{1.29}$$

This completes the proof of Lemma 1.4.

Remark 1.1 The statements of Corollary 1.1 for Lemma 1.3, and Lemma 1.4 are valid (with constants $\eta_2' = \frac{\eta_2 64\eta}{1+64\eta}$ and $\eta_3' = \frac{\eta_2 64\eta}{1+64\eta}$ respectively). In that case, $Lu \geq MC(x)$, where $M = \sup\limits_{D} u(x)$ provided the function $u(x)$ is not L -subelliptic in D.

Prove the statement of Lemma 1.4. For that consider the function

$$W(x) = u(x) + Mh\sum_{i=1}^{n}\frac{(x_i - x_i^0)}{(\omega_i^{-1}(R))^2}$$

where a positive constant h will be chosen later. We have

$$LW(x) = Lu(x) + 2Mh\sum_{i=1}^{n}\frac{1}{(\omega_i^{-1}(R))^2} + 2Mh\sum_{i=1}^{n}b_i(x)\frac{x_i - x_i^0}{(\omega_i^{-1}(R))^2}$$

$$+c(x)Mh\sum_{i=1}^{n}\frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2} \geq Mc(x) + 2Mhb_0\left(\sum_{i=1}^{n}\frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2}$$

$$\times\left(\sum_{i=1}^{n}\frac{1}{(\omega_i^{-1}(R))^2}\right)^{1/2} - C_0Mh64 \geq -MC_0 + 2C_1nhM\gamma - 16Mhb_0\sqrt{n}$$

$$= -MC_0 + M\left(2C_1nh\gamma - 16hb_0\sqrt{n} - 64C_0h\right).$$

We have chosen a fixed number

$$h = \frac{C_0}{2C_1n\gamma - 16b_0\sqrt{n} - 64C_0}$$

Then the function $W(x)$ is L -subelliptic in D. Applying Lemma 1.4 to it , we get

$$\sup_D W(x) \geq (1+\eta_3)\sup_{D\cap B^6} W(x)$$

or

$$\sup_D\left(u(x) + 64Mh\right) \geq (1+\eta_3)\sup_{D\cap B^6}(x),$$

$$M(1+64h) \geq (1+\eta_3)\sup_{D\cap B^6} u(x).$$

Thus

$$\sup_D u \geq \frac{1+\eta_3}{1+64h}\sup_{D\cap B^6} u = \left(1 + \frac{\eta_3 - 64h}{1+64h}\right)\sup_{D\cap B^6} u$$

$$= \left(1+\eta_3^{'}\right)\sup_{D\cap B^6} u, \quad \text{where} \quad \eta_3^{'} = \frac{\eta_3 - 64h}{1+64h}.$$

□

Corollary 1.2 *Let the assumptions of Lemma 1.3 be fulfilled with respect to the domain D and a positive L -superelliptic function $v(x)$ is continuous in $\overline{D}$ and equals unit on $\partial D \cap B^5$. Then*

$$\inf_{D\cap B^5} v(x) \geq \eta_2^{''} = \frac{\eta_2^{'}}{1+\eta_2^{'}}.$$

Proof. Let $D^{'} = \{x, x \in D,\ v(x) < 1\}$. Consider the function $u(x) = 1 - v(x)$ for $x \in D'$. Applying to this function the corollary of Lemma 1.3 (according to remark to Lemma 1.4), we get

$$\sup_{D''} u(x) \geq (1+\eta_2^{'}) \sup_{D\cap B^6} u(x)$$

or

$$1 - \inf_{D'} v(x) \geq (1+\eta_2^{'}) \left(1 - \inf_{D\cap B^6} v(x)\right),$$

i.e.

$$(1+\eta_2^{'}) \left(1 - \inf_{D'\cap B^6} v(x)\right) \leq 1$$

$$(1+\eta_2^{'}) - (1+\eta_2^{'}) \inf_{D'\cap B^6} v(x) \leq 1$$

$$\inf_{D'\cap B^6} v(x) \geq \frac{\eta_2^{'}}{1+\eta_2^{'}}.$$

On the other hand,

$$\inf_{D\backslash D'} v(x) \geq 1.$$

Inserting $\eta_2^{''} = \frac{\eta_2^{'}}{1+\eta_2^{'}}$ this completes the proof of Corollary 1.2.

The following Corollary 1.3 is proved in the same way.

Corollary 1.3 *Let the conditions of Lemma 1.4 be fulfilled with respect to the domain D and a positive L -subelliptic function $v(x)$ that is continuous in D and equals to unit on $\partial D \cap B^5$. Then*

$$\inf_{D'\cap B^6} v(x) \geq \eta_3^{''} = \frac{\eta_3^{'}}{1+\eta_3^{'}}.$$

Lemma 1.5 *Let D be a domain with limit points on ∂B^5 and having nonempty intersection with B^6 is contained in B^5. A positive L -superelliptic function $v(x)$ nonnegative in D, be continuous in $\overline{D}$ and equals unit on that part $D^{'} \cap B^5$. If conditions* (1.2)-(1.4) *are fulfilled and $mesH \geq \sigma_1 mesB^6$ then*

$$\inf_{D'\cap B^6} v(x) \geq \eta_4(\gamma, n, b_0, C_0, \sigma_0),$$

where $\eta > 0$ and $H = B^6 \backslash D$.

Proof. Consider the ellipsoid

$$B^7 = B^7(\rho_0) = \ni_{R:1-\rho_0}(x^0) = \left\{ x : \sum_{i=1}^{n} \frac{(x_i - x_i^0)^2}{\left(\omega_i^{-1}(R)\right)^2} < (1-\rho_0)^2 \right\},$$

where $\rho_0 \in \left(0,\ \frac{1}{4}\right)$. Choose a fixed $\rho_0 > 0$ such that

$$mes\left(B^6 \setminus B^7\right) = \frac{\sigma_1}{2} mes\, B^6. \tag{1.30}$$

It is known that

$$mes\,(B^6 \setminus B^7) = (1 - (1-\rho_0)^n) \mathop{\Pi}_{i=1}^{n} \omega_i^{-1}(R).$$

Inserting this in (1.29) we get

$$(1-(1-\rho_0)^n)=\frac{\sigma_1}{2}.$$

Hence, it follows

$$\rho_0=1-\sqrt[n]{1-\frac{\sigma_1}{2}}.$$

It is clear that ρ_0 depends only on n and σ_1.

Denote by E^0 the set of inner points $H\cap B^7$. Since

$$H\backslash(B^6\backslash B^7)\subset E^0,$$

it follows

$$mesE^0\geq mesH-mes(B^6\backslash B^7)\geq\sigma_1 mesB^6-\frac{\sigma_1}{2}mesB^6=\frac{\sigma_1}{2}mesB^6. \quad (1.31)$$

Now let x' be an arbitrary point from E^0, set

$$B^5_\nu(x')=\ni_{R:8\nu}(x'),\ \ B^6_\nu(x')=\ni_{R:\nu}(x'),$$

where $\nu\in(0,\rho_0]$ is such that $\overline{B}^5_\nu(x')\subset B^6$.

Denote by $\nu(x')$ the least upper bound of $\nu\in(0,\rho_0]$, for which

$$mes\left(D\cap B^5_\nu(x')\right)\leq\delta mesB^5_\nu(x'), \quad (1.32)$$

where δ is the constant of Lemma 1.4. Two cases are possible: 1) $\nu(x')=\rho_0,\ 2)\nu(x')<\rho_0$.

Let case 1) holds. According to Corollary 1.2 and Lemma 1.4, either $D\cap B^6_{\rho_0}(x')=\emptyset$ or

$$\inf_{D\cap B^6_{\rho_0}(x')}\upsilon(x)\geq\eta''_3.$$

Let $D=\{x:x\in D,\upsilon(x)<\eta''_3\}$. From the above assumption it follows $B^6\setminus D'$ contains the ellipsoid $B^6_{\rho_0}(\times')$. Applying to the function $\frac{\nu(x)}{\eta''_3}$ Corollary 1.1 of Lemma 1.4, we get

$$\inf_{D\cap B^6}\frac{\upsilon(x)}{\eta''_3}\geq\eta''_2,$$

where the constant η''_2 depends only on γ,n,b_0,C_0 and σ_1. Taking into account $\upsilon(x)\geq\eta''_3$ for $x\in D\backslash D'$ we conclude

$$\inf_{D\cap B^6}\upsilon(x)\geq\eta''_2\eta''_3.$$

Thus, in case 1) the Lemma 1.5 has been proved. Consider case 2). If for a point $x''\in E^0$ case 1) takes place the assertion of Lemma 1.5 takes place as above. So, it remains to consider the salutation when for any point $x\in E^0$ it holds case 2).

Cover E^0 with ellipsoids $B^6_{\nu(x)}(x)$ and choose from theirs subcover of countable bolls $\{B^6_{\nu(k)}(x^k)\},k=1,2,...$, with finite multiplicity $q_1(n)$ so that

$$mes\left[\bigcup_{k=1}^{\infty}\left(D\cap B^6_{\nu_k}(x^k)\right)\right]\leq q_2(n)mes\left[\bigcup_{k=1}^{\infty}\left(D\cap B^6_{\nu_k}(x^k)\right)\right].$$

According to Corollary 1.2 and Lemma 1.4, for every natural k it follows $D\cap B^6_{\nu_k}(x^k)=\emptyset$ or

$$\inf_{D\cap B^6_{\nu_k}(x^k)}\upsilon(x)\geq\eta''_3. \quad (1.33)$$

Furthermore, inserting (1.31) one gets

$$mes\left[\bigcup_{k=1}^{\infty}\left(D\cap B^6_{\nu_k}(x^k)\right)\right]\le\frac{1}{q_2}mes\left[\bigcup_{k=1}^{\infty}\left(D\cap B^5_{\nu_k}(x^k)\right)\right]$$

$$\ge\frac{1}{q_1q_2}\sum_{k=1}^{\infty}mes\left(D\cap B^5_{\nu_k}(x^k)\right)=\frac{\delta}{q_1q_2}\sum_{k=1}^{\infty}mesB^5_{\nu_k}(x^k)$$

$$\ge\frac{\delta}{q_1q_2}\sum_{k=1}^{\infty}mesB^5_{\nu_k}(x^k)\ge\frac{\delta}{q_1q_2}mesE^0\ge\frac{\sigma\delta}{q_1q_2}mesB^6. \tag{1.34}$$

Denote $\frac{\upsilon(x)}{\eta_3''}$ by $\upsilon_1(x)$ and set $D_1=\{x: x\in D,\, \upsilon_1(x)<1\}$. Let $E^1=B^6\setminus D_1$. According to (1.33) it is seen that

$$E^1\supset\left(B^6\setminus D\right)\cup\left(\bigcup_{k=1}^{\infty}\left(D\cap B^5_{\nu_k}(x^k)\right)\right).$$

Consequently, regarding to the condition on H and (1.34) it follows

$$mes\,E^1\ge mes\,H+mes\left[\bigcup_{k=1}^{\infty}\left(D\cap B^5_{\nu_k}(x^k)\right)\right]$$

$$\ge\sigma_1 mes\,B^6+\frac{\delta\sigma_1}{q_1q_2}mes\,B^6\ge\sigma_1\left(1+\frac{\delta}{2q_1q_2}\right)mes\,B^6.$$

Apply to the function $\upsilon_1(x)$ the same argues used for $\upsilon(x)$ above. The last estimate proves assertion of Lemma or yields a domain $D_2=\left\{x: x\in D, \frac{\upsilon(x)}{\eta_3''}<1\right\}$ with $E^2=B^6\backslash D_2$ such that

$$mes\,E^2\ge\sigma_1\left(1+\frac{\delta}{2q_1q_2}\right)^2 mes\,B^6.$$

Let $l_0=l_0(n,\delta,\sigma_1)$ is a least integer for which

$$\sigma_1\left(1+\frac{\delta}{2q_1q_2}\right)^{l_0}>1.$$

Repeating the above used argues l_0 times for the function $\frac{\upsilon(x)}{\eta_3''}$ we get an estimate

$$\inf_{D\cap B^6}\frac{\upsilon(x)}{(\eta_3'')^{l_0}}\ge\eta_3'',$$

i.e.

$$\inf_{D\cap B^6}\upsilon(x)\ge\left(\eta_3''\right)^{l_0+1}.$$

This completes the proof of Lemma 1.5.

Corollary 1.4 *Let all assumptions of Lemma 1.5 be fulfilled for domain D and a positive L -subelliptic function $u(x)$, assume that, it is also continuous in $\overline{D}$ and vanishes on $\partial D\cap B^5$. Then*

$$\sup_D u\ge(1+\eta_4)\sup_{D\cap B^6}u. \tag{1.35}$$

Proof. Let $\upsilon(x) = 1 - \frac{u(x)}{M}$, where $M = \sup\limits_{D\cap B^6} u(x)$. Applying Lemma 1.5 for the function $\upsilon(x)$, we get

$$\inf_{D\cap B^6} \upsilon(x) \geq \eta_4$$

or

$$1 - \frac{1}{M} \sup_{D\cap B^6} u \geq \eta_4,$$

Therefore, the required estimation (1.35) has been derived.

Lemma 1.6 *Let D be a domain located in B^3 and be such that it have a limit points on the boundaries of both ellipses B^1 and B^2. Let $u(x)$ be an L -subelliptic continuous function in $\overline{D}$ and vanishes on $\partial D \cap B^3$. Then*

$$\sup_D u \geq (1 + \eta(\gamma, n, b_0, C_0, \tau) \sup_{D\cap B^4} u \tag{1.36}$$

provided that,

$$mes\, H \geq \sigma\, mes\, B^8$$

for $H = B^8 \setminus D$ and fixed $\sigma > 0$.

Proof. Without loss of generality, we can assume $\sup\limits_{D\cap B^4} u(x) = 1$.

Let $x^* \in D \cap B^4$ be a point wherein $u(x^*) = 1$.

Choose on ∂B^4 minimal number of points $x^1, ...x^m$ so that

1) $\overline{B^4} \subset \bigcup\limits_{k=1}^{\infty} B^6(x^i)$, where $B^6(x^i) = \ni_{R:1}(x^i)$;

2) One of the points $x^i, i = 1, ..., m$ coincides with x^*.

3) for any $i, 1 \leq i \leq m$, there will be found j, $1 \leq j \leq m$ such that $x^j \in \partial \ni_{\frac{R}{A^m}:1}(x^i)$, where the constant $A(n) > 1$ will be chosen later.

It is clear, the number m depends on n. From the properties of the covering it follows: for any $i_0, 1 \leq i_0 \leq m$, there exist a chain $x^{i_1}, ..., x^{i_k}$ such that $x^{i_k} = x^*$ and $x^{i_{l+1}} \in \partial \ni_{\frac{R}{A^m}:1}(x^{i_l})$, $l = 0, 1, ..., k-1$. The condition of H implies, there exists a $0 \leq i_0 \leq m$ such that

$$mes\, \left(H \cup B^6(x^{i_0}\right) \geq \frac{mes\, H}{m} \geq \frac{\sigma}{m} mes\, B^8 \tag{1.37}$$

It is easily seen that, $mes\, B^8 \leq mes\, B^6(x^{i_0})$. Therefore and from (1.37) it follows

$$mes\, \left(H \cup B^6(x^{i_0}\right) \leq \frac{\sigma}{m} mes B^6(x^{i_0}). \tag{1.38}$$

Let $\delta_1 = \frac{\eta_4}{2(1+\eta_4})$, with η_4 taken from Lemma 1.5 and $\sigma_1 = \frac{\sigma}{m}$. Assume that

$$\sup_{D\cap B^6(x^{i_0})} \leq 1 - \delta_1.$$

Then according to Corollary 1.4 of Lemma 1.5 and (1.38) one gets

$$\sup_D u(x) \geq \sup_{D\cap B^5(x^{i_0})} u(x) \geq (1 + \eta_4) \sup_{D\cap B^6(x^{i_0})} u(x) \geq (1 + \eta_4)(1 - \delta_1) =$$

$$= \left(1 + \frac{\eta_4}{2}\right) 1 = \left(1 + \frac{\eta_4}{2}\right) \sup_{D\cap B^4} u(x),$$

which completes the proof of Lemma in this case.

Now, let

$$\sup_{D\cap B^6(x^{i_0})} u(x) < 1-\delta_1$$

and consider the function $\upsilon_1(x) = u(x) - 1 + \delta_1$. Since $\sigma k\delta_1 < 1$ it easily seen that $\upsilon_1(x)$ is L- -subelliptic in D.

Let

$$D_1 = \{x :\, x \in D,\, \upsilon_1(x) < 0\}\,.$$

By assumptions, the ellipsoid $B^6(x^{i_0})$ is contained in the complement of D. Let $x' \in \partial B^4$. Insert the ellipsoids $B^i(x')$, $i = 5, 6$ and find such $A > 1$ that $B^5_R(x') \subset B^6_{AR}(x')$. It is clear that, for the validity of this inclusion it suffices

$$8\omega_i^{-1}(R) \le \omega_i^{-1}(AR).$$

From condition (1.4) it easily seen that the last inequalities are fulfilled if $A \ge \alpha > 1$.

Let $x^{i_1}, ..., x^{i_k}$ be the above-mentioned chain of points. By construction, $B^6_{\frac{R}{A}}(x^{i_1}) \setminus D_1$ contains ellipsoid $\ni_{\frac{R}{A}:\rho_1}(x^{i_1})$, where ρ_1 depends only on n. Let $\sigma_0 = \frac{\eta_2}{2(1+\eta_2)}$, where the constant η_2 is taken from Corollary 1.1 of Lemma 1.3 and $\rho = \rho_1$. Assume that

$$\sup_{D_1\cap B^6_{\frac{R}{A}}(x^{i_1})} \upsilon_1(x) \ge \delta_1(1-\sigma_0)\,,\ \text{i.e.} \sup_{D_1\cap B^6_{\frac{R}{A}}(x^{i_1})} u(x_0) \ge 1-\delta_1\sigma_0.$$

Applying the Corollary 1.1 of Lemma 1.3 again we get

$$\sup_{D_1\cap B^6_R(x^{i_1})} \upsilon_1 \ge (1+\eta_2) \sup_{D_1\cap B^6_{\frac{R}{A}}(x^{i_1})} \upsilon_1 \ge (1+\eta_2)\delta_1(1-\sigma_0).$$

Thus,

$$\begin{aligned}\sup_D u(x) &\ge \sup_{D_1\cap B^6_R(x^{i_1})} u(x) \ge 1-\delta_1+(1+\eta_2)\delta_1(1-\sigma_0)\\ &= \left(1+\frac{\delta_1\eta_2}{2}\right)1 = \left(1+\frac{\delta_1\eta_2}{2}\right)\sup_{D\cap B^4} u(x),\end{aligned}$$

which completes the proof of Lemma in this case.

Let

$$\sup_{D\cap B^6_{\frac{R}{A}}(x^{i_1})} \upsilon_1(x) < \eta_1(1-\sigma_0), \quad \text{i.e.} \quad \sup_{D\cap B^6_{\frac{R}{A}}(x^{i_1})} u(x) < 1-\delta_1\sigma_0).$$

We consider in D a subelliptic function $\upsilon_2(x) = u(x) - 1 + \delta_1\sigma_0$. Let

$$D_2 = \{x : x \in D, \upsilon_2(x) > 0\}\,.$$

By assumptions, the ellipsoid $B^6_{\frac{R}{A}}(x^{i_1})$ is contained in complement of D_2. If

$$\sup_{D_2\cap B^6_{\frac{R}{A^2}}(x^{i_2})} \upsilon_2(x) \le \delta_1\sigma_0(1-\sigma_0)\,,\ \text{i.e.} \sup_{D_2\cap B^6_{\frac{R}{A^2}}(x^{i_2})} u(x) \le 1-\delta_1\sigma_0^2,$$

then we apply the Corollary 1.1 to Lemma 1.3 in order to get

$$\sup_{D_2\cap B^6_{\frac{R}{A}}(x^{i_2})} \upsilon_2(x) \ge (1+\eta_2) \sup_{D_2\cap B^6_{\frac{R}{A^2}}(x^{i_2})} \upsilon_2(x) \ge (1+\eta_2)\delta_1\sigma_0(1-\sigma_0).$$

Therefore,

$$\sup_D u(x) \geq \sup_{D_2 \cap B^6_{\frac{R}{A}}(x^{i_2})} u \geq 1 - \delta_1\sigma_0 + (1+\eta_2)\delta_1\sigma_0(1-\sigma_0)$$

$$= 1 + \frac{\delta_1\sigma_0\eta_2}{2} = \left(1 + \frac{\delta_1\sigma_0\eta_2}{2}\right) \sup_{D\cap\partial B^4} u(x),$$

which completes the proof of lemma in this case.

Now, let

$$\sup_{D_2 \cap B^6_{\frac{R}{A^2}}(x^{i_2})} \upsilon_2(x) < \delta_1\sigma_0(1-\sigma_0)\ , \text{ i.e. } \sup_{D_2 \cap B^6_{\frac{R}{A^2}}(x^{i_2})} u(x) < 1 - \delta_1\sigma_0^2,$$

In this case, we set $\upsilon_3(x) = u(x) - 1 + \delta_1\sigma_0^2$ and $D_3 = \{x : x \in D,\ \upsilon_3(x) > 0\}$. By assumptions, the ellipsoid $B^6_{\frac{R}{A^2}}(x^{i_2})$ is located in the complement to D_3. Continue the process in the similar way. No later then in the k-th step we will attain

$$\sup_{D_2 \cap B^6_{\frac{R}{A^k}}(x^{i_k})} u(x) \geq 1 - \delta_1\sigma_0^k,$$

showing that $u(x^{2k}) = u(u^*) = 1$.

This completes the proof of Lemma in this case.

Theorem 1.1 *Let the domain D located in B^1 has a limit point on ∂B^1 and intersects B^4. Let a positive L-subelliptic function $u(x)$ be continuous in D and vanishes on $\bar{D}$. i If the coefficients of the operator L satisfy conditions* (1.2)-(1.4)*, then*

$$\sup_D u \geq (1, \eta, \gamma, n, b_0 C_0, \sigma) \sup_{D\cap B^4} u$$

provided that $mesH \leq \sigma mesB^8$ and $R \leq 1$.

The statement of the theorem follows from Lemma 1.6 and the maximum principle.

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