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Privalov type estimates for high order Riesz-Bessel transforms

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Abstract. *In this paper, we consider the Privalov type estimates for the high order Riesz-Bessel transforms.*

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1 Introduction

The singular integral operators that have been considered by Mihlin [14], and Calderon and Zygmund [3] are playing an important role in the theory of partial differential equations. Klyuchantsev [13] and Kipriyanov and Klyuchantsev [12] have firstly introduced multi-dimensional singular integrals generated by the generalized shift operator. Aliev, Gadjiev, Guliyev, Ekincioglu and Serbetci have studied the boundedness of certain singular integrals generated by the generalized shift operator in $L_{p,\gamma}$-spaces with radial weights [2, 5–8]. In this paper, we obtain the Privalov type estimates for the high order Riesz transforms, generated by the Bessel generalized shift operator (high order Riesz-Bessel transform). It is an extension of the classical inequalities of Hölder, Korn, Lichtenstein and Giraud for the convolution type integrals.

Let $\mathbb{R}^n_+$ denote the part of Euclidean space $\mathbb{R}^n$ consisting of points x for which $x_1 > 0, \dots, x_n > 0$. We shall denote by $L_{p,\gamma}(\mathbb{R}^n_+, d\mu_\gamma(x))$-spaces (Lebesgue space with respect

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to the measure μ_γ) the set of all measurable functions f on $\mathbb{R}^n_+$ such that the norm

$$\|f\|_{L_{p,\gamma}} \equiv \|f\|_{p,\gamma} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p d\mu_\gamma(x) \right)^{1/p}, \quad 1 \le p < \infty$$

is finite, where $d\mu_\gamma(x) = x^\gamma dx = \prod_{i=1}^n x_i^{\gamma_i} dx$ and $\gamma = (\gamma_1, \dots, \gamma_n)$ is a multi index consisting of fixed positive numbers.

By $\mathcal{C}^{(2)}(\mathbb{R}^n_+)$, we denote the set of second order differentiable functions even with respect to each of the variables $x_1, x_2, \dots, x_k$, where $0 \le k \le n$. We determine the Bessel generalized shift $T^y\varphi(x) = u(x)$, $x, y \in \mathbb{R}^n_+$ of a function $\varphi(x) \in \mathcal{C}^{(2)}(\mathbb{R}^n_+)$ as the solution to the following initial value problem:

$$\begin{cases} \mathcal{B}_{x_j} u(x,y) = \mathcal{B}_{y_j} u(x,y), \\ u(x,0) = f(x), \quad u_y(x,0) = 0, \end{cases} \tag{1.1}$$

where $j = 1, 2, \dots, n$ and

$$\mathcal{B}_j \equiv \mathcal{B}_{x_j} = \sum_{j=1}^n \Big(\frac{\partial^2}{\partial x_j^2} + \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j} \Big) \quad , \gamma_j > 0$$

is the Bessel differential operator. The solution of the initial value problem (1.1) exists, is unique and can be written explicitly as

$$T^y f(x) := c_\gamma \int_0^\pi \dots \int_0^\pi f\big((x_1, y_1)_{\alpha_1}, \dots, (x_n, y_n)_{\alpha_n}\big) d\mu_\gamma(\alpha) \tag{1.2}$$

where $c_\gamma = \prod_{i=1}^n \Gamma\big(\frac{\gamma_i + 1}{2}\big)\big[\Gamma(\frac{1}{2})\Gamma(\frac{\gamma_i}{2})\big]^{-1}$ and $(x_i, y_i)_{\alpha_i} = \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos \alpha_i}$ and $d\mu_\gamma(\alpha) = \big(\prod_{i=1}^n \sin^{\gamma_i - 1} \alpha_i\big) d\alpha_1 \dots d\alpha_n$ [4–6,9]. By formula (1.2) the operator T^y can be extended to all functions $L_{p,\gamma}(\mathbb{R}^n_+)$. The shift T^y generates the corresponding convolution

$$(f * g)(x) = \int_{\mathbb{R}^n_+} f(y) T^y g(x) d\mu_\gamma(y)$$

which satisfies the property $(f * g) = (g * f)$. A homogenous polynomial $P_k(x)$, $x \in \mathbb{R}^n_+$, of order k which is even with respect to all x_i, $i = 1, \dots, n$, and satisfies the Bessel equation $\mathcal{B}_j P_k(x) = 0$ is called a $\mathcal{B}$-spherical polynomial, where

$$\Delta_{\mathcal{B}} = \sum_{j=1}^n \big(\frac{\partial^2}{\partial x_j^2} + \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j} \big) \quad , \gamma_j > 0.$$

We give some typical examples of kernels on $\mathbb{R}^n_+$. Let $H_{k,\gamma}$ be the space of all B-spherical polynomials of degree k. A weighted spherical function, or a spherical B-harmonic, of order k is a quotient

$$Y_k^\gamma(\theta) = \frac{Y_k^\gamma(x)}{|x|^k} = Y_k^\gamma\big(\frac{x}{|x|}\big) = \frac{P_k(x)}{|x|^k},$$

where $\theta = \dfrac{x}{|x|}$, and $P_k(x) \in H_{k,\gamma}$ [10,11].

The aim of this paper is to derive some theorems which are called of Privalov type for the high order Riesz-Bessel transformations. Therefore, we introduce the high order Riesz-Bessel transformations, which is closely related to the Bessel generalized shift operator as

$$R_\gamma^{(k)} f(x) = p.v. c_k(n,\gamma) \int_{\mathbb{R}_+^n} T^y \big[\tfrac{Y_k^\gamma(\theta)}{|x|^{n+|\gamma|}} \big] f(y) d\mu_\gamma(y), \tag{1.3}$$

where $\mathcal{S}_+^{n-1} = \{x \in \mathbb{R}_+^n : |x| = 1\}$ is the hemisphere and the cancellation condition holds

$$\int\limits_{\mathcal{S}_+^{n-1}} Y_k^\gamma(\theta) d\mu_\gamma(\theta) = 0,$$

and

$$c_k(n,\gamma) = 2^{\frac{n+|\gamma|}{2}} \Gamma\big(\tfrac{n+k+|\gamma|}{2}\big) \big[\Gamma\big(\tfrac{k}{2}\big)\big]^{-1}, \quad (k = 1,2,\ldots,n).$$

Note that the operator (1.3) can be written as

$$R_\gamma^{(k)} f(x) = p.v. c_k(n,\gamma) \int_{\mathbb{R}_+^n} \tfrac{Y_k^\gamma(\theta)}{|y|^{n+|\gamma|}} T^y f(x) d\mu_\gamma(y),$$

since generalized shift operator is self-adjoint. This operator was firstly introduced by [10, 11] for $k = 1$. In [4–6] Ekincioglu and Serbetci showed that the operator $R_\gamma^{(k)}$ is bounded in $L_{p,\gamma}$-spaces with radial weights.

First, we start to give the following lemma that we will use later.

Lemma 1.1 *Let $x, y \in \mathbb{R}_+^n$ and $\tilde{y} = (y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_{2n})$. Then the following equality holds*

$$\int\limits_{\{s \in \mathbb{R}_+^n : |s| > \epsilon\}} |s|^{-n-|\gamma|} Y_k^\gamma\big(\tfrac{s}{|s|}\big) \big[T^s f(x)\big] d\mu_\gamma(s) = c_\gamma \int\limits_{\{\tilde{y} \in \mathbb{R}_+^{2n} : |\tilde{y}| > \epsilon\}} |\tilde{y}|^{-n-|\gamma|} Y_k^\gamma(\tilde{\theta})$$
$$\times f\big([(x_1 - y_1)^2 + y_{n+1}^2]^{\frac{1}{2}}, \ldots, [(x_n - y_n)^2 + y_{2n}^2]^{\frac{1}{2}}\big) d\mu_\gamma(\tilde{y}),$$

where $\tilde{\theta} = \big(|\tilde{y}|^{-1}(y_1^2 + y_{n+1}^2)^{\frac{1}{2}}, \ldots, |\tilde{y}|^{-1}(y_n^2 + y_{2n}^2)^{\frac{1}{2}}\big)$.

Proof. By the definition of the Bessel generalized shift operator, we have

$$\int_{\{s \in \mathbb{R}_+^n :\ |s| > \epsilon\}} |s|^{-n-|\gamma|} Y_k^\gamma\big(\tfrac{s}{|s|}\big) \big[T^s f(x)\big] d\mu_\gamma(s) = c_\gamma \int_{|s|>\epsilon} |s|^{-n-|\gamma|} Y_k^\gamma\big(\tfrac{s}{|s|}\big)$$
$$\int\limits_0^\pi \ldots \int\limits_0^\pi f\big(\sqrt{x_1^2 + s_1^2 - 2x_1 s_1 \cos\alpha_1}, \ldots, \sqrt{x_n^2 + s_n^2 - 2x_n s_n \cos\alpha_n}\big)$$
$$\times \prod_{i=1}^n \big[s_i^{\gamma_i} \sin^{\gamma_i - 1} \alpha_i d\alpha_i\big] ds = c_\gamma \int_{|s|>\epsilon} |s|^{-n-|\gamma|} Y_k^\gamma\big(\tfrac{s}{|s|}\big) \int\limits_0^\pi \ldots \int\limits_0^\pi$$
$$\times f\big(\sqrt{x_1^2 - 2x_1 s_1 \cos\alpha_1 + (s_1\cos\alpha_1)^2 + (s_1 \sin\alpha_1)^2},$$
$$\ldots, \sqrt{x_n^2 - 2x_n s_n \cos\alpha_n + (s_n \cos\alpha_n)^2 + (s_n\sin\alpha_n)^2}\big) \prod_{i=1}^n s_i^{\gamma_i} d\mu_\gamma(\alpha) ds.$$

Changing of variables in the last integral, $s_i \cos\alpha_i = y_i, s_i \sin\alpha_i = y_{n+i}, i = 1,2,\ldots,n$, then we obtain the required equality.

2 Privalov Type Estimates For The Operator $R_\gamma^{(k)}$

In this section, we establish an estimate of the same type as the inequalities of Hölder, Korn, Lichtenstein and Giraud (see [1]) for the singular integral operator $R_\gamma^{(k)}$. We follow [1,13, 14] and find some results which are called theorems of I.I.Privalov type. We consider an integral operator mapping functions $f(y)$ of n variables $y = (y_1, y_2, \dots, y_n),\ y_i \geq i = 1, \dots, n$, into functions $U(x,t)$ of $n+1$ variables, where $x_i \geq 0, i = 1, \dots, n$ and $t \geq 0$. Let K be a kernel defined in the subspace $t \geq 0$ and homogeneous of degree $-n-|\gamma|$

$$K(x,t) = |P|^{-n-|\gamma|} Y_k^\gamma\big(\tfrac{x}{|P|}, \tfrac{t}{|P|}\big),$$

where $P = (x,t),\ |P|^2 = |x|^2 + t^2$. Unless stated otherwise we will assume throughout that $Y_k^\gamma\big(\frac{x}{|P|}, \frac{t}{|P|}\big)$ satisfies a uniform Bessel-Hölder condition on $|P| = 1$, i.e., if P and Q are unit vectors, then

$$T^Q Y_k^\gamma(P) \equiv T^{(y,s)} Y_k^\gamma(x,t) = T^y Y_k^\gamma(x, t-s) \tag{2.1}$$

and for some positive constants A and $\alpha' \leq 1$,

$$|T^Q Y_k^\gamma(P) - Y_k^\gamma(P)| \leq A|Q|^{\alpha'}, \tag{2.2}$$

where $\max\limits_k |Y_k^\gamma| \leq A$. In addition we make the basic assumption

$$\int_{\mathcal{S}_+} Y_k^\gamma(x,0) d\mu_\gamma(x) = 0. \tag{2.3}$$

Now, we consider the integral operator

$$U(x,t) = \int_{\mathbb{R}_+^n} T^y K(x,t) f(y) d\mu_\gamma(y).$$

Since the operator T^y is formally self-adjoint, we may write

$$U(x,t) = \int_{\mathbb{R}_+^n} K(y,t) T^y f(x) d\mu_\gamma(y).$$

Thus, the kernel K can be written as a sum of

$$\begin{aligned} K(x,t) &= |P|^{-n-|\gamma|} Y_k^\gamma\big(\tfrac{x}{|P|}, 0\big) + |P|^{-n-|\gamma|}\big[Y_k^\gamma\big(\tfrac{x}{|P|}, \tfrac{t}{|P|}\big) - Y_k^\gamma\big(\tfrac{x}{|P|}, 0\big)\big] \\ &= K_1 + K_2. \end{aligned}$$

By the Bessel-Hölder condition (2.2), we obtain

$$\big|Y_k^\gamma\big(\tfrac{x}{|P|}, \tfrac{t}{P}\big) - Y_k^\gamma\big(\tfrac{x}{|x|}, 0\big)\big| \leq A\, c(\alpha') |P|^{-\alpha'}\, t^{\alpha'}$$

and

$$\big|K_2(x,t)\big| \leq A\, c(\alpha')\, |P|^{-n-|\gamma|-\alpha'}\, t^{\alpha'}.$$

It follows from that

$$\begin{aligned} \int_{\mathbb{R}_+^n} \big|K_2(x,t)\big| d\mu_\gamma(x) &\leq A\, c(\alpha') \int_{\mathbb{R}_+^n} \big(|\xi|^2 + 1\big)^{-\frac{n+|\gamma|+\alpha'}{2}} d\mu_\gamma(\xi) \\ &\leq c_1 A. \end{aligned} \tag{2.4}$$

where $x = t\xi$. For $\alpha < \alpha'$, we obtain

$$\int\limits_{\mathbb{R}^n_+} \big|K_2(x,t)\big||x|^{\alpha} d\mu_\gamma(x) \le c(\alpha') A t^{\alpha} \int\limits_{\mathbb{R}^+_n} \left(|\xi|^2+1\right)^{\frac{-n-|\gamma|}{2}} d\mu_\gamma(\xi) \le c_1\, A\, t^{\alpha} \tag{2.5}$$

where c_1 depends only on $\alpha,\ \alpha'$ and n. Then by (2.1) and (2.2) the following inequality holds

$$\big|T^z K_1(x,t) - K_1(x,t)\big| \le c_1\, A\, |P|^{-n-\gamma}\, |z|^{\alpha'}.$$

Now we introduce the following semi-norms:

$$\big[U\big]_{\alpha'} = \sup \tfrac{|T^Q U(P) - U(P)|}{|Q|^{\alpha'}}, \quad [f]_\alpha = \sup \tfrac{|T^y f(x) - f(x)|}{|y|^{\alpha}}, \tag{2.6}$$

where $P = P(x,t),\ Q = Q(y,t),\ f \in L_{p,\gamma}$, and $0 < \alpha < \alpha'$.

Theorem 2.1 *Under the assumptions (2.2), (2.3), if $f \in L_{p,\gamma}$ and $[f]_\alpha < \infty$ for some positive α, then the inequality*

$$[U]_\alpha \le c_2 A [f]_\alpha$$

holds for the integral operator $U(x,t)$, where c_2 depends only on $\alpha \le 1$, and $(n + |\gamma|)$.

Proof. It is sufficient to show that the following inequality hold

$$\frac{\big|T^z U(x,t) - U(x,t)\big|}{|z|^{\alpha}} \le c_4 A [f]_\alpha. \tag{2.7}$$

We first assume that $K_1 = 0$. Then we have

$$\begin{aligned} I &= T^z U(x,t) - U(x,t) \\ &= \int_{\mathbb{R}^n_+} \big[T^z T^y K_2(x,t) - T^y K_2(x,t)\big] f(y) d\mu_\gamma(y) \\ &= \int_{\mathbb{R}^n_+} T^y K_2(x,t)\ \big[T^z f(y) - f(y)\big] d\mu_\gamma(y). \end{aligned}$$

It follows from that

$$\begin{aligned} |I| &\le c_\gamma\, [f]_\alpha\, |z|^\alpha \int_{\mathbb{R}^n_+} T^y |K_2(x,t)| d\mu_\gamma(y) \\ &= c_\gamma\, [f]_\alpha\, |z|^\alpha \int_{\mathbb{R}^n_+} |K_2(y,t)| d\mu_\gamma(y). \end{aligned}$$

From the equation (2.4), we have

$$|I| \le c_3\, A\, [f]_\alpha\, |z|^\alpha.$$

Let us consider the case $K_2 = 0$. Then

$$\begin{aligned} II &= T^z U(x,t) - U(x,t) \\ &= \int_{\mathbb{R}^n_+} \big[T^z T^y K_1(x,t) - T^y K_1(x,t)\big] f(y) d\mu_\gamma(y) \\ &= \int_{\mathbb{R}^n_+} T^y K_1(x,t)\ \big[T^z f(y) - f(y)\big] d\mu_\gamma(y). \end{aligned} \tag{2.8}$$

It follows from that

$$\begin{aligned}|II| &\le c_\gamma\,[f]_\alpha\,|z|^\alpha \int_{\mathbb{R}^n_+} T^y|K_1(x,t)|d\mu_\gamma(y)\\ &= c_\gamma\,[f]_\alpha\,|z|^\alpha \int_{\mathbb{R}^n_+} |K_1(y,t)|d\mu_\gamma(y)\\ &= c_5\,[f]_\alpha\,|z|^\alpha \int_{\mathbb{R}^n_+} |(y,t)|^{-n-|\gamma|}d\mu_\gamma(y)\\ &\le c_6\,A\,[f]_\alpha\,|z|^\alpha.\end{aligned}$$

Combining these estimates for $I,\ II$, we obtain the required result. This finishes the proof.

The following theorem follows immediately from Theorem 2.1.

Theorem 2.2 *Under the hypothesis (2.2) and (2.3) above, if* $f\in L_{p,\gamma}$, *and* f *satisfies the Bessel-Hölder condition*

$$|T^y f(x)-f(x)|\le M|y|^\alpha, \tag{2.9}$$

for some $0<\alpha\le 1$, *then for the integral operator (1.3) the limit*

$$R^{(k)}_\gamma f(x)=\lim_{\epsilon\to 0}\left(R^{(k)}_\gamma\right)_\epsilon f(x)$$

exists, and satisfies the following inequality

$$|T^y R^{(k)}_\gamma f(x)-R^{(k)}_\gamma f(x)|\le c\,M\,A\,|y|^\alpha. \tag{2.10}$$

Proof. First it is easy to show that the limit $\left(R^{(k)}_\gamma\right)_\epsilon f(x)$ exists as $\epsilon\to 0$ and equal to $R^{(k)}_\gamma f(x)$. With the aid of the condition (2.3), the operator (1.3) can be written as

$$\begin{aligned}\left(R^{(k)}_\gamma\right)_\epsilon f(x) &= \int\limits_{|y|>1}\frac{Y^\gamma_k(\theta)}{|y|^{n+|\gamma|}}T^y f(x)d\mu_\gamma(y)\\ &+\int\limits_{\epsilon<|y|<1}\frac{Y^\gamma_k(\theta)}{|y|^{n+|\gamma|}}\left[T^y f(x)-f(x)\right]d\mu_\gamma(y)\\ &= I_1+I_2.\end{aligned}$$

Let us start to estimate I_2. By (2.9), we obtain

$$I_2\le c_\gamma MA\int_{\epsilon<|y|<1}\frac{|y|^\alpha}{|y|^{n+|\gamma|}}d\mu_\gamma(y)\le\frac{c_\gamma MA}{\alpha}(1-\epsilon^\alpha),$$

whence it follows the existence of the limit as $\epsilon\to 0$ of the function $\left(R^{(k)}_\gamma\right)_\epsilon f(x)$. Let

$$U(x,t)=\int_{\mathbb{R}^n_{k,+}}\frac{Y^\gamma_k(\theta)}{|P|^{n+|\gamma|}}T^y f(x)d\mu_\gamma(y),$$

where $\theta=\frac{y}{|y|}$, and $P=(y,t),\ y\in\mathbb{R}^n_+$. By Theorem 2.1 for any t, we have

$$\left|T^y U(x,t)-U(x,t)\right|\le cMA|y|^\alpha. \tag{2.11}$$

Therefore, it is sufficient to show that

$$\lim_{t\to 0} U(x,t) = \lim_{\epsilon\to 0} \left(R_\gamma^{(k)}\right)_\epsilon f(x).$$

For $t \to 0$, (2.11) in turn implies that (2.10). Then we choose a arbitrarily small fixed $\delta > 0$ and set $\epsilon < \delta$. So we obtain

$$\begin{aligned}
&\left(R_\gamma^{(k)}\right)_\epsilon f(x) - U(x,\epsilon)\\
&= \int\limits_{|y|>\delta} Y_k^\gamma(\theta)\Big[\frac{1}{|y|^{n+|\gamma|}} - \frac{1}{(|y|^2+\epsilon^2)^{\frac{n+|\gamma|}{2}}}\Big] T^y f(x) d\mu_\gamma(y)\\
&+ \int\limits_{\epsilon<|y|<\delta} Y_k^\gamma\Big[\frac{1}{|y|^{n+|\gamma|}} - \frac{1}{(|y|^2+\epsilon^2)^{\frac{n+|\gamma|}{2}}}\Big]\big[T^y f(x) - f(x)\big] d\mu_\gamma(y)\\
&- \int\limits_{|y|<\epsilon} \frac{Y_k^\gamma(\theta)}{(|y|^2+\epsilon^2)^{\frac{n+|\gamma|}{2}}}\big[T^y f(x) - f(x)\big] d\mu_\gamma(y) = I_1 + I_2 + I_3.
\end{aligned}$$

Since integral tends to zero, the term I_1 tends to zero for $\epsilon \to 0$. By using Lemma 2.1 and the condition (2.9) we have

$$\begin{aligned}
|I_2| &\le cMA \int_\epsilon^\delta \left| \frac{1}{\rho^{n+|\gamma|}} - \frac{1}{(\rho^2+\epsilon^2)^{\frac{n+|\gamma|}{2}}} \right| \rho^{n+|\gamma|+\alpha-1} d\rho\\
|I_3| &\le cMA\epsilon^\alpha \int_0^1 \frac{\rho^{n+|\gamma|+\alpha-1}}{(\rho^2+1)^{\frac{n+|\gamma|}{2}}} d\rho.
\end{aligned}$$

It follows from that

$$|I_2| \le 2cMA\delta^\alpha \int_0^1 \frac{\rho^{n+|\gamma|+\alpha-1}}{\rho^{n+|\gamma|}} d\rho = c_1\delta^\alpha,$$

and

$$|I_3| \le cMA\epsilon^\alpha \int_0^1 \frac{\rho^{n+|\gamma|+\alpha-1}}{(\rho^2+1)^{\frac{n+|\gamma|}{2}}} d\rho = c_2\epsilon^\alpha.$$

Hence the difference $\left(R_\gamma^{(k)}\right)_\epsilon f(x) - U(x,t)$ can be made arbitrarily small. This completes the proof of the theorem.

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References

1. Agmon, S., Douglis, A., Nirenberg, L.: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions,* Comm. Pure and Appl. Math. **12** (4) (1959), 623–727.
2. Aliev, I.A., Gadzhiev, A.D.: *Weighted estimates for multidimensional singular integrals generated by the generalized shift operator,* Mat. Sb. **183** (9) (1992), 45–66; translation in Russian Acad. Sci. Math. **77** (1994), 37–55.

3. Calderon, A.P., Zygmund, A.: *On singular integrals,* Amer. J. Math. **78** (2) (1956), 289–309.
4. Ekincioglu, I.: *The boundedness of high order Riesz-Bessel transformations generated by the generalized shift operator in weighted $L_{p,w,\gamma}$-spaces with general weights,* Acta Appl.Math. **109** (2) (2010), 591–598.
5. Ekincioglu, I., Ozkin, I.K.: *On High order Riesz Transformations Generated by Generalized Shift Operator,* Turkish Journal of Mathematics. **21** (EK) (1997), 51–60.
6. Ekincioglu, I., Serbetci, A.: *On weighted estimates of high order Riesz-Bessel transformations generated by the generalized shift operator,* Acta Math. Sin. Eng. Ser. **21** (1) (2005), 53–64.
7. Guliyev, E.V.: *Weighted inequality for singular integrals in Lebesgue spaces, associated with the Laplace-Bessel differential operators,* Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **25** (2006), 27–34.
8. Guliev, V.S., Narimanov, A.Kh.: *On the L_p^γ-boundedness of the anisotropic Fourier-Bessel singular integrals,* Translated by E. T. Aliyeva. Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **19** (5) (1999), Math. Mech. (2000), 32–41.
9. Levitan, B.M.: *Bessel function expansions in series and Fourier integrals,* Uspekhi Mat. Nauk. (Russian) **6** (2) (1951), 102–143.
10. Lyakhov, L.N.: *On a class of spherical functions and singular pseudodifferential operators,* Dokl. Akad. Nauk. **272** (4) (1983), 781–784; translation in Soviet Math.Dokl. **28** (1983), 431–434.
11. Lyakhov, L.N.: *Spherical weighted harmonic functions and singular pseudodifferential operators,* Differents. Uravn. **21** (6) (1985), 1020–1032; translation in Diff. Equ. **21** (1985), 693–703.
12. Kipriyanov, I.A., Klyuchantsev, M.I.: *On singular integrals generated by the generalized shift operator,* Sibirsk. Mat. Zh. **11** (1970), 1060–1083; translation in Siberian Math. **11** (1970), 787–804.
13. Klyuchantsev, M.I.: *On singular integrals generated by the generalized shift operator,* Sibirsk. Math. **11** (1970), 810–821; translation in Zh. Siberian Math. J. **11** (1970), 612–620.
14. Mihlin, S.G.: Multidimensional singular integrals and integral equations. *Fizmatgiz, Moscow* (1962); *english transl. Pergamon Press, NY* (1965).