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Fourier series analysis of a time-dependent perfusion coefficient determination in a 2D bioheat transfer process

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Abstract. *In this work, an inverse problem of determining the time-dependent perfusion coefficient of two-dimensional (2D) heat equation with a classical and total energy integral overdetermination condition is considered. The existence and uniqueness of the problem is obtained by generalized Fourier method combined with the unique solvability of the second kind Volterra integral equation. Moreover, the proof of the continuous dependence upon the data of the inverse problem is given.*

Keywords. 2D bioheat equation, Volterra integral equation, Generalized Fourier method

Mathematics Subject Classification (2010): Primary 35R30 · Secondary 35K20

1 Introduction

The inverse coefficient problems for heat equation are of importance in some engineering, industrial and medical point of views. Among these inverse problems, much attention is given to the determination of the lowest order coefficient in multidimensional heat equation, in particular, when this coefficient depends solely on time [4,7,9,12–15,21], although many researchers have reported its difficulties.

The bioheat model is composed of a partial differential equation [20]:

$$\rho c U_\tau - \kappa \Delta U + w_b c_b (U - U_a) = h_m + h_e, \tag{1.1}$$

where κ is the thermal conductivity of the tissue, U is the temperature of the tissue, w_b is mass flow rate of blood, c_b is specific heat of the blood, U_a is the temperature of the arterial blood, h_m is a volumetric rate of metabolic heating generation, h_e volumetric rate of external heat, ρ is the density, and c is specific heat of the tissue.

In two-dimensional (2D) case, (1.1) takes the following dimensionless form

$$u_t = u_{xx} + u_{yy} - p(t)u(x,y,t) + f(x,y,t), \quad (x,y,t) \in D_T, \tag{1.2}$$

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by using the change of variables, letting g_0 as a reference source of heating generation

$$u = \frac{\kappa(U - U_a)}{g_0}, \quad t = \frac{\kappa\tau}{\rho c}, \quad p = \frac{w_b c_b}{\kappa}, \quad f = \frac{h_e + h_m}{g_0},$$

where the domain $D_T = D \times (0 < t \le T)$ with $D = \{(x, y) : 0 < x, y < 1\}$, the parameter $p(t)$ is blood perfusion coefficient, and f is the dimensionless source of heating generation.

Inverse blood perfusion determination techniques for one dimensional (1D) case based on known initial and classicial boundary conditions (Dirichlet, Neumann, Robin) and additional classical measurements have been described in [6,16–18,21]. Moreover, these inverse problems in multidimensional cases had been studied in [3,22]. The determination of the coefficient for one dimensional case based on nonlocal or nonclassical and additional integral boundary conditions have been described in [8,10,11].

The determination of a time-dependent blood perfusion coefficient in a 2D bioheat equation is formulated as an inverse problem of finding the coefficient of the lowest term in a 2D heat equation with classical boundary conditions (Dirichlet boundary conditions) will be considered in the present paper.

We consider (1.2) with the initial condition

$$u(x, y, 0) = \varphi(x, y), \quad 0 \le x, y \le 1, \tag{1.3}$$

Dirichlet boundary conditions

$$\begin{aligned} u(0, y, t) = u(1, y, t) = 0, \quad 0 \le y \le 1, \quad 0 \le t \le T, \\ u(x, 0, t) = u(x, 1, t) = 0, \quad 0 \le x \le 1, \quad 0 \le t \le T. \end{aligned} \tag{1.4}$$

In the mathematical model of this problem, we consider a rectangular perfused tissue with both thickness and length are 1. The boundary conditions (1.4) include 0° temperature at $x = 0$, $x = 1$, on the upper skin surface at $y = 1$ and on a wall between the tissue and an adjoint large blood vessel at $y = 0$.

The model which is considered a rectangular perfused tissue with length and thickness equal to some value L and 1, respectively, including (some) given temperature on the upper skin $y = 1$, adiabatic conditions at $x = 0$ and $x = L$, and convective heat transfer between the tissue and an adjoint large blood vessel at $y = 0$ as a space-dependent bioheat transfer problem presented in [1,2].

The problem of finding a pair $\{p(t), u(x, y, t)\} \in C[0, T] \times C^{2,2,1}(\overline{D}_T)$ satisfying the equation (1.2), initial condition (1.3), boundary conditions (1.4) and overdetermination condition

$$\int_0^1 \int_0^1 u(x, y, t) dx dy = E(t). \tag{1.5}$$

will be called an inverse problem.

The paper is organized as follows. In Section 2, we recall some necessary results on basisness of root functions concerning the two-dimensional spectral problem with classical boundary condition. Then the well-posedness of the inverse problem (1.2)-(1.5) for global T is showed by using generalized Fourier method combined with the unique solvability of the second kind of Volterra integral equation in Section 3. Finally, the concluding remarks are presented in Section 4.

2 Spectral problem

Consider the following spectral problem:

$$\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} + \mu Z = 0, \quad x, y \in D, \tag{2.1}$$

$$Z(0, y) = Z(1, y) = 0, \quad Z(x, 0) = Z(x, 1) = 0, \quad 0 \leq x, y \leq 1, \tag{2.2}$$

where μ is the separation parameter. We present the solution as follows

$$Z(x, y) = X(x)V(y). \tag{2.3}$$

Substituting this expression into (2.1) and (2.2), we obtain the following problems

$$X''(x) + \gamma X(x) = 0, \quad 0 < x < 1, \quad X(0) = X(1) = 0, \tag{2.4}$$

$$V''(y) + \lambda V(y) = 0, \quad 0 < y < 1, \quad V(0) = V(1) = 0, \tag{2.5}$$

where $\gamma = \mu - \lambda$. It is known that the solutions of (2.4) and (2.5) are

$$\gamma_m = (\pi m)^2, \quad X_m(x) = \sqrt{2}\sin(\pi m x),$$

$$\lambda_k = (\pi k)^2, \quad V_k(y) = \sqrt{2}\sin(\pi k y),$$

for $k, m = 1, 2, \ldots$, respectively. Here, we give the constants multiplying the eigenfunctions suitable for normalization conditions.

Consequently, the eigenvalues and eigenfunctions of problem (2.1), (2.2), with the representation (2.3), are

$$\mu_{m,k} = \gamma_m + \lambda_k = (\pi m)^2 + (\pi k)^2, \quad Z_{m,k}(x, y) = X_m(x)V_k(y), \quad m, k = 1, 2, \ldots. \tag{2.6}$$

The problem (2.1)-(2.2) is self-adjoint in the sense of the following inner product

$$(\psi, \xi) = \int_0^1 \int_0^1 \psi(x, y)\xi(x, y)dxdy.$$

Additionally, since $X_m(x)$ and $V_k(y)$ are complete orthonormal system on $[0, 1]$, then the set of eigenfunctions $Z_{m,k}(x, y)$ is complete in $L_2(D)$ and form an orthonormal system of functions on $\overline{D}$; i.e., for any admissible indices m, k, l, and p, we have $(Z_{m,k}, Z_{l,p}) = 1$ if $m = l$ and $k = p$ and $(Z_{m,k}, Z_{l,p}) = 0$ otherwise.

For any integrable function $\varphi(x, y)$ in $\overline{D}$, the Parseval's relation is

$$\iint_D \varphi^2(x, y)dxdy = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \varphi_{m,k}^2,$$

where $\varphi_{m,k} = \iint_D \varphi(x, y)Z_{m,k}(x, y)dxdy$, (see for example [19]).

3 Global well-posedness of the inverse problem

We have the following assumptions on $\varphi(x,y)$, $f(x,y,t)$ and $E(t)$.

$$(A_1)\ \begin{array}{ll} (A_1)_1 & \varphi(x,y)\in C^{2,2}(\overline{D}), \\ (A_1)_2 & \varphi(0,y)=\varphi(1,y)=0, \quad \varphi(x,0)=\varphi(x,1)=0, \\ (A_1)_3 & \varphi_{1,1}>0, \quad \varphi_{2m-1,2k-1}\geq 0, \quad m,k=2,3,\ldots, \end{array}$$

$$(A_2)\ \begin{array}{ll} (A_2)_1 & f(x,y,t)\in C(\overline{D}_T),\ \ f(x,y,t)\in C^{2,2}(\overline{D}),\ \ \forall t\in[0,T] \\ (A_2)_2 & f(0,y,t)=f(1,y,t)=0, \quad f(x,0,t)=f(x,1,t)=0, \\ (A_2)_3 & f_{2m-1,2k-1}(t)\geq 0, \quad m,k=1,2,\ldots, \end{array}$$

$$(A_3)\ \begin{array}{ll} (A_3)_1 & E(t)\in C^1[0,T], \\ (A_3)_2 & E(0)=\int_0^1\int_0^1\varphi(x,y)dxdy, \\ (A_3)_3 & E(t)>0, \quad \forall t\in[0,T], \end{array}$$

where

$$f_{m,k}(t)=\iint_D f(x,y,t)Z_{m,k}(x,y)dxdy, \quad m,k=1,2,\ldots.$$

The main result on existence and uniqueness of the solution of the inverse problem (1.2)-(1.5) is presented as follows.

Theorem 3.1 (Existence and uniqueness) *Under the conditions (A_1)-(A_3) the inverse problem* (1.2)-(1.5) *has a unique solution.*

Proof. Since (2.6) is a basis in $L_2(\overline{D})$, we present the solution of (1.2)-(1.4) in the following form for arbitrary $p(t)\in C[0,T]$:

$$u(x,y,t)=\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\alpha_{m,k}(t)Z_{m,k}(x,y), \tag{3.1}$$

where

$$\alpha_{m,k}(t)=\varphi_{m,k}e^{-\left[(\pi m)^2+(\pi k)^2\right]t-\int_0^t p(s)ds}+\int_0^t f_{m,k}(\tau)e^{-\left[(\pi m)^2+(\pi k)^2\right](t-\tau)-\int_\tau^t p(s)ds}d\tau.$$

Under conditions (A_1) and (A_2), the series (3.1), its $t-$partial derivative, the $xx-$second order and $yy-$second order partial derivatives converge uniformly in $\overline{D}_T$ that their majorizing sums absolutely convergence. Thus $u(x,y,t)\in C^{2,2,1}(\overline{D}_T)$. By considering (3.1) and the overdetermination condition (1.5), we obtain the following Volterra integral equation of the second kind with respect to $r(t)=e^{\int_0^t p(s)ds}$:

$$r(t)=F(t)+\int_0^t K(t,\tau)r(\tau)d\tau, \tag{3.2}$$

where

$$\begin{aligned} F(t)=&\frac{8}{\pi^2E(t)}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_{2m-1,2k-1}}{(2m-1)(2k-1)}e^{-\left[((2m-1)\pi)^2+((2k-1)\pi)^2\right]t}, \\ K(t,\tau)=&\frac{8}{\pi^2E(t)}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{f_{2m-1,2k-1}(\tau)}{(2m-1)(2k-1)}e^{-\left[((2m-1)\pi)^2+((2k-1)\pi)^2\right](t-\tau)}. \end{aligned}$$

In the case that (3.2) has a positive solution in $C^1[0,T]$, the function $p(t)$ can be determined from the equation

$$p(t) = \frac{r'(t)}{r(t)}. \tag{3.3}$$

Under the assumptions $(A_1)_{1,2} - (A_3)_{1,2}$, the right-hand side of $F(t)$ and the kernel $K(t,\tau)$ are continuously differentiable functions in $[0,T]$ and $[0,T]\times[0,T]$, respectively. Moreover, under the conditions $(A_1)_3 - (A_3)_3$, $F(t) > 0$ in $[0,T]$ and $K(t,\tau) \geq 0$ in $[0,T]\times[0,T]$. In addition, applying the Gronwall inequality [5] to (3.2), the following inequality holds:

$$\|r\|_{C[0,T]} \leq \|F\|_{C[0,T]} e^{T\|K\|_{C([0,T]\times[0,T])}}. \tag{3.4}$$

Then we obtain a unique positive function $r(t) \in C^1[0,T]$, which the function (3.3) together with the solution of the problem (1.2)-(1.4) given by the Fourier series (3.1), form the unique solution of the inverse problem (1.2)-(1.5). Theorem 3.1 is proved.

The following result on continuously dependence on the data of the solution of the inverse problem (1.2)-(1.5) holds.

Theorem 3.2 (Stability) *Let $\Im$ be the class of triples in the form $\{\varphi, f, E\}$ which satisfy the assumptions (A_1) - (A_3) and*

$$\|f\|_{C^{2,2,0}(\overline{D}_T)} \leq N_0,\ \|\varphi\|_{C^{2,2}(\overline{D})} \leq N_1,\ \|E\|_{C^1[0,T]} \leq N_2,$$
$$0 < N_3 \leq \min_{t\in[0,T]} |E(t)|,$$

for some positive constants N_i, $i = 0,1,2,3$. Then the solution (p,u) of the inverse problem (1.2)-(1.5) *depends continuously upon the data in $\Im$.*

Proof. Let $\Phi = \{\varphi, E, f\}$, $\overline{\Phi} = \{\overline{\varphi}, \overline{E}, \overline{f}\} \in \Im$ be two sets of data and (p,u) and $(\overline{p},\overline{u})$ be the solutions of inverse problem (1.2)-(1.5) corresponding the data Φ and $\overline{\Phi}$, respectively. Denote $\|\Phi\| = \|\varphi\|_{C^{2,2}(\overline{D})} + \|E\|_{C^1[0,T]} + \|f\|_{C^{2,2,0}(\overline{D}_T)}$.

According to (3.1) and (3.2), we get

$$r(t) = F(t) + \int_0^t K(t,\tau) r(\tau) d\tau, \quad \text{and} \quad \overline{r}(t) = \overline{F}(t) + \int_0^t \overline{K}(t,\tau)\overline{r}(\tau) d\tau,$$

where

$$F(t) = \frac{8}{\pi^2 E(t)} \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\varphi_{2m-1,2k-1}}{(2m-1)(2k-1)} e^{-\left[((2m-1)\pi)^2 + ((2k-1)\pi)^2\right]t},$$
$$K(t,\tau) = \frac{8}{\pi^2 E(t)} \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{f_{2m-1,2k-1}(\tau)}{(2m-1)(2k-1)} e^{-\left[((2m-1)\pi)^2 + ((2k-1)\pi)^2\right](t-\tau)},$$

and

$$\overline{F}(t) = \frac{8}{\pi^2 \overline{E}(t)} \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\overline{\varphi}_{2m-1,2k-1}}{(2m-1)(2k-1)} e^{-\left[((2m-1)\pi)^2 + ((2k-1)\pi)^2\right]t},$$
$$\overline{K}(t,\tau) = \frac{8}{\pi^2 \overline{E}(t)} \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\overline{f}_{2m-1,2k-1}(\tau)}{(2m-1)(2k-1)} e^{-\left[((2m-1)\pi)^2 + ((2k-1)\pi)^2\right](t-\tau)}.$$

At first, let us estimate the difference $r - \overline{r}$. We obtain from (3.2)

$$r(t)-\overline{r}(t) = F(t)-\overline{F}(t)+\int_0^t \left[K(t,\tau) - \overline{K}(t,\tau)\right] r(\tau)d\tau+\int_0^t \overline{K}(t,\tau) \left[r(\tau) - \overline{r}(\tau)\right] d\tau. \tag{3.5}$$

Let $\gamma = \|F-\overline{F}\|_{C[0,T]}+T\|K-\overline{K}\|_{C([0,T]\times[0,T])}\|r\|_{C[0,T]}$ and denote $R(t) = |r(t)-\overline{r}(t)|$. Then (3.5) implies the inequality

$$R(t) \leq \gamma + \int_0^t |\overline{K}(t,\tau)|R(\tau)d\tau \tag{3.6}$$

By applying the Gronwall inequality, we obtain from (3.6)

$$R(t) \leq \gamma \cdot e^{\int_0^t \sup_{s\in[\tau,t]} |\overline{K}(s,\tau)|d\tau}.$$

Moreover, by using Schwarz and Bessel's inequalities, we get

$$|F(t)| \leq \frac{8}{\pi^2|E(t)|}\left|\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_{2m-1,2k-1}}{(2m-1)(2k-1)}\right| \leq \frac{8}{\pi^2 N_3}C_1C_2\|\varphi\|_{C^{2,2}(\overline{D})} \leq \frac{8N_1}{\pi^2 N_3}C_1C_2, \tag{3.7}$$

and similarly

$$|K(t,\tau)| \leq \frac{8N_0}{\pi^2 N_3}C_1C_2, \tag{3.8}$$

where

$$C_1 = \left(\sum_{m=1}^{\infty}\left(\frac{1}{2m-1}\right)^2\right)^{1/2}, \quad C_2 = \left(\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^2\right)^{1/2}.$$

Finally, from (3.4), (3.7) and (3.8), we have

$$\|r-\overline{r}\|_{C[0,T]} \leq C_3(\|F-\overline{F}\|_{C[0,T]} + C_4\|K-\overline{K}\|_{C([0,T]\times[0,T])}),$$

where $C_3 = e^{T(8N_0/\pi^2 N_3)C_1C_2}$ and $C_4 = T(8N_1/\pi^2 N_3)C_1C_2C_3$. Therefore, r continuously depends upon F and K.

Now let us see that F and K continuously depend upon the data. It is easy to derive the following inequalities:

$$|F(t) - \overline{F}(t)| \leq \frac{8}{\pi^2}C_1C_2\left(\frac{1}{N_3}\|\varphi-\overline{\varphi}\|_{C^{2,2}(\overline{D})} + \frac{N_1}{N_3^2}\|E-\overline{E}\|_{C^1[0,T]}\right),$$

$$|K(t,\tau) - \overline{K}(t,\tau)| \leq \frac{8}{\pi^2}C_1C_2\left(\frac{1}{N_3}\|f-\overline{f}\|_{C^{2,2,0}(\overline{D}_T)} + \frac{N_0}{N_3^2}\|E-\overline{E}\|_{C^1[0,T]}\right),$$

These inequalities imply that

$$\begin{aligned}\|F-\overline{F}\|_{C[0,T]} &\leq M_1(\|f-\overline{f}\|_{C^{2,2,0}(\overline{D}_T)} + \|\varphi-\overline{\varphi}\|_{C^{2,2}(\overline{D})} + \|E-\overline{E}\|_{C^1[0,T]}) \\ &\leq M_1\|\Phi-\overline{\Phi}\|,\end{aligned}$$

and similarly

$$\|K-\overline{K}\|_{C([0,T]\times[0,T])} \leq M_2\|\Phi-\overline{\Phi}\|,$$

where M_1 and M_2 are constants obtained from the constants C_1, C_2 and N_k, $k = 0,1,2,3$. Thus, F and K continuously depend upon the data. So, r also continuously depends upon the data.

We also need to show that r' depends continuously upon the data in order to show that p depends continuously upon the data. Differentiating both sides of (3.2) with respect to t, we get the followings:

$$r'(t) = F'(t) + K(t,t)r(t) + \int_0^t K_t(t,\tau)r(\tau)d\tau,$$
$$\overline{r}'(t) = \overline{F}'(t) + \overline{K}(t,t)\overline{r}(t) + \int_0^t \overline{K}_t(t,\tau)\overline{r}(\tau)d\tau.$$

Then, the following estimation holds:

$$\begin{aligned}\|r' - \overline{r}'\|_{C[0,T]} \leq &\|F' - \overline{F}'\|_{C[0,T]} \\ &+ \left(\|K - \overline{K}\|_{C([0,T]\times[0,T])} + T\|K_t - \overline{K}_t\|_{C([0,T]\times[0,T])}\right)\|r\|_{C[0,T]} \\ &+ \left(\|\overline{K}\|_{C([0,T]\times[0,T])} + T\|\overline{K}_t\|_{C([0,T]\times[0,T])}\right)\|r - \overline{r}\|_{C[0,T]}.\end{aligned} \tag{3.9}$$

Since we know that r, F, and K are continuously dependent upon the data, we need to show continuously dependence of the remaining statements. Since the following inequality holds by two times over the $x-$axis and two times over the $y-$axis integration by parts

$$\begin{aligned}&|\overline{K}_t(t,\tau)| \\ &\leq \frac{8}{\pi^2}\left|\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\overline{f}_{2m-1,2k-1}(\tau)\left[\frac{\pi^2}{\overline{E}(t)}\left(\frac{2m-1}{2k-1} + \frac{2k-1}{2m-1}\right) + \frac{\overline{E}'(t)}{\overline{E}^2(t)}\frac{1}{(2m-1)(2k-1)}\right]\right|, \\ &\leq \frac{8}{\pi^2}\|\overline{f}\|_{C^{2,2,0}(\overline{D}_T)}\left(\frac{2C_1C_2}{N_3} + \frac{N_2C_1C_2}{N_3^2}\right) \leq \frac{8}{\pi^2}N_0C_1C_2\left(\frac{2}{N_3} + \frac{N_2}{N_3^2}\right),\end{aligned}$$

r' depends continuously upon F' and K_t, we can easily obtain the following inequalities similarly:

$$\begin{aligned}|F'(t) - \overline{F}'(t)| \leq \frac{8}{\pi^2}C_1C_2 &\left[\left(\frac{N_2 + 2N_3}{N_3^2}\right)\|\varphi - \overline{\varphi}\|_{C^{2,2}(\overline{D})}\right. \\ &\left. + N_1\left(\frac{2N_2 + 3N_3}{N_3^3}\right)\|E - \overline{E}\|_{C^1[0,T]}\right], \\ |K_t(t,\tau) - \overline{K}_t(t,\tau)| \leq \frac{8}{\pi^2}C_1C_2 &\left[\left(\frac{N_2 + 2N_3}{N_3^2}\right)\|f - \overline{f}\|_{C^{2,2,0}(\overline{D}_T)}\right. \\ &\left. + N_0\left(\frac{2N_2 + 3N_3}{N_3^3}\right)\|E - \overline{E}\|_{C^1[0,T]}\right],\end{aligned}$$

Hence, by using the last inequalities, from (3.9)

$$\|r' - \overline{r}'\|_{C[0,T]} \leq M_3\|\Phi - \overline{\Phi}\|,$$

where M_3 is a constant which is determined from C_1, C_2 and N_k, $k = 0,1,2,3$. This means that r' depends continuously upon the data as well. Thus, from (3.3) p also depends continuously upon the data.

Similarly, we obtain the estimate the difference $u - \overline{u}$ from (3.1):

$$\|u - \overline{u}\|_{C(\overline{D}_T)} \leq M_4\|\Phi - \overline{\Phi}\|.$$

Theorem 3.2 is proved.

4 Conclusion

The paper considers the problem of determining the lowest coefficient that depends on time only, for a two-dimensional parabolic equation with classical boundary conditions and the total energy measurement. The existence and uniqueness of the solution of such an inverse problem and global well-posedness of this problem are examined by using the method of series expansion in terms of eigenfunctions of corresponding spatial differential operator which is self-adjoint and hence the system of eigenfunctions is complete.

The numerical method of the inverse problem (1.2)-(1.5) will be considered with a suitable combination of the finite difference scheme and numerical integration as a future work.

References

1. Bazan, F.S.V., Bedin, L., Borges, L.S.: *Space-dependent perfusion coefficient estimation in a 2D bioheat transfer problem*, Comput. Phys. Comm., **214** (2017), 18–30.
2. Bedin, L., Bazan, F.S.V.: *On the 2D bioheat equation with convective boundary conditions and its numerical realization via a highly accurate approach*, Appl. Math. Comput., **236** (2014), 422–436.
3. Cannon, J.R., Lin, Y., Wang, S.: *Determination of source parameter in parabolic equations*, Meccanica, **27** (2) (1992), 85–94.
4. Choulli, M.: *An Inverse Problem for a Semilinear Parabolic Equation*, Inverse Problems, **10** (1994), 1123–1132.
5. Dragomir, S.S.: Some Gronwall Type Inequalities and Applications, *RGMIA Monographs, Australia: Victoria University* (2002).
6. Grabski, J.K., Lesnic, D., Johansson, B.T.: *Identification of a time-dependent bio-heat blood perfusion coefficient*, International Communications in Heat and Mass Transfer, **75** (2016), 218–222.
7. Isakov, V.: Inverse Problems for Partial Differential Equations, *Springer* (1998).
8. Ismailov, M.I., Kanca, F.: *An Inverse Coefficient Problem for a Parabolic Equation in the Case of Nonlocal Boundary and Overdetermination Conditions*, Math. Meth. Appl. Sci., **34** (2011), 692–702.
9. Kamynin, V.L.: *The Inverse Problem of Determining the Lower-Order Coefficient in Parabolic Equations with Integral Observation*, Mathematical Notes, **94** (1) (2013), 205–213.
10. Kerimov, N.B., Ismailov, M.I.: *An inverse coefficient problem for the heat equation in the case of nonlocal boundary conditions*, J. Math. Anal. Appl., **396** (2) (2012), 546–554.
11. Kerimov, N.B., Ismailov, M.I.: *Direct and inverse problems for the heat equation with a dynamic-type boundary condition*, IMA J. Appl. Math., **80** (5) (2015), 1519–1533.
12. Kostin, A.B.: *Inverse Problem of Finding the Coefficient of u in a Parabolic Equation on the Basis of a Nonlocal Observation Condition*, Differential Equations, **51** (5) (2015), 605–619.
13. Kozhanov, A.I.: *Parabolic Equations with an Unknown Absorption Coefficient*, Doklady Mathematics, **74** (1) (2006), 573–576.
14. Kozhanov, A.I.: *On Solvability of an Inverse Problem with an Unknown Coefficient and Right-Hand Side for a Parabolic Equation*, J. Inv. Ill-Posed Problems, **10** (6) (2002), 611–627.
15. Lavrent'ev, M.M., Romanov, V. G., Shishatskii, S. P.: Ill-posed Problems of Mathematical Physics and Analysis, *American Mathematical Society* (1986).
16. Lesnic, D.: *Identification of the time-dependent perfusion coefficient in bio-heat conduction equation*, J. Inverse and Ill-Posed Prob., **17** (2009), 1–12.

17. Lesnic, D., Ivanchov, M.: *Determination of the time-dependent perfusion coefficient in the bio-heat equation*, Applied Mathematics Letters, **39** (2015), 96–100.
18. Lin, Y.: *An inverse problem for a class of quasi-linear parabolic equations*, SIAM J. Math. Anal., **22** (1991), 146–156.
19. Lo, C.Y.: Boundary Value Problems, *World Scientific* (2000).
20. Pennes, H.H.: *Analysis of tissue and arterial blood temperatures in the resting human forearm*, J. Appl. Physiol, **1** (2) (1948), 93–122.
21. Prilepko, A.I., Solov'ev, V.V.: *On the solvability of inverse boundary value problems for the determination of the coefficient preceding the lower derivative in a parabolic equation*, (Russian) Differ. Uravn., **23** (1) (1987), 136–143.
22. Pyatkov, S.G.: *Solvability of Some Inverse Problems for Parabolic Equations*, J. Inv. Ill-Posed Problems, **12** (4) (2004), 397–412.