Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
38 (2018), no. 4, Mathematics, 79-92.

Commutators of the Marcinkiewicz integral on generalized weighted Morrey spaces

Afag F. Ismayilova

Received: 02.12.2017 / Revised: 09.05.2018 / Accepted: 20.06.2018

Abstract. *In this paper, we study the boundedness of the commutators of the Marcinkiewicz operator $\mu_{b,\Omega}$ on generalized weighted Morrey spaces $M_{p,\varphi}(w)$. We find the sufficient conditions on the pair (φ_1, φ_2) with $b \in BMO(\mathbb{R}^n)$ and $w \in A_p$ which ensures the boundedness of the operators $\mu_{\Omega,b}$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1 < p < \infty$. In all cases the conditions for the boundedness of the operator $\mu_{b,\Omega}$ is given in terms of Zygmund-type integral inequalities on (φ_1, φ_2) and w, which do not assume any assumption on monotonicity of $\varphi_1(x, r)$, $\varphi_2(x, r)$ in r.*

Keywords. Marcinkiewicz operator; generalized weighted Morrey spaces; commutator; A_p weights.

Mathematics Subject Classification (2010): 42B20, 42B35

1 Introduction

The classical Morrey spaces were originally introduced by Morrey in [36] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [17, 19, 22, 36]. Recently, Komori and Shirai [33] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [23] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L^{p,\kappa}(w)$. In [23] Guliyev also studied the boundedness of the classical operators and its commutators in these spaces $M_{p,\varphi}(w)$, see also Guliyev et al. (see also [3, 18, 26–30].

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{1.1}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

A.F. Ismayilova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, AZ1141 Baku, Azerbaijan
Azerbaijan University of Cooperation, Baku, Azerbaijan
E-mail: afaismayilova28@gmail.com

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.2}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega f(x) = \left(\int_0^\infty |F_{\Omega,t} f(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t} f(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

It is well known that the Littlewood-Paley g-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley g-function. In this paper, we will also consider the com- mutator $\mu_{\Omega,b}$ which is given by the following expression

$$\mu_{\Omega,b} f(x) = \left(\int_0^\infty |F^b_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F^b_{\Omega,t}(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y)dy.$$

On the other hand, the study of Schrödinger operator $L = -\Delta + V$ recently attracted much attention. In particular, Shen [38] considered Lp estimates for Schrödinger operators L with certain potentials which include Schrödinger Riesz transforms $R_j^L = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \dots, n$. Then, Dziubanński and Zienkiewicz [16] introduced the Hardy type space $H^1_L(\mathbb{R}^n)$ associated with the Schrödinger operator L, which is larger than the classical Hardy space $H^1(\mathbb{R}^n)$.

Similar to the classical Marcinkiewicz function, we define the Marcinkiewicz functions $\mu_{j,\Omega}$ associated with the Schrödinger operator L by

$$\mu^L_{j,\Omega} f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| K^L_j(x,y) f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $K^L_j(x,y) = \widetilde{K^L_j}(x,y)|x-y|$ and $\widetilde{K^L_j}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \dots, n$. In particular, when $V = 0$, $K^\Delta_j(x,y) = \widetilde{K^\Delta_j}(x,y)|x-y| = \frac{(x-y)_j/|x-y|}{|x-y|^{n-1}}$ and $\widetilde{K^\Delta_j}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} \Delta^{-\frac{1}{2}}$, $j = 1, \dots, n$. In this paper, we write $K_j(x,y) = K^\Delta_j(x,y)$ and

$$\mu_{j,\Omega} f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| K_j(x,y) f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Obviously, $\mu_{j,\Omega}$ are classical Marcinkiewicz functions with rough kernel. Therefore, it will be an interesting thing to study the property of $\mu^L_{j,\Omega}$. The main purpose of this paper is to show that Marcinkiewicz operators with rough kernel associated with Schrödinger operators $\mu^L_{j,\Omega}$, $j = 1, \dots, n$ are bounded from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$, $1 < p < \infty$.

The commutator of the classical Marcinkiewicz function with rough kernel is defined by

$$\mu_{j,\Omega,b}f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| K_j(x,y)[b(x)-b(y)]f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

The commutator $\mu^L_{j,\Omega,b}$ formed by $b \in BMO(\mathbb{R}^n)$ and the Marcinkiewicz function with rough kernel $\mu^L_{j,\Omega}$ is defined by

$$\mu^L_{j,\Omega,b}f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} |\Omega(x-y)| K^L_j(x,y)[b(x)-b(y)]f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$. The maximal operator with rough kernel M_Ω is defined by

$$M_\Omega f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |\Omega(x-y)|\,|f(y)|dy.$$

It is obvious that when $\Omega \equiv 1$, M_Ω is the Hardy-Littlewood maximal operator M. For $b \in L_1^{\text{loc}}(\mathbb{R}^n)$ the commutator of the maximal operator $M_{\Omega,b}$ is defined by

$$M_{\Omega,b}f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |b(x)-b(y)|\,|\Omega(x-y)|\,|f(y)|dy.$$

We find the sufficient conditions on the pair (φ_1, φ_2) with $b \in BMO(\mathbb{R}^n)$ and $w \in A_p$ which ensures the boundedness of the operators $\mu^L_{j,\Omega,b}$, $j = 1, \dots, n$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $\Omega \in L_\infty(S^{n-1})$ and $1 < p < \infty$.

2 Preliminaries

We say that $\Omega \in \text{Lip}_\alpha(S^{n-1})$, $0 < \alpha \le 1$ if there exists a constant $C > 0$ such that $|\Omega(x') - \Omega(y')| \le C|x'-y'|^\alpha$ for all $x', y' \in S^{n-1}$.

The operator μ_Ω was first defined by Stein [40]. And Stein proved that if is continuous and satisfies a $\text{Lip}_\alpha(S^{n-1})(0 < \alpha \le 1)$ condition, then μ_Ω is an operator of type (p,p) $(1 < p \le 2)$ and of weak type $(1,1)$. In [4], Benedek, Calderón and Panzone proved that if $\Omega \in C^1(S^{n-1})$, then μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The L_p boundedness of μ_Ω has been studied extensively. See [4,31,40,41], among others. A survey of past studies can be found in [12]. Ding, Fan and Pan [13] proved the weighted $L_p(\mathbb{R}^n)$ boundedness with A_p weighs for a class of rough Marcinkiewicz integrals. Recently, Ding, Fan and Pan [14] improved the results mentioned above and showed that if $\Omega \in H^1(S^{n-1})$, the Hardy space on the unit sphere, then μ_Ω is still a bounded operator on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. In [42], Xu, Chen and Ying proved the same result as [14] using a different method.

Theorem 2.1 *([15]) Suppose that Ω be satisfies the conditions* (1.1)*,* (1.2) *and $\Omega \in L_\infty(S^{n-1})$. Let also $b \in BMO(\mathbb{R}^n)$. Then for every $1 < p < \infty$ and $w \in A_p$, there is a constant $C > 0$ independent of f such that*

$$\left\|\mu_{\Omega,b}f\right\|_{L_{p,w}} \le C\|f\|_{L_{p,w}}.$$

Note that a nonnegative locally L_q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)dy\right)^{1/q} \le C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)dy\right) \qquad (2.1)$$

holds for every ball $x \in \mathbb{R}^n$ and $r > 0$, where $B(x,r)$ denotes the open ball centered at x with radius r; see [38]. It is worth pointing out that the B_q class is that, if $V \in B_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only n and the constant C in (2.1), such that $V \in B_{q+\varepsilon}$. Throughout this paper, we always assume that $0 \ne V \in B_n$.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem in the case $w = 1$ was proved in [5].

Theorem 2.2 *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w^* g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \qquad (2.2)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty. \qquad (2.3)$$

Moreover, the value $C = B$ is the best constant for (2.2).

Remark 2.1 In (2.2) and (2.3) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

3 Generalized weighted Morrey spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [36] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [21,34].

We recall that a weight function w is in the Muckenhoupt's class A_p [37], $1 < p < \infty$, if

$$\begin{aligned}[w]_{A_p} :&= \sup_B [w]_{A_p(B)} \\ &= \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1} \end{aligned} \qquad (3.1)$$

where the sup is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls B by Hölder's inequality

$$[w]_{A_p(B)}^{1/p} = |B|^{-1} \|w\|_{L_1(B)}^{1/p} \|w^{-1/p}\|_{L_{p'}(B)} \ge 1. \qquad (3.2)$$

For $p = 1$, the class A_1 is defined by the condition $Mw(x) \leq Cw(x)$ with $[w]_{A_1} = \sup\limits_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$ $A_\infty = \bigcup_{1\leq p<\infty} A_p$ and $[w]_{A_\infty} = \inf\limits_{1\leq p<\infty} [w]_{A_p}$.

We define the generalized weighed Morrey spaces as follows.

Definition 3.1 *[23] Let $1 \leq p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL^{\rm loc}_{p,w}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\, |f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

Remark 3.1 (1) If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(v,w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(5) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

Suppose that T represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|Tf(x)| \leq c_0 \int\limits_{\mathbb{R}^n} |x-y|^{-n}\, |f(y)|dy, \tag{3.3}$$

where c_0 is independent of f and x.

For a function b, suppose that the commutator operator T_b represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_b f(x)| \leq c_0 \int_{\mathbb{R}^n} |b(x)-b(y)|\, |x-y|^{-n} |f(y)|dy, \tag{3.4}$$

where c_0 is independent of f and x.

We point out that the condition (3.3) was first introduced by Soria and Weiss in [39]. The condition (3.3) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operators, Carleson's maximal operator, Hardy–Littlewood maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, the Bochner–Riesz means and so on (see [35], [39] for details).

The following statement, was proved in [32], see also [23,26].

Theorem 3.1 *Let $1 \leq p < \infty$, $w \in A_p$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leq C\, \varphi_2(x,r), \tag{3.5}$$

where C does not depend on x and r. Let T be a sublinear operator satisfying condition (3.3) *bounded on $L_{p,w}(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_{1,w}(\mathbb{R}^n)$ to $WL_{1,w}(\mathbb{R}^n)$. Then the operator T is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.*

The following statement, was proved in [26], see also [23].

Theorem 3.2 *Let $1 < p < \infty$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leq C\, \varphi_2(x,r), \tag{3.6}$$

where C does not depend on x and r. Let T_b be a sublinear commutator operator satisfying condition (3.4) *bounded on $L_{p,w}(\mathbb{R}^n)$. Then the operator T_b is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.*

Note that, in the case $w = 1$ Theorem 3.1 was proved in [24] and for the Hardy–Littlewood maximal operator M and Calderón-Zygmund operators K in [1].

Theorem 3.3 *[27] Suppose that Ω be satisfies the conditions* (1.1), (1.2) *and $\Omega \in L_\infty(S^{n-1})$. Let also, $1 \leq p < \infty$, $w \in A_p$ and the pair (φ_1, φ_2) satisfy the condition* (3.5)*. Then the operator μ_Ω is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.*

4 Commutators of Marcinkiewicz operator with rough kernels $\mu_{\Omega,b}$ in the spaces $M_{p,\varphi}(w)$

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [6,7] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [10] states that the commutator operator $[b, K]f = K(bf) - b\,Kf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [8]-[9], [11], [17], [19]).

Remark 4.1

(1) The John-Nirenberg inequality : There are constants $C_1, C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B \,:\, |b(x) - b_B| > \beta\}| \le C_1 |B| e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy \right)^{\frac{1}{p}} \tag{4.1}$$

for $1 < p < \infty$.

(3) Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left| b_{B(x,r)} - b_{B(x,t)} \right| \le C\|b\|_* \ln \frac{t}{r} \ \text{ for } \ 0 < 2r < t, \tag{4.2}$$

where C is independent of b, x, r and t.

Lemma 4.1 *Suppose that Ω be satisfies the conditions* (1.1), (1.2), $1 < p < \infty$, *and* $\Omega \in L_\infty(S^{n-1})$. *Let also* $b \in BMO(\mathbb{R}^n)$ *and* $w \in A_p$ *the inequality*

$$\|\mu_{\Omega,b} f\|_{L_{p,w}(B(x_0,r))} \lesssim \|b\|_* w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1{+}\ln \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$ *and for all* $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

Proof. Let $p \in (1,\infty)$ and $b \in BMO(\mathbb{R}^n)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \ \ f_1(y) = f(y)\chi_{2B}(y), \ \ f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \ \ r > 0 \tag{4.3}$$

and have

$$\|\mu_{\Omega,b} f\|_{L_{p,w}(B)} \le \|\mu_{\Omega,b} f_1\|_{L_{p,w}(B)} + \|\mu_{\Omega,b} f_2\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_p(w)$, $\mu_{\Omega,b} f_1 \in L_p(w)$ and from the boundedness of $\mu_{\Omega,b}$ in $L_p(w)$ for $w \in A_p$ and $1 < p < \infty$ (see Theorem 2.1) it follows that

$$\begin{aligned}\|\mu_{\Omega,b} f_1\|_{L_{p,w}(B)} &\le \|\mu_{\Omega,b} f_1\|_{L_{p,w}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L_\infty(S^{n-1})} \, [w]_{A_p}^{\frac{1}{p}} \, \|b\|_* \, \|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &= \|\Omega\|_{L_\infty(S^{n-1})} \, [w]_{A_p}^{\frac{1}{p}} \, \|b\|_* \, \|f\|_{L_{p,w}(2B)}.\end{aligned}$$

For $x \in B$ we have

$$\mu_{\Omega,b}(f_2(x)) \lesssim \int_{\complement_{(2B)}} |b(y) - b(x)| |\Omega(x-y)| \frac{|f(y)|}{|x_0 - y|^n} dy.$$

Then

$$\begin{aligned}
\|\mu_{\Omega,b}f_2\|_{L_{p,w}(B)} \lesssim & \left(\int_B \left(\int_{\complement_{(2B)}} |b(y)-b(x)||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy\right)^p w(x)dx\right)^{\frac{1}{p}} \\
\lesssim & \left(\int_B \left(\int_{\complement_{(2B)}} |b(y)-b_{B,w}||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy\right)^p w(x)dx\right)^{\frac{1}{p}} \\
& + \left(\int_B \left(\int_{\complement_{(2B)}} |b(x)-b_{B,w}||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy\right)^p w(x)dx\right)^{\frac{1}{p}} \\
= & I_1+I_2.
\end{aligned}$$

Let us estimate I_1.

$$\begin{aligned}
I_1 &= w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} |b(y)-b_{B,w}||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy \\
&\approx w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} |b(y)-b_{B,w}||\Omega(x-y)||f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}}\,dy \\
&\approx w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \int_{2r\le|x_0-y|\le t} |b(y)-b_{B,w}||\Omega(x-y)||f(y)|\,dy\frac{dt}{t^{n+1}} \\
&\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y)-b_{B,w}||\Omega(x-y)||f(y)|dy\frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality and by (4.2), we get

$$\begin{aligned}
I_1 &\lesssim \|b\|_*\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \|\Omega(x-\cdot)\|_{L_q(B(x_0,t))}\, \|f\|_{L_{q'(B(x_0,t))}}\, \frac{dt}{t^{n+1}} \\
&\lesssim \|\Omega\|_{L_\infty(S^{n-1})} \|b\|_*\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))}\, \|w^{-1/p}\|_{L_{p'}B(x_0,t)}\, \frac{dt}{t^{n+1}} \\
&\lesssim \|\Omega\|_{L_\infty(S^{n-1})} [w]_{A_p}^{\frac{1}{p}} \|b\|_*\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \|f\|_{L_{p,w}(B(x,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\, \frac{dt}{t}.
\end{aligned}$$

In order to estimate I_2 note that

$$I_2 = \left(\int_B |b(x)-b_{B,w}|^p w(x)dx\right)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}\,dy.$$

By Fubini's theorem we have

$$\begin{aligned}
\int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}dy &\approx \int_{\complement_{(2B)}} |\Omega(x-y)|\,|f(y)| \int_{|x-y|}^{\infty} \frac{dt}{t^{n+1}}dy \\
&= \int_{2r}^{\infty} \int_{2r\le|x_0-y|<t} |\Omega(x-y)|\,|f(y)|dy\frac{dt}{t^{n+1}} \\
&\lesssim \|\Omega\|_{L_\infty(S^{n-1})} \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|dy\frac{dt}{t^{n+1}}\;.
\end{aligned}$$

By applying Hölder's inequality for $w \in A_p$, we get

$$\int_{\complement_{(2B)}} \frac{|\Omega(x-y)|\,|f(y)|}{|x_0-y|^n} dy \lesssim \int_{2r}^{\infty} \|\Omega(x-\cdot)\|_{L_\infty(B(x_0,t))}\, \|f\|_{L_1(B(x_0,t))}\, \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_\infty(S^{n-1})} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, \|w^{-1/p}\|_{L_{p'}(B(x_0,t))}\, \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_\infty(S^{n-1})}\, [w]_{A_p}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\, |B(x_0,t)|\, \frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_\infty(S^{n-1})}\, [w]_{A_p}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}}\, \frac{dt}{t}. \tag{4.4}$$

By (4.2) and (4.4), we get

$$I_2 \lesssim \|b\|_*\, w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}\, dy$$
$$\lesssim \|\Omega\|_{L_\infty(S^{n-1})}[w]_{A_p}^{\frac{1}{p}}\|b\|_*\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\, \frac{dt}{t}.$$

Summing up I_1 and I_2, for all $p \in (1,\infty)$ we get

$$\|\mu_{\Omega,b} f_2\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_\infty(S^{n-1})}[w]_{A_p}^{\frac{1}{p}}\|b\|_*\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}.$$

Thus

$$\|\mu_{\Omega,b} f\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_\infty(S^{n-1})}\, [w]_{A_p}^{\frac{1}{p}}\, \|b\|_* \Big(\|f\|_{L_{p,w}(2B)}$$
$$+\; w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}}\frac{dt}{t}\Big). \tag{4.5}$$

On the other hand,

$$\|f\|_{L_{p,w}(2B)} \approx |B|\, \|f\|_{L_{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}}$$
$$\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, \frac{dt}{t^{n+1}}$$
$$\lesssim w(B)^{\frac{1}{p}}\, \|w^{-1/p}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, \frac{dt}{t^{n+1}}$$
$$\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, \|w^{-1/p}\|_{L_{p'}(B(x_0,t))}\, \frac{dt}{t^{n+1}}$$
$$\lesssim [w]_{A_p}^{\frac{1}{p}}\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\, \frac{dt}{t}. \tag{4.6}$$

Then, by (4.5) and (4.6) we get

$$\|\mu_{\Omega,b} f\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_\infty(S^{n-1})}\, [w]_{A_p}^{\frac{1}{p}}\, \|b\|_*\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\, \frac{dt}{t}$$
$$\lesssim \|b\|_*\, w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\, \frac{dt}{t}.$$

Theorem 4.1 *Suppose that Ω be satisfies the conditions* (1.1), (1.2), $1 < p < \infty$, *and* $\Omega \in L_\infty(S^{n-1})$. *Let also* $b \in BMO(\mathbb{R}^n)$, $w \in A_p$ *the pair* (φ_1, φ_2) *satisfy the condition* (3.6). *Then the operator* $\mu_{\Omega,b}$ *is bounded from* $M_{p,\varphi_1}(w)$ *to* $M_{p,\varphi_2}(w)$. *Moreover*

$$\|\mu_{\Omega,b} f\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. By Lemma 4.1 and Theorem 2.2 with $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,t))^{-\frac{1}{p}}$, $\nu_2(r) = \varphi_2(x,r)^{-1}$ and $w(r) = w(B(x,t))^{-\frac{1}{p}}$ we have

$$\begin{aligned}
&\|\mu_{\Omega,b} f\|_{M_{p,\varphi_2}(w)} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|\mu_{\Omega,b} f\|_{L_{p,w}(B(x,r))} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \Big(1 + \ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x,t))}\, w(B(x,t))^{-\frac{1}{p}}\, \frac{dt}{t} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w} B(x,r)} \\
&= \|b\|_* \|f\|_{M_{p,\varphi_1}(w)}.
\end{aligned}$$

Remark 4.2 Note that, in the case $w = 1$ the boundedness of the parametric Marcinkiewicz operator on generalized Morrey spaces were study in [25], see also [2].

5 Commutators of Marcinkiewicz operator $\mu^L_{j,\Omega,b}$ in the spaces $M_{p,\varphi}(w)$

In this section, we prove the boundedness of the Marcinkiewicz operator μ^L_j on $M_{p,\varphi}(w)$ spaces. For $x \in \mathbb{R}^n$, the function $m_V(x)$ is defined by

$$\rho(x) = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V\,(y) dy\ \le 1 \right\}.$$

Lemma 5.1 *[38] Let* $V \in B_q$ *with* $q \ge n/2$. *Then there exists* $l_0 > 0$ *such that*

$$\frac{l}{C}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-l_0} \le \frac{\rho(y)}{\rho(x)} \le C\left(1 + \frac{|x-y|}{\rho(x)}\right)^{l_0/(l_0+1)}.$$

In particular, $\rho(x) \approx \rho(y)$, *if* $|x-y| < C\rho(x)$.

Lemma 5.2 *[38] Let* $V \in B_q$ *with* $q \ge n/2$. *For any* $l > 0$, *there exists* $C_l > 0$ *such that*

$$\left|K^L_j(x,y)\right| \le \frac{C_l}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^l} \frac{1}{|x-y|^{n-1}},$$

and

$$\left|K^L_j(x,y) - K_j(x-y)\right| \le\ C\frac{\rho(x)^{-1}}{|x-y|^{n-2}}.$$

The following theorem in the case $w = 1$ was proved in [2].

Theorem 5.1 *Suppose that* $\Omega \in L_\infty(S^{n-1})$ *satisfies the conditions* (1.1), (1.2), $1 < p < \infty$ *and* $V \in B_n$. *Let also* $b \in BMO(\mathbb{R}^n)$ *and for every* $w \in A_p$ *there is a constant* C *independent of* f *such that*

$$\|\mu^L_{j,\Omega,b} f\|_{L_{p,w}} \le C\|f\|_{L_{p,w}}.$$

Proof. In the proof we used the idea in [20]. It suffices to show that

$$\mu^L_{j,\Omega,b}f(x) \le \mu_{j,\Omega,b}f(x) + CM_{\Omega,b}f(x), a.e.\ x \in \mathbb{R}^n, \tag{5.1}$$

where $M_{\Omega,b}$ denotes the commutator of the commutator of Hardy-Littlewood operator with rough kernel.

Fixing $x \in \mathbb{R}^n$ and let $r = \rho(x)$.

$$\begin{aligned}
\mu^L_{j,b}f(x) \le & \left(\int_0^r \left|\int_{|x-y|\le t} |\Omega(x-y)|K^L_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& + \left(\int_r^\infty \left|\int_{|x-y|\le r} |\Omega(x-y)|K^L_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& + \left(\int_r^\infty \left|\int_{r<|x-y|\le t} |\Omega(x-y)|K^L_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& \le \left(\int_0^r \left|\int_{|x-y|\le t} |\Omega(x-y)|[K^L_j(x,y)-K_j(x,y)][b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& + \left(\int_0^r \left|\int_{|x-y|\le t} |\Omega(x-y)|K_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& + \left(\int_r^\infty \left|\int_{|x-y|\le r} |\Omega(x-y)|K^L_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& + \left(\int_r^\infty \left|\int_{r<|x-y|\le t} |\Omega(x-y)|K^L_j(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
& := E_1 + E_2 + E_3 + E_4.
\end{aligned}$$

For E_1, by Lemma 5.2, we have

$$\begin{aligned}
E_1 &\le C\left(\int_0^r \left|\frac{1}{r}\int_{|x-y|\le t} |\Omega(x-y)|[b(x)-b(y)]\frac{|f(y)|}{|x-y|^{n-2}}dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\le CM_{\Omega,b}f(x).
\end{aligned}$$

Obviously,

$$E_2 \lesssim \mu_{j,\Omega,b}f(x).$$

For E_3, using Lemma 5.2 again, we get

$$
\begin{aligned}
E_3 &\leq \left(\int_r^\infty \left|\frac{1}{r}\int_{|x-y|\leq r} |\Omega(x-y)||[b(x)-b(y)]\frac{|f(y)|}{|x-y|^{n-2}}dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\leq CM_{\Omega,b}f(x)
\end{aligned}
$$

It remains to estimate E_4. From Lemma 5.2, we obtain

$$
\begin{aligned}
E_4 &\leq C\left(\int_r^\infty \left|r\int_{r<|x-y|\leq t} |\Omega(x-y)||[b(x)-b(y)]\frac{|f(y)|}{|x-y|^{n}}dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\leq C_r\left(\int_r^\infty \left|\sum_{k=0}^{[\log_2 t/r]+1} (2^k r)^n \int_{|x-y|\leq 2^k r} |\Omega(x-y)||[b(x)-b(y)]|f(y)|dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\leq C_r\left(\int_r^\infty |([\log_2 t/r]+1)M_{\Omega,b}f(x)|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&\leq C_r\left(\int_r^\infty \frac{t}{r}M_{\Omega,b}f(x)^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \leq CM_{\Omega,b}f(x).
\end{aligned}
$$

Thus, Theorem 5.1 is proved.

In the proof of the Theorem 5.1 the validity of the following inequality is proved (5.1). Note that the operators $M_{\Omega,b}$ and $\mu_{j,\Omega,b}$ which are sublinear operators satisfies the condition (3.3) and bounded on $L_{p,w}(\mathbb{R}^n)$ for $p>1$. Statements of the Lemma 4.1 for the operators $M_{\Omega,b}$ and $\mu_{j,\Omega,b}$ is provided. Then we get that the statements of the Lemma 4.1 also true for the operators $\mu^L_{j,\Omega,b}$, $j=1,\ldots,n$. From Lemma 4.1 and Theorem 5.1 the following corollary is obtained.

Corollary 5.1 *Suppose that $\Omega \in L_\infty(S^{n-1})$ satisfies the conditions* (1.1), (1.2)*,* $1<p<\infty$ *and $V \in B_n$. Let also $b \in BMO(\mathbb{R}^n)$ and for every $w \in A_p$ the inequality*

$$
\begin{aligned}
&\|\mu^L_{j,\Omega,b}f\|_{L_{p,w}(B(x_0,r))} \\
&\lesssim ||b||_* w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^\infty \left(1+\ln\frac{t}{r}\right)\|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}
\end{aligned}
$$

holds for any ball $B(x_0,r)$, and for all $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

Corollary 5.2 *Suppose that $\Omega \in L_\infty(S^{n-1})$ satisfies the conditions* (1.1), (1.2) *and $V \in B_n$. Let also $b \in BMO(\mathbb{R}^n)$, $w \in A_p$ the pair (φ_1,φ_2) satisfies the condition* (3.5)*. Then the operator $\mu^L_{j,\Omega,b}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p>1$. Moreover*

$$
\|\mu^L_{j,\Omega,b}f\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.
$$

Acknowledgements. The author would like to express their gratitude to the referees for his very valuable comments and suggestions.

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