Approximation by linear means of Fourier series in Orlicz spaces

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Received: 08.01.2018 / Revised: 13.05.2018 / Accepted: 04.07.2018

Abstract. In this work the approximation of functions by linear means of Fourier series in reflexive Orlicz spaces was studied. This result was applied to the approximation of the functions by linear means of Faber series in Smirnov-Orlicz classes defined on simply connected domain of the complex plane

Keywords. Young function, Orlicz space, Boyd indices, modulus of smoothness, best approximation.


1 Introduction and main results

Let $M(u)$ be a continuous increasing convex function on $[0, \infty)$ such that $M(u)/u \to 0$ if $u \to 0$, and $M(u)/u \to \infty$ if $u \to \infty$. We denote by $N$ the complementary of $M$ in Young’s sense, i.e. $N(u) = \max \{uv - M(v) : v \geq 0\}$ if $u \geq 0$. We will say that $M$ satisfies the $\Delta_2$-condition if $M(2u) \leq cM(u)$ for any $u \geq u_0$ with some constant $c$ independent of $u$.

Let $T$ denote the interval $[-\pi, \pi]$, $\mathbb{C}$ the complex plane, and $L_p(T), 1 \leq p \leq \infty$, the Lebesgue space of measurable complex-valued functions on $T$.

For a given Young function $M$, let $\tilde{L}_M(T)$ denote the set of all Lebesgue measurable functions $f : T \to \mathbb{C}$ for which

$$\int_T M(\|f(x)\|) \, dx < \infty.$$ 

Let $N$ be the complementary Young function of $M$. It is well-known [23, p. 69], [30, pp. 52-68] that the linear span of $\tilde{L}_M(T)$ equipped with the Orlicz norm

$$\|f\|_{L_M(T)} := \sup \left\{ \int_T |f(x)g(x)| \, dx : g \in \tilde{L}_N(T), \int_T N(|g(x)|) \, dx \leq 1 \right\},$$

The author would like to thank referee for all precious advices and very helpful remarks. This work was supported by Scientific Research Projects Coordination Unit of Muş Alparslan University. The title of the project "Approximation of the function in the Orlicz spaces" and number: BAP-18-EMF-4901-07.

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or with the Luxembourg norm

$$\|f\|^{*}_{L_{M}(T)} := \inf \left\{ k > 0 : \int_{T} M \left( \frac{|f(x)|}{k} \right) dx \leq 1 \right\}$$

becomes a Banach space. The space is denoted by $L_{M}(T)$ and is called an Orlicz space [23, p. 26]. The Luxembourg norm is equivalent to the Orlicz norm as

$$\|f\|^{*}_{L_{M}(T)} \leq \|f\|_{L_{M}(T)} \leq 2 \|f\|_{L_{M}(T)}, \quad f \in L_{M}(T)$$

holds true [23, p. 80].

If we choose $M(u) = u^{p}/p$ $(1 < p < \infty)$ then the complementary function is $N(u) = u^{q}/q$ with $1/p + 1/q = 1$ and we have the relation

$$p^{-1/p} \|u\|_{L_{p}(T)} = \|u\|^{*}_{L_{M}(T)} \leq \|u\|_{L_{M}(T)} \leq q^{1/q} \|u\|_{L_{q}(T)},$$

where $\|u\|_{L_{N}(T)} = \left( \int_{T} |u(x)|^{p} dx \right)^{1/p}$ denotes the usual norm of the $L_{p}(T)$-space.

If $N$ is complementary to $M$ in Young’s sense and $f \in L_{M}(T)$, $g \in L_{N}(T)$ then the so-called strong H"{o}lder inequalities [23, p. 80]

$$\int_{T} |f(x)g(x)| dx \leq \|f\|_{L_{M}(T)} \|g\|^{*}_{L_{N}(T)},$$

$$\int_{T} |f(x)g(x)| dx \leq \|f\|^{*}_{L_{M}(T)} \|g\|_{L_{N}(T)}.$$ 

are satisfied.

The Orlicz space $L_{M}(T)$ is reflexive if and only if the $N-$function $M$ and its complementary function $N$ both satisfy the $\Delta_{2}-$condition [30, p. 113].

Let $M^{-1} : [0, \infty) \to [0, \infty)$ be the inverse function of the $N-$function $M$. The lower and upper indices $\alpha_{M}, \beta_{M}$ [5, p. 350]

$$\alpha_{M} := \lim_{t \to +\infty} -\frac{\log h(t)}{\log t}, \quad \beta_{M} := \lim_{t \to 0^{+}} -\frac{\log h(t)}{\log t}$$

of the function

$$h : (0, \infty) \to (0, \infty], \quad h(t) := \lim_{y \to \infty} \sup_{t > 0} \frac{M^{-1}(y)}{M^{-1}(ty)}$$

first considered by Matuszewska and Orlicz [27], are called the Boyd indices of the Orlicz spaces $L_{M}(T)$.

It is known that the indices $\alpha_{M}$ and $\beta_{M}$ satisfy $0 \leq \alpha_{M} \leq \beta_{M} \leq 1$, $\alpha_{N} + \beta_{M} = 1$, $\alpha_{M} + \beta_{N} = 1$ and the space $L_{M}(T)$ is reflexive if and only if $0 < \alpha_{M} \leq \beta_{M} < 1$. The detailed information about the Boyd indices can be found in [4], [5], [6], [20] and [26].

Let $L_{M}(T)$ be an Orlicz space. Suppose that $x, h$ are real, and let us take into account the following series

$$\Delta^{\alpha}_{h} f(x) = \sum_{k=0}^{\infty} (-1)^{k} \binom{\alpha}{k} f(x + (\alpha - k)h), \quad \alpha > 0, \quad f \in L_{M}(T).$$
Then, by [24, Theorem 11, p.135] the last series converges absolutely almost everywhere (a.e.) on \( T \). Hence the operator \( \Delta_h^\alpha \) by [24, Theorem 10, p.134] is bounded in the space \( L_M(T) \). Namely,

\[
\Delta_h^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + (\alpha - k)h) = \sum_{k=0}^{\alpha} (-1)^{\alpha-k} \binom{\alpha}{k} f(x + kh).
\]

for \( \alpha \in \mathbb{Z}^+ \). Let \( L_M(T, \omega) \) be a weighted Orlicz space.

The function

\[
\omega_\alpha(f, \delta)_M := \sup_{|h| \leq \delta} \| \Delta_h^\alpha f \|_{L_M(T)}, \alpha \in \mathbb{Z}^+
\]

is called \( \alpha \)-th modulus of smoothness of \( f \in L_M(T) \).

It can easily be shown that \( \omega_\alpha(f, \delta)_M \) is a continuous, nonnegative and nondecreasing function satisfying the conditions

\[
\lim_{\delta \to 0} \omega_\alpha(f, \delta)_M = 0, \quad \omega_\alpha(f + g, \delta)_M \leq \omega_\alpha(f, \delta)_M + \omega(g, \delta)_M
\]

for \( f, g \in L_M(T) \).

Let

\[
a_0^2 + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx)
\]

be the Fourier series of the function \( f \in L_1(T) \), where \( \alpha_k(f) \) are \( b_k(f) \) the Fourier coefficients of the function \( f \).

Let (1.1) be the Fourier series of the function \( f \). For \( f \in L_M(T) \) we define the summability method by the triangular matrix \( \Lambda = \{ \lambda_{ij} \}_{i,j=0}^{\infty} \) by the linear means

\[
U_n(x, f) = \lambda_0 a_0^2 + \sum_{i=1}^{n} \lambda_i (a_i(f) \cos ix + b_i(f) \sin ix)
\]

If the Fourier series of \( f \) is given by (1.1), then Zygmund-Riesz means of order \( k \) is defined as

\[
Z_n^k(x, f) = \frac{a_0}{2} + \sum_{i=1}^{n} \left( 1 - \frac{i^k}{(n+1)^k} \right) (a_i(f) \cos ix + b_i(f) \sin ix).
\]

We denote by \( E_n(f)_M \) the best approximation of \( f \in L_M(T) \) by trigonometric polynomials of degree not exceeding \( n \), i.e.,

\[
E_n(f)_M = \inf \left\{ \| f - T_n \|_{L_M(T)} : T_n \in \Pi_n \right\},
\]

where \( \Pi_n \) denotes the class of trigonometric polynomials of degree at most \( n \).

Let \( T_n \in \Pi_n \)

\[
T_n = \frac{c_0}{2} + \sum_{i=1}^{n} (c_i \cos ix + d_i \sin ix).
\]
The conjugate polynomial $\tilde{T}_n$ is defined by

$$\tilde{T}_n = \sum_{i=1}^{n} (c_i \sin ix - d_i \cos ix).$$

We will say that the method of summability by the matrix $\Lambda$ satisfies condition $b_{k,M}$ (respectively $b_{s,k,M}$) if for $T_n \in \Pi_n$ the inequality

$$\|T_n - U_n(T_n)\|_{L_M(T)} \leq c(n + 1)^{-k} \left\|T_n^{(k)}\right\|_{L_M(T)}$$

holds and the norms

$$\|\Lambda\|_1 := \int_0^{\frac{2\pi}{2}} \left| \frac{\lambda_{0n}}{2} + \sum_{i=1}^{n} \lambda_{in} \cos it \right| dt$$

are bounded.

We use the constants $c, c_1, c_2, \ldots$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest.

The problems of approximation theory in the weighted and non-weighted Orlicz spaces have been investigated by several authors (see, for example, [1]-[3], [11], [12], [15]-[19], [21] and [29]).

In the present paper necessary and sufficient condition about the relationship between the approximation of functions by linear means of Fourier series and by Zygmund-Riesz means of order $k$ was investigated in reflexive Orlicz spaces. Also, we investigate the approximation of functions by linear means of Fourier series in terms of the modulus of smoothness of these functions in Orlicz spaces. This result was applied to the approximation of the functions by linear means of Faber series in Smirnov-Orlicz classes defined on simply connected domain of the complex plane. The similar problems in different spaces were investigated in [7],[8],[13],[14],[22],[31], and [33]-[35].

Main results in the present work are the following theorems:

**Theorem 1.1** Let $L_M(T)$ be a reflexive Orlicz space. In order that for $f \in L_M(T)$

$$\|f - U_n(\cdot, f)\|_{L_M(T)} \leq c_1 \left\|f - Z_n^{k}(\cdot, f)\right\|_{L_M(T)}$$

(1.2)

it is sufficient and necessary that for $f \in L_M(T)$

$$\|T_n - U_n(\cdot, T_n)\|_{L_M(T)} \leq c_2 \left\|T_n - Z_n^{k}(\cdot, T_n)\right\|_{L_M(T)}.$$ 

(1.3)

**Theorem 1.2** Let $L_M(T)$ be a reflexive Orlicz space. In order that for every $f \in L_M(T)$

$$\|f - U_n(\cdot, f)\|_{L_M(T)} \leq c_1 \left\|f - Z_n^{k}(\cdot, f)\right\|_{L_M(T)}$$

(1.4)

it is sufficient and necessary that

(i) $\|U_n(\cdot, f)\|_{L_M(T)} = O(1)$;

(ii) if $k$ is even, $U_n(\cdot, f)$ satisfies the condition $(b_{k,M})$; if $k$ is odd $U_n(\cdot, f)$ satisfies the condition $(b_{s,k,M})$. 


Theorem 1.3 Let $L_M(\mathbb{T})$ be a reflexive Orlicz space. If the summability method with the matrix $\Lambda$ satisfies the condition $(b_{k,M})$ or $\left(b_{k,M}^\ast\right)$, then for $f \in L_M(\mathbb{T})$ the inequality

$$\|f - U_n(\cdot, f)\|_{L_M(\mathbb{T})} \leq c_3 \omega_\alpha\left(f, \frac{1}{n}\right)_M$$

holds with a constant $c_3 > 0$ independent of $n$.

Theorem 1.4 Let $L_M(\mathbb{T})$ be a reflexive Orlicz space. If the summability method with the matrix $\Lambda$ satisfies the condition $(b_{k,M})$ or $\left(b_{k,M}^\ast\right)$, then for $f \in L_M(\mathbb{T})$ the estimate

$$\omega_\alpha\left(U_n(\cdot, f), \delta\right)_M \leq c_4 \omega_\alpha\left(f, \delta\right)_M,$$

holds with a constant $c_4 > 0$ not depend on $n$, $f$ and $\delta$.

Corollary 1.1 This results obtained in Theorems 1.1 and 1.3 are valid for the Zygmund-Riesz means of order $k$.

Note that Theorem 1.3 in weighted Orlicz spaces for different modulus of continuity and Zygmund-Riesz means of order 2 was proved in [12].

Let $\mathbb{G}$ be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve $\Gamma$, and let $\Gamma^- := ext\Gamma$. Further let

$$\mathbb{T} := \{ w \in \mathbb{C} : |w| = 1 \}, \mathbb{D} := int\mathbb{T} \text{ and } \mathbb{D}^- := ext\mathbb{T}.$$

Let $w = \phi(z)$ be the conformal mapping of $\Gamma^-$ onto $\mathbb{D}^-$ normalized by

$$\phi(\infty) = \infty, \lim_{z \to \infty} \frac{\phi(z)}{z} > 0,$$

and let $\psi$ denote the inverse of $\phi$.

Let $w = \phi_1(z)$ denote a function that maps the domain $\mathbb{G}$ conformally onto the disk $|w| < 1$. The inverse mapping of $\phi_1$ will be denoted by $\psi_1$. Let $\Gamma_r$ denote circular images in the domain $\mathbb{G}$, that is, curves in $\mathbb{G}$ corresponding to circle $|\phi_1(z)| = r$ under the mapping $z = \psi_1(w)$.

Let us denote by $E_p$, where $p > 0$, the class of all functions $f(z) \neq 0$ which are analytic in $\mathbb{G}$ and have the property that the integral

$$\int_{\Gamma_r} |f(z)|^p |dz|$$

is bounded for $0 < r < 1$. We shall call the $E_p$-class the Smirnov class. If the function $f(z)$ belongs to $E_p$, then $f(z)$ has definite limiting values $f(z')$ almost every where on $\Gamma$, over all nontangential paths; $|f(z')|$ is summable on $\Gamma$; and

$$\lim_{r \to 0} \int_{\Gamma_r} |f(z)|^p |dz| = \int_{\Gamma} |f(z')|^p |dz|.$$

It is known that $\varphi' = E_1(\Gamma^-)$ and $\psi' \in E_1(\mathbb{D}^-)$. Note that the general information about Smirnov classes can be found in the books [9, pp. 168-185] and [10, pp. 483-453].

Let $L_M(\mathbb{T})$ is a Orlicz space defined on $\Gamma$. We define also the Smirnov-Orlicz class $E_M(\mathbb{G})$ as

$$E_M(\mathbb{G}) := \{ f \in E_1(\mathbb{G}) : f \in L_M(\Gamma) \}.$$
With every weight function \( \omega \) on \( \Gamma \), we associate another weight \( \omega_0 \) on \( \mathbb{T} \) defined by
\[
\omega_0 (t) := \omega (\psi (t)), \ t \in \mathbb{T}.
\]
For \( f \in L_M (\Gamma, \omega) \) we define the function
\[
f_0(t) := f (\psi (t)), \ t \in \mathbb{T}.
\]
Let \( h \) be a continuous function on \([0, 2\pi]\). Its modulus of continuity is defined by
\[
\omega (t, h) := \sup \{ |h (t_1) - h (t_2)| : t_1, t_2 \in [0, 2\pi], |t_1 - t_2| \leq t \}, \ t \geq 0.
\]
The curve \( \Gamma \) is called Dini-smooth if it has a parameterization
\[
\Gamma : \varphi_0 (s), \ 0 \leq s \leq 2\pi
\]
such that \( \varphi_0 (s) \) is Dini-continuous, i.e.
\[
\int_0^\pi \frac{\omega (t, \varphi_0')}t dt < \infty
\]
and \( \varphi_0 (s) \neq 0 \) [28, p. 48].

If \( \Gamma \) is a Dini-smooth curve, then there exist [36] the constants \( c_5 \) and \( c_6 \) such that
\[
0 \leq c_5 \leq |\psi' (t)| \leq c_6 < \infty, \ |t| > 1.
\]
(1.7)

Note that if \( \Gamma \) is a Dini-smooth curve, then by (1.6) we have \( f_0 \in L_M (\mathbb{T}) \) and \( f \in L_M (\Gamma) \).

Let \( \Gamma \) be a rectifiable Jordan curve and \( f \in L_1 (\Gamma) \). Then the function \( f^+ \) defined by
\[
f^+ (z) := \frac{1}{2\pi i} \int_{T} \frac{f(s)ds}{s-z}, \ z \in G
\]
is analytic in \( G \). Note that if \( 0 < \alpha_M \leq \beta_M < 1 \) and \( f \in L_M (\Gamma) \), then by [2] \( f^+ \in E_M (G) \).

Let \( \phi_k (z), \ k = 0, 1, 2, ... \) be the Faber polynomials for \( G \). The Faber polynomials \( \phi_k (z) \), associated with \( G \cup \Gamma \), are defined through the expansion
\[
\frac{\psi' (w)}{\psi (w) - z} = \sum_{k=0}^{\infty} \frac{\phi_k (z)}{w^{k+1}}, \ z \in G, \ w \in D^- (1.8)
\]
and the equalities
\[
\phi_k (z) = \frac{1}{2\pi i} \int_{T} \frac{t^k \psi' (t)}{\psi (t) - z} dt, \ z \in G, (1.9)
\]
\[
\phi_k (z) = \phi^k (z) + \frac{1}{2\pi i} \int_{T} \frac{\phi^k (s)}{s-z} ds, \ z \in G^- \]
hold [32, p. 33-48].

Let \( f \in E_M (G, \omega) \). Since \( f \in E_1 (G) \), we obtain
\[
f(z) := \frac{1}{2\pi i} \int_{T} \frac{f(s)ds}{s-z} = \frac{1}{2\pi i} \int_{T} \frac{f (\psi (t)) \psi' (t)}{\psi (t) - z} dt,
\]
f(z) \sim \sum_{i=0}^{\infty} a_i(f) \phi_i(z), \; z \in G,

(1.10)

where

\begin{align*}
a_i(f) := \frac{1}{2\pi i} \int_{T} \frac{f'(t)}{t^{i+1}} \, dt, \; i = 0, 1, 2, \ldots
\end{align*}

This series is called the Faber series expansion of \( f \), and the coefficients \( a_i(f) \) are said to be the Faber coefficients of \( f \).

Let (1.9) be the Faber series of the function \( f \in E_M(G) \). For the function \( f \) we define the summability method by the triangular matrix \( \Lambda = \{ \lambda_{ij} \}_{i,j=0}^{\infty} \) by the linear means

\begin{align*}
U_n(z, f) = \sum_{i=0}^{n} \lambda_{in} a_i(f) \phi_i(z),
\end{align*}

The \( n \)-the partial sums and Zygmund means of order \( k \) of the series (1.9) are defined, respectively, as

\begin{align*}
S_n(z, f) = \sum_{k=0}^{n} a_k(f) \phi_k(z),
\end{align*}

\begin{align*}
Z_n^k(z, f) = \sum_{i=0}^{n} \left( 1 - \frac{x^k}{(n+1)^k} \right) a_i(f) \phi_i(z).
\end{align*}

Let \( \Gamma \) be a Dini-smooth curve. Using the nontangential boundary values of \( f_0^+ \) on \( T \) we define the \( \alpha \)-th modulus of smoothness of \( f \in L_M(\Gamma) \) as

\begin{align*}
\omega_{\alpha, \Gamma}(f, \delta)_M := \omega_{\alpha, \Gamma}(f_0^+, \delta), \; \delta > 0,
\end{align*}

for \( \alpha = 1, 2, 3, \ldots \)

The following theorem holds.

**Theorem 1.5** Let \( \Gamma \) be a Dini-smooth curve, and let \( L_M(\Gamma) \) be a reflexive Orlicz space. If the summability method with the matrix \( \Lambda \) satisfies the condition \( (b_k)_M \) or \( (b_k^*)_M \), then for \( f \in E_M(G) \) the estimate

\begin{align*}
\| f - U_n(\cdot, f) \|_{L_M(\Gamma)} \leq c_7 \omega_{\alpha, \Gamma} \left( f_0^+, \frac{1}{n} \right)_M
\end{align*}

holds with a constant \( c_7 > 0 \), independent of \( n \).

Let \( \mathcal{P} \) be the set of all algebraic polynomials (with no restriction on the degree), and let \( \mathcal{P}(\mathbb{D}) \) be the set of traces of members of \( \mathcal{P} \) on \( \mathbb{D} \). We define the operator

\begin{align*}
T : \mathcal{P}(\mathbb{D}) &\rightarrow E_M(G, \omega)
\end{align*}

as

\begin{align*}
T(P)(z) := \frac{1}{2\pi i} \int_{T} \frac{P(w)\psi'(w)}{\psi(w) - z} \, dw, \; z \in G.
\end{align*}
Then from (1.8) we have

\[ T \left( \sum_{k=0}^{n} \beta_k w^k \right) = \sum_{k=0}^{n} \beta_k \phi_k(z). \]

The following result hold for the linear operator \( T \) [2].

**Theorem 1.6** Let \( \Gamma \) be a Dini- smooth curve and \( L_M(\Gamma) \) be a reflexive Orlicz space. Then linear operator \( T : P(D) \rightarrow E_M(G) \) is bounded.

**Theorem 1.7** If \( \Gamma \) is a Dini- smooth curve. Then the operator

\[ T : E_M(D) \rightarrow E_M(G) \]

is one-to-one and onto.

### 2 Proof of main results

**Proof of Theorem 1.1.** Necessity. It is clear that the inequality (1.3) follows from the inequality (1.2).

Sufficiency. Let \( f \in L_M(T) \) and let \( T_n \in \Pi_n \) \((n = 0, 1, 2, \ldots)\) be the polynomial of best approximation to \( f \). We obtain

\[
\| f - U_n(\cdot, f) \|_{L_M(T)} \\
\leq \| f - T_n \|_{L_M(T)} - \| T_n - U(\cdot, T_n) \|_{L_M(T)} + \| U_n(\cdot, f - T_n) \|_{L_M(T)} \\
\leq E_n(f) + c_2 \left( \| T_n - Z_n^k(\cdot, f) \|_{L_M(T)} + \| Z_n(\cdot, f - T_n) \|_{L_M(T)} \right) \\
\leq c_9 E_n(f) + c_2 E_n(f) + c_2 \left( \| f - Z_n^k(\cdot, f) \|_{L_M(T)} \right) \\
\leq c_1 E_n(f) + c_2 \left( \| f - Z_n^k(\cdot, f) \|_{L_M(T)} \right) \leq c_{12} \left( \| f - Z_n^k(\cdot, f) \|_{L_M(T)} \right)
\]

and Theorem 1.1 is proved.

**Proof of Theorem 1.2.** Necessity. According to [33] \( Z_n^k(f) \) is bounded in \( L_p(\mathbb{T}) \) and \( L_q(\mathbb{T}) \). Then by [16, Theorem 7] (see also [4, p.153]) \( \| Z_n^k(\cdot, f) \|_{L_M(T)} = O(1) \). Taking account of (1.4) we have \( \| U_n(\cdot, f) \|_{L_M(T)} = O(1) \). Let \( f \in L_M(T) \). Then the following inequality holds:

\[
\| f - Z_n^k(\cdot, f) \|_{L_M(T)} \\
\leq \| f - S_n(\cdot, f) \|_{L_M(T)} + (n + 1)^{-k} \| \sum_{p=1}^{n} \nu^k A_\nu(\cdot, f) \|_{L_M(T)} \\
= U_1 + U^{(k)} (2.1)
\]

It is well known from [33], [14] that

\[
U_1 = \| f - S_n(\cdot, f) \|_{L_M(T)} \leq c_17(M) E_n(f) \\
\leq \frac{c}{(n + 1)^k} \left\| f^{(k)} \right\|_{L_M(T)} \leq (2.2)
\]
We note that if $k$ is even
\[ \sum_{\nu=1}^{n} \nu^k A_\nu(x, f) = (-1)^{k/2} S_n^{(k)}(x, f), \]
if $k$ is odd
\[ \sum_{\nu=1}^{n} \nu^k A_\nu(x, f) = (-1)^{(k+3)/2} \tilde{S}_n^{(k)}(x, f), \]
where $g(x)$ is the function that is trigonometricaly conjugate to $g(x)$. Then
\[
U^{(k)}_2 = \begin{cases} 
(n+1)^{-k} \left\| S_n^{(k)}(\cdot, f) \right\|_{L_M(T)}, & k \text{ even} \\
(n+1)^{-k} \left\| \tilde{S}_n^{(k)}(\cdot, f) \right\|_{L_M(T)}, & k \text{ odd}.
\end{cases}
\]
(2.3)

Using (2.3) if $k$ is even, we have
\[
U^{(k)}_2 = (n+1)^{-k} \left\| S_n^{(k)}(\cdot, f) \right\|_{L_M(T)} \leq c_{20} (n+1)^{-k} \left\| f^{(k)} \right\|_{L_M(T)},
\]
(2.4)

if $k$ is odd, we find that
\[
U^{(k)}_2 = (n+1)^{-k} \left\| \tilde{S}_n^{(k)}(\cdot, f) \right\|_{L_M(T)} \leq c_{25} (n+1)^{-k} \left\| \tilde{f}^{(k)} \right\|_{L_M(T)}.
\]
(2.5)

Taking into account the relations (2.1), (2.2), (2.4) and (2.5), if $k$ is even for $f \in L_M(\mathbb{T})$ and $f^{(k)} \in L_M(\mathbb{T})$ we obtain the inequality
\[
\| f - Z_{n,k}(\cdot, f) \|_{L_M(\mathbb{T})} \leq \frac{c}{(n+1)^k} \left\| f^{(k)} \right\|_{L_M(\mathbb{T})},
\]
and if $k$ is odd for $f \in L_M(\mathbb{T})$ and $\tilde{f}^{(k)} \in L_M(\mathbb{T})$ we reach
\[
\| f - Z_{n,k}(\cdot, f) \|_{L_M(\mathbb{T})} \leq \frac{c}{(n+1)^k} \left\| \tilde{f}^{(k)} \right\|_{L_M(\mathbb{T})}.
\]

**Sufficiency.** We note that for $T_n \in H_n$ we get
\[
\| T_n - Z_{n,k}(\cdot, T_n) \|_{L_M(\mathbb{T})} = \frac{\left\| T_n^{(k)} \right\|_{L_M(\mathbb{T})}}{(n+1)^k}, \text{ if } k \text{ is even}, \quad (2.6)
\]
\[
\| T_n - Z_{n,k}(\cdot, T_n) \|_{L_M(\mathbb{T})} = \frac{\left\| \tilde{T}_n^{(k)} \right\|_{L_M(\mathbb{T})}}{(n+1)^k}, \text{ if } k \text{ is odd}, \quad (2.7)
\]

Use of (2.6), (2.7) and condition (ii) gives us
\[
\| T_n - U(\cdot, T_n) \|_{L_M(\mathbb{T})} \leq c \| T_n - Z_{n,k}(\cdot, T_n) \|_{L_M(\mathbb{T})}.
\]
The last inequality and Theorem 1.1 imply that (1.4). Theorem 1.2 is completely proved.

**Proof of Theorem 1.3.** We suppose that the condition \( b^{*}_{k,M} \) is satisfied. Let \( f \in L_{M}(\mathbb{T}) \) and \( T_{n} \in H_{n} \) be the polynomial of best approximation to \( f \). Note that \( U_{n}(f) = \Lambda_{n} * f \). The operator \( U_{n}(f) \) is bounded in \( L_{q}(\mathbb{T}) \) and \( L_{q}(\mathbb{T}) \) [26]). Using the method of proof of Lemma 1 in [16] we can show that the operator \( U_{n}(f) \) is bounded in \( L_{M}(\mathbb{T}) \), i.e. \( \|U_{n} (\cdot, f)\|_{L_{M}(\mathbb{T})} \leq c_{3}p\|f\|_{L_{M}(\mathbb{T})} \). Then we get

\[
\|f - U_{n} (\cdot, f)\|_{L_{M}(\mathbb{T})} \leq \|f - T_{n}\|_{L_{M}(\mathbb{T})} + \|T_{n} - U (\cdot, T_{n})\|_{L_{M}(\mathbb{T})} + \|U (\cdot, T_{n}) - U (\cdot, f)\|_{L_{M}(\mathbb{T})} \leq c_{13}E_{n}(f)M_{\omega} + c_{7}E_{n}(f)M + c_{14}n^{-k}\|\tilde{T}_{n}^{(k)}\|_{L_{M}(\mathbb{T})} \leq c_{15}E_{n}(f)M_{\omega} + c_{16}n^{-k}\|\tilde{T}_{n}^{(k)}\|_{L_{M}(\mathbb{T})} .
\]

(2.8)

Taking the Bernstein inequality and the boundedness of the linear operator \( f \to \tilde{f} \) in \( L_{M}(\mathbb{T}) \) into account [16, Lemma 3 and (15)] and [3] we have

\[
n^{-k}\|\tilde{T}_{n}^{(k)}\|_{L_{M}(\mathbb{T})} \leq c_{17}n^{-k}\|T_{n}^{(k)}\|_{L_{M}(\mathbb{T})} \leq c_{18}\omega_{\alpha}(f, \frac{1}{n})M .
\]

(2.9)

where \( \tilde{f} \) is the conjugate function of \( f \in L_{M}(\mathbb{T}) \).

Note that according to the direct theorem of approximation in \( L_{M}(\mathbb{T}) \) given in [2] following inequality holds:

\[
E_{n}(f)M \leq c_{19}\omega_{\alpha}(f, \frac{1}{n})M .
\]

(2.10)

Taking into account the relations (2.8), (2.9), and (2.10) we have

\[
\|f - U_{n} (\cdot, f)\|_{L_{M}(\mathbb{T})} \leq c_{20}\omega_{\alpha}(f, \frac{1}{n})M .
\]

If the summability method with the matrix \( A \) satisfies condition \( (b^{*}_{k,M}) \), the proof is made analogously to the above.

The proof of Theorem 1.3 is completed.

**Proof of Theorem 1.4.** By [3] the following inequality holds:

\[
\omega_{\alpha}(U_{n}(f) - f, \delta)M \leq c_{21}\|U_{n} (\cdot, f) - f\|_{L_{M}(\mathbb{T})} .
\]

(2.11)

Let \( \delta \geq (n + 1)^{-1} \). Using Theorem 1.3 and (2.11) we have

\[
\omega_{\alpha}(U_{n}(f), \delta)M \leq \omega_{\alpha}(f, \delta)M + \omega_{\alpha}(U_{n} (\cdot, f) - f, \delta)M \leq \omega_{\alpha}(f, \delta)M + c_{21}\|U_{n} (\cdot, f) - f\|_{L_{M}(\mathbb{T}, \omega)} \leq \omega_{\alpha}(f, \delta)M + c_{18}\omega_{\alpha}(f, \frac{1}{n})M \leq c_{22}\omega_{\alpha}(f, \delta)M .
\]

(2.12)
Now we suppose that $\delta < (n + 1)^{-1}$. Then by virtue of [3, Lemma 3.2] and [16, Lemma 3] we obtain
\[
\omega_\alpha(U_n(f), \delta)_M \leq c_{23} \delta^\alpha \left\| U_n^{(\alpha)}(\cdot, f) \right\|_{LM(T)} \\
\leq c_{24} \delta^\alpha n^\alpha \left\| U_n(\cdot, f) \right\|_{LM(T)} \leq c_{25} \delta^\alpha n^\alpha \omega_\alpha(U_n(f), \frac{1}{n})_M \\
\leq c_{26} \delta^\alpha (n + 1)^\alpha \omega_\alpha(U_n(f), \frac{1}{n})_M \leq c_{27} \omega_\alpha(f, \delta)_M. \tag{2.13}
\]

Now combining (2.12) and (2.13) we obtain the inequality (1.6) of Theorem 1.3.

**Proof of Theorem 1.5.** Let $f \in L_M(G)$. Then by virtue of Theorem 1.7 the operator $T : E_M(D) \rightarrow E_M(G)$ is bounded one-to-one and onto and $T(f_0^+) = f$. The function $f$ has the following Faber series
\[
f(z) \sim \sum_{m=0}^\infty a_m(f) \phi_m(z).
\]
Using Lemma 1 in [15, p.760] we conclude that $f_0^+ \in E_M(D)$. For the function $f_0^+$ the following Taylor series holds:
\[
f_0^+(w) = \sum_{m=0}^\infty a_m(f) w^m.
\]
Note that $f_0^+ \in E_1(D)$ and boundary function $f_0^+ \in L_M(T)$. Then by [9, Theorem, 3.4] for the function $f_0^+$ we have the following Fourier expansion:
\[
f_0^+(w) \sim \sum_{m=0}^\infty a_m(f) e^{imt}.
\]
Hence, if we consider boundedness of the operator $T : E_M(D) \rightarrow E_M(G)$ and Theorem 1.3, we obtain
\[
\| f - U_n(\cdot, f) \|_{LM(T)} \leq c_{35} \left\| f_0^+ - U_n(\cdot, f_0^+) \right\|_{LM(T)} \leq c_{36} \omega_\alpha(f_0^+, \frac{1}{n})_M = c_{37} \omega_\alpha(f, \frac{1}{n})_M,
\]
and (1.11) is proved.

**Remark 2.1.** Let $L_M(T)$ be the Orlicz space. Then by virtue of Theorem 2 in [2] for $f \in L_M(T)$ the inequality
\[
\omega_\alpha(f, \frac{1}{n + 1})_M \leq \frac{c_{38}}{(n + 1)^\alpha} \sum_{m=0}^n (m + 1)^{\alpha-1} E_m(f)_M, \tag{2.14}
\]
holds with a constant $c$ independent of $n$. If the summability method with the matrix $\Lambda$ satisfy the condition $(b_{k,M})$ or $(b^*_{k,M})$ then for $f \in L_M(\mathbb{T})$ relation (1.5) and inequality (2.14) immediately yield

$$\|f - U_n(f, f)\|_{L_M(\mathbb{T})} \leq \frac{c}{(n+1)^{\alpha}} \sum_{m=0}^{n} (m+1)^{\alpha-1} E_m(f)_M.$$ 

The inequality (2.15) holds for Zygmund-Riesz means of order $k$. Note that in the Lebesgue spaces $L_p(\mathbb{T})$, $1 < p \leq \infty$ the inequality (2.15) was proved in [33].

References


