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Commutators of maximal operator associated with the Dunkl operators on Orlicz spaces

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Abstract. *On the real line, the Dunkl operators are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. In this paper we prove the boundedness of the commutator of D-maximal operator $M_{D,b}$ in D-Orlicz spaces $L_{\Phi,\alpha}(\mathbb{R})$.*

Keywords. D-maximal operator; Orlicz space, commutator, BMO

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35

1 Introduction

It is well known that the maximal operator, associated with Λ_α differential-difference Dunkl operators (D-maximal operator) play an important role in Dunkl harmonic analysis (D-harmonic analysis), differentiation theory and PDE's. The harmonic analysis of the one-dimensional Dunkl operator and Dunkl transform was developed in [3,4,13]. The Dunkl operator and Dunkl transform considered here are the rank-one case of the general Dunkl theory, which is associated with a finite reflection group acting on a Euclidean space. The Dunkl theory provides a useful framework for the study of multivariable analytic structures and has gained considerable interest in various fields of mathematics and in physical applications (see, for example, [6]). The D-maximal operator and related topics associated with the Dunkl differential-difference operator have been research areas for many mathematicians such as C. Abdelkefi and M. Sifi [1], C.F. Dunkl [5], V.S. Guliyev and Y.Y. Mammadov [3,4], Y.Y. Mammadov [11], L. Kamoun [9], M.A. Mourou [14], F. Soltani [18,19] and others.

On the real line, the Dunkl operators Λ_α are differential-difference operators introduced in 1989 by Dunkl [5]. For a real parameter $\alpha \geq -1/2$, we consider the Dunkl operator, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha(f)(x) := \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right).$$

Note that $\Lambda_{-1/2} = d/dx$.

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It is well known that maximal operator play an important role in harmonic analysis (see [20]). Harmonic analysis associated to the Dunkl transform and the Dunkl differential-difference operator gives rise to convolutions with a relevant generalized translation. In the framework of this analysis we study commutator of maximal functions (D-maximal functions) in the relevant D-Orlicz space. We prove the boundedness of the commutator of Dunkl-type maximal operator M_b (D-maximal operator) in D-Orlicz spaces $L_{\Phi,\alpha}(\mathbb{R})$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1}\, dx.$$

Let $B(x,r) = \{y \in \mathbb{R} : |y| \in]\max\{0, |x|-r\}, |x|+r[\ \}$ and $r > 0$. Then $B(0,r) =]-r, r[$ and

$$\mu_\alpha B(0,r) = b_\alpha\, r^{2\alpha+2},$$

where $b_\alpha = \left[2^{\alpha+1}(\alpha+1)\,\Gamma(\alpha+1)\right]^{-1}$.

For every $1 \le p \le \infty$, we denote by $L_{p,\alpha}(\mathbb{R}) = L_p(d\mu_\alpha)(\mathbb{R})$ the spaces of complex-valued functions f, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} \equiv \|f\|_{L_{p,\alpha}} = \left(\int_{\mathbb{R}} |f(x)|^p\, d\mu_\alpha(x)\right)^{1/p} < \infty \quad \text{if} \quad p \in [1,\infty),$$

and

$$\|f\|_{\infty,\alpha} \equiv \|f\|_{L_{\infty,\alpha}} = ess\sup_{x\in\mathbb{R}} |f(x)| \quad \text{if} \quad p = \infty.$$

For $1 \le p < \infty$ we denote by $WL_{p,\alpha}(\mathbb{R})$, the weak $L_{p,\alpha}(\mathbb{R})$ spaces defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$ with the finite norm

$$\|f\|_{WL_{p,\alpha}} = \sup_{r>0} r \left(\mu_\alpha \{x \in \mathbb{R} : |f(x)| > r\}\right)^{1/p}.$$

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha} \quad \text{and} \quad \|f\|_{WL_{p,\alpha}} \le \|f\|_{p,\alpha} \text{ for all } f \in L_{p,\alpha}(\mathbb{R}).$$

For all $x,\ y,\ z \in \mathbb{R}$, we put

$$W_\alpha(x,y,z) = (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x})\Delta_\alpha(x,y,z)$$

where

$$\sigma_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x,\ y \in \mathbb{R}\setminus 0, \\ 0 & \text{otherwice} \end{cases}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x,y,z) = \begin{cases} d_\alpha\, \frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\alpha-1/2}}{|xyz|^{2\alpha}} & \text{if } |z| \in A_{x,y}, \\ 0 & \text{otherwice,} \end{cases}$$

where $d_\alpha = (\Gamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2}))$ and $A_{x,y} = [||x|-|y||, |x|+|y|]$.

Proposition 2.1 *(see Rösler [21]) The signed kernel W_α is even and satisfies the following properties*

$$W_\alpha(x,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y),$$
$$W_\alpha(x,y,z) = W_\alpha(-z,y,-x) = W_\alpha(-x,-y,-z)$$

and

$$\int_{\mathbb{R}} |W_\alpha(x,y,z)|\, d\mu_\alpha(z) \le 4.$$

In the sequel we consider the signed measure $\nu_{x,y}$, on $\mathbb{R}$, given by

$$\nu_{x,y} = \begin{cases} W_\alpha(x,y,z)\, d\mu_\alpha(z) & \text{if } x,\ y \in \mathbb{R}\setminus 0, \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0. \end{cases}$$

Definition 2.1 *For $x,\ y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, we put*

$$\tau_x f(y) = \int_{\mathbb{R}} f(z)\, d\nu_{x,y}(z).$$

The operators τ_x, $x \in \mathbb{R}$, are called Dunkl translation operators on $\mathbb{R}$ and it can be expressed in the following form (see ref. [21,23])

$$\tau_x f(y) = c_\alpha \int_0^\pi f_e\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right) h_1(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta$$
$$+ c_\alpha \int_0^\pi f_o\left(\sqrt{x^2+y^2-2|xy|\cos\theta}\right) h_2(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta,$$

where $f = f_e + f_o$, f_o and f_e being respectively the odd and the even parts of f, with $c_\alpha = \Gamma(\alpha+1)/(\sqrt{\pi}\,\Gamma(\alpha+1/2))$,

$$h_1(x,y,\theta) = 1 - sgn(xy)\cos\theta \ \text{ and } \ h_2(x,y,\theta) = \begin{cases} \frac{(x+y)\,[1-sgn(xy)\cos\theta]}{\sqrt{x^2+y^2-2|xy|\cos\theta}} & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

3 On Young Functions and Orlicz Spaces

Orlicz space was first introduced by Orlicz in [15,16] as a generalizations of Lebesgue spaces L^p. Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L^1 space when L^1 space does not work.

First, we recall the definition of Young functions.

Definition 3.1 *A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.*

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0, \tag{3.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty) \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\Phi(r), \qquad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2C}\Phi(Cr), \qquad r \ge 0$$

for some $C > 1$.

Definition 3.2 *(Orlicz Space). For a Young function Φ, the set*

$$L_{\Phi,\alpha}(\mathbb{R}) = \left\{ f \in L_{1,\alpha}^{\rm loc}(\mathbb{R}) : \int_{\mathbb{R}} \Phi(k|f(x)|)\, d\mu_\alpha(x) < \infty \textit{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L_{\Phi,\alpha}(\mathbb{R}) = L_{p,\alpha}(\mathbb{R})$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L_{\Phi,\alpha}(\mathbb{R}) = L_\infty(\mathbb{R})$. The space $L_{\Phi,\alpha}^{\rm loc}(\mathbb{R})$ is defined as the set of all functions f such that $f\chi_B \in L_{\Phi,\alpha}(\mathbb{R})$ for all balls $B \subset \mathbb{R}$.

$L_{\Phi,\alpha}(\mathbb{R})$ is a Banach space with respect to the norm

$$\|f\|_{L_{\Phi,\alpha}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, d\mu_\alpha(x) \le 1\right\}.$$

For a measurable function f on $\mathbb{R}$ and $t > 0$, let

$$m(f,t)_\alpha = \mu_\alpha\{x \in \mathbb{R} : |f(x)| > t\}.$$

Definition 3.3 *The weak Orlicz space*

$$WL_{\Phi,\alpha}(\mathbb{R}) = \{f \in L_{1,\alpha}^{\rm loc}(\mathbb{R}) : \|f\|_{WL_{\Phi,\alpha}} < \infty\}$$

is defined by the norm

$$\|f\|_{WL_{\Phi,\alpha}} = \inf\Big\{\lambda > 0 \,:\, \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\, t\Big)_\alpha \le 1\Big\}.$$

We note that $\|f\|_{WL_{\Phi,\alpha}} \le \|f\|_{L_{\Phi,\alpha}}$,

$$\sup_{t>0} \Phi(t) m(f,t)_\alpha = \sup_{t>0} t\, m(f, \Phi^{-1}(t))_\alpha = \sup_{t>0} t\, m(\Phi(|f|), t)_\alpha$$

and

$$\int_{\mathbb{R}} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_{\Phi,\alpha}}}\Big)\, d\mu_\alpha(x) \le 1, \qquad \sup_{t>0} \Phi(t) m\Big(\frac{f}{\|f\|_{WL_{\Phi,\alpha}}},\, t\Big)_\alpha \le 1. \tag{3.2}$$

The following analogue of the Hölder's inequality is well known (see, for example, [17]).

Theorem 3.1 *Let the functions f and g measurable on $\mathbb{R}$. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_{\mathbb{R}} |f(x)g(x)|\, d\mu_\alpha(x) \le 2\|f\|_{L_{\Phi,\alpha}} \|g\|_{L_{\widetilde{\Phi},\alpha}}.$$

By elementary calculations we have the following property.

Lemma 3.1 *Let Φ be a Young function and B be a balls in $\mathbb{R}$. Then*

$$\|\chi_B\|_{L_{\Phi,\alpha}} = \|\chi_B\|_{WL_{\Phi,\alpha}} = \frac{1}{\Phi^{-1}\left(\left(\mu_\alpha(B)\right)^{-1}\right)}.$$

By Theorem 3.1, Lemma 3.1 and (3.1) we get the following estimate.

Lemma 3.2 *For a Young function Φ and for the balls $B = B(x,r)$ the following inequality is valid:*

$$\int_B |f(y)|\, d\mu_\alpha(y) \le 2\mu_\alpha(B)\Phi^{-1}\left(\left(\mu_\alpha(B)\right)^{-1}\right) \|f\|_{L_{\Phi,\alpha}(B)}.$$

Now we define the Dunkl-type maximal function (see [1,11,19]) by

$$M_\alpha f(x) = \sup_{r>0} \left(\mu_\alpha B(0,r)\right)^{-1} \int_{B(0,r)} \tau_x |f|(y)\, d\mu_\alpha(y).$$

In [12] the boundedness of the maximal operator M_α in Orlicz spaces $L_{\Phi,\alpha}(\mathbb{R})$ have been obtained.

Theorem 3.2 *[12] Let Φ any Young function. Then the D-maximal operator M_α is bounded from $L_{\Phi,\alpha}(\mathbb{R})$ to $WL_{\Phi,\alpha}(\mathbb{R})$ and for $\Phi \in \nabla_2$ bounded in $L_{\Phi,\alpha}(\mathbb{R})$.*

Remark 3.1 Note that Theorem 3.2 in the case $\Phi(t) = t^p$, $1 \le p < \infty$ were proved in [1, 11,19].

4 Boundedness of the commutator of D-maximal operator in the D-Orlicz spaces $L_{\Phi,\alpha}(\mathbb{R})$

In this section we investigate the boundedness of the maximal commutator $M_{b,\alpha}$ in Orlicz spaces.

We recall the definition of the space of $BMO_\alpha(\mathbb{R})$.

Definition 4.1 *Suppose that* $b \in L^1_{\rm loc}(\mathbb{R})$, *let*

$$\|b\|_{*,\alpha} = \sup_{x\in\mathbb{R}, r>0} \frac{1}{\mu_\alpha B(0,r)} \int_{B(0,r)} |\tau_x b(y) - b_{B(0,r)}(x)| \, d\mu_\alpha(y),$$

where

$$b_{B(0,r)}(x) = \frac{1}{\mu_\alpha B(0,r)} \int_{B(0,r)} \tau_x b(y) \, d\mu_\alpha(y).$$

Define

$$BMO_\alpha(\mathbb{R}) = \{b \in L^1_{\rm loc}(\mathbb{R}) \ : \ \|b\|_{*,\alpha} < \infty\}.$$

Modulo constants, the space $BMO_\alpha(\mathbb{R})$ is a Banach space with respect to the norm $\|\cdot\|_{D,*}$.

Before proving our theorems, we need the following lemmas and theorem.

Lemma 4.1 *[8] Let* $b \in BMO_\alpha(\mathbb{R})$. *Then there is a constant* $C > 0$ *such that*

$$\left|b_{E(0,r)}(x) - b_{E(0,t)}(x)\right| \le C\|b\|_{*,\alpha} \ln\frac{t}{r} \quad \textit{for} \quad 0 < 2r < t, \tag{4.1}$$

where C *is independent of* b, x, r, *and* t.

Lemma 4.2 *[7] Let* $b \in BMO_\alpha(\mathbb{R})$ *and* Φ *be a Young function with* $\Phi \in \Delta_2$, *then*

$$\|b\|_{*,\alpha} \approx \sup_{x\in\mathbb{R}, r>0} \Phi^{-1}\big(|E(0,r)|_\alpha^{-1}\big) \left\|\tau_x b(\cdot) - E_{B(0,r)}(x)\right\|_{L_{\Phi,\alpha}(E(0,r))}. \tag{4.2}$$

By Theorem 3.2 and Theorem 1.13 in [2] we get the following theorem.

Theorem 4.1 *Let* $b \in BMO_\alpha(\mathbb{R})$ *and* $\Phi \in \nabla_2$.

Then the operator $M_{b,\alpha}$ *is bounded on* $L_{\Phi,\alpha}(\mathbb{R})$, *and the inequality*

$$\|M_{b,\alpha} f\|_{L_{\Phi,\alpha}} \le C_0 \|b\|_* \|f\|_{L_{\Phi,\alpha}} \tag{4.3}$$

holds with constant C_0 *independent of* f.

The following theorem is valid.

Theorem 4.2 *Let* $b \in BMO_\alpha(\mathbb{R})$ *and* Φ *be a Young function. Then the condition* $\Phi \in \nabla_2$ *is necessary for the boundedness of* $M_{b,\alpha}$ *on* $L_{\Phi,\alpha}(\mathbb{R})$.

Proof. Assume that (4.3) holds. For the particular symbol $b(x) = \log|x| \in BMO_\alpha(\mathbb{R})$ and $f(x) = \chi_{B(0,r)}(x)$, (4.3) becomes

$$\left\|M_{b,\alpha}\chi_{B(0,r)}\right\|_{L_{\Phi,\alpha}} \le C_1 \left\|\chi_{B(0,r)}\right\|_{L_{\Phi,\alpha}}, \tag{4.4}$$

where $r = (a_1 uv)^{-1/(2\alpha+2)}$, $B = B(0,r)$, $a_r = \mu_\alpha B(0,r)$, $u > 0$ and $v > 1$. By Lemma 3.1 and (3.1), we have

$$\begin{aligned}\left\|\chi_{B(0,r)}\right\|_{L_{\Phi,\alpha}} &= \frac{1}{\Phi^{-1}\big(\big(\mu_\alpha B(0,r)\big)^{-1}\big)} \\ &= \frac{1}{\Phi^{-1}\big(r^{-2\alpha-2}\big(\mu_\alpha B(0,1)\big)^{-1}\big)} = \frac{1}{\Phi^{-1}(uv)} \le \frac{1}{uv}\widetilde{\Phi}^{-1}(uv).\end{aligned}$$

On the other hand, if $x \notin B(0,r)$ then $B(0,r) \subset B(x, 2|x|)$ since for $y \in B(0,r)$ we have

$$|x-y| \le |x| + |y| \le |x| + r \le 2|x|.$$

Also for each $y \in B(0,r)$, we have

$$b(x) - b(y) \geq \log\Big(\frac{|x|}{r}\Big).$$

Therefore

$$\begin{aligned} M_{b,\alpha}\chi_{B(0,r)}(x) &\geq \frac{1}{\mu_\alpha B(x,2|x|)} \int_{B(x,2|x|)\cap B(0,r)} |b(x)-b(y)|\, d\mu_\alpha(y) \\ &\geq \Big(\frac{r}{2|x|}\Big)^{2\alpha+2} \log\Big(\frac{|x|}{r}\Big). \end{aligned}$$

Following the ideas of [10], for $g = \widetilde{\Phi}^{-1}(u)\chi_{B(0,s)}$ with $s = (a_1 u)^{-1/(2\alpha+2)}$ we obtain

$$\int_{\mathbb{R}} \widetilde{\Phi}(|g(x)|)\, d\mu_\alpha(x) \leq u\, \mu_\alpha B(0,s) = u\, s^{2\alpha+2} \mu_\alpha B(0,1) = 1.$$

Since the Luxemburg-Nakano norm is equivalent to the B_n-Orlicz norm

$$\|f\|^*_{\Phi,\alpha} := \sup\Big\{ \int_{\mathbb{R}} |f(x)g(x)|\, d\mu_\alpha(x) : \|g\|_{L_{\widetilde{\Phi},\alpha}} \leq 1 \Big\}$$

(more precisely, $\|f\|_{L_{\Phi,\alpha}} \leq \|f\|^*_{\Phi,\alpha} \leq 2\|f\|_{L_{\Phi,\alpha}}$), it follows that

$$\begin{aligned} \big\|M_{b,\alpha}\chi_{B(0,r)}\big\|^*_{L_{\Phi,\alpha}} &= \sup\Big\{ \int_{\mathbb{R}} |M_{b,\alpha}\chi_{B(0,r)}(x) g(x)|\, d\mu_\alpha(x) : \int_{\mathbb{R}} \widetilde{\Phi}(|g(x)|)\, d\mu_\alpha(x) \leq 1 \Big\} \\ &\geq \widetilde{\Phi}^{-1}(u) \int_{B(0,s)} M_{b,\alpha}\chi_{B(0,r)}(x)\, d\mu_\alpha(x) \\ &\geq \widetilde{\Phi}^{-1}(u) \int_{B(0,s)\setminus B(0,r)} \Big(\frac{r}{2|x|}\Big)^{2\alpha+2} \log\big(\frac{|x|}{r}\big)\, d\mu_\alpha(x) \\ &= \frac{\widetilde{\Phi}^{-1}(u)}{2^{2\alpha+2} a_1 uv} \int_{B(0,s)\setminus B(0,r)} \frac{1}{|x|^{2\alpha+2}} \log\big(\frac{|x|}{r}\big)\, d\mu_\alpha(x) \\ &= \frac{\widetilde{\Phi}^{-1}(u)}{2^{2\alpha+3} a_1 uv} (2\alpha+2) a_1 \big(\log\frac{s}{r}\big)^2 = \frac{\widetilde{\Phi}^{-1}(u)}{2^{2\alpha+3}(2\alpha+2)uv} (\log v)^2. \end{aligned}$$

Hence (4.4), implies that

$$\frac{\widetilde{\Phi}^{-1}(u)}{2^{2\alpha+3}(2\alpha+2)uv} (\log v)^2 \leq \frac{2C_1}{uv}\, \widetilde{\Phi}^{-1}(uv)$$

for $u > 0$ and $v > 1$. Thus, taking $v = \exp(\sqrt{(2\alpha+2)C_1} \cdot 2^{\frac{2\alpha+5}{2}})$ we obtain $2\widetilde{\Phi}^{-1}(u) \leq \widetilde{\Phi}^{-1}(u \exp(\sqrt{(2\alpha+2)C_1} \cdot 2^{\frac{2\alpha+5}{2}}))$ for $u > 0$ or $\widetilde{\Phi}(2t) \leq \exp(\sqrt{(2\alpha+2)C_1} \cdot 2^{\frac{2\alpha+5}{2}})\widetilde{\Phi}(t)$ for every $t > 0$, and so $\widetilde{\Phi}$ satisfies the Δ_2 condition.

By Theorems 4.1 and 4.2 we have the following result.

Corollary 4.1 *Let $b \in BMO_\alpha(\mathbb{R})$ and $\Phi \in \mathcal{Y}$. Then the condition $\Phi \in \nabla_2$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ on $L_{\Phi,\alpha}(\mathbb{R})$.*

Theorem 4.3 *$b \in L_1^{\mathrm{loc}}(\mathbb{R})$ and Φ be a Young function. The condition $b \in BMO_\alpha(\mathbb{R})$ is necessary for the boundedness of $M_{b,\alpha}$ on $L_{\Phi,\alpha}(\mathbb{R})$.*

Proof. Suppose that $M_{b,\alpha}$ is bounded from $L_{\Phi,\alpha}(\mathbb{R})$ to $L_{\Phi,\alpha}(\mathbb{R})$. Choose any ball $B = B(0,r)$ in $\mathbb{R}$, by (3.1)

$$\begin{aligned}\frac{1}{\mu_\alpha B}\int_B |\tau_x b(y) - b_B|\, d\mu_\alpha(y) &\le \frac{1}{\mu_\alpha B}\int_B \frac{1}{\mu_\alpha B}\int_B |\tau_x b(y) - \tau_x b(z)|\chi_B(z)\, d\mu_\alpha(z)\, d\mu_\alpha(y)\\ &\le \frac{1}{\mu_\alpha B}\int_B M_{b,\alpha}(\chi_B)(y)\, d\mu_\alpha(y)\\ &\le \frac{2}{\mu_\alpha B}\|M_{b,\alpha}(\chi_B)\|_{L_{\Phi,\alpha}(B)}\|1\|_{L_{\widetilde{\Phi},\alpha}(B)}\\ &\le \frac{2}{\mu_\alpha B}\|\chi_B\|_{L_{\Phi,\alpha}}\|\chi_B\|_{L_{\widetilde{\Phi},\alpha}} \le C.\end{aligned}$$

Thus $b \in BMO_\alpha(\mathbb{R})$.

By Theorems 4.1 and 4.3 we have the following result.

Corollary 4.2 *Let Φ be a Young function with $\Phi \in \nabla_2$. Then the condition $b \in BMO_\alpha(\mathbb{R})$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ on $L_{\Phi,\alpha}(\mathbb{R})$.*

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