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Bases from generalized Faber polynomials in weighted Smirnov spaces

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Abstract. *In the paper it is considered the generalized Faber polynomials defined inside and outside a regular curve on the complex plane. The weighted Smirnov spaces corresponding to bounded and unbounded regions are defined. It is proved that the generalized Faber polynomials forms a basis in weighted Smirnov spaces, if the weight function satisfies the Muckenhoupt condition on the regular curve.*

Keywords. Faber polynomials · weighted Smirnov's classes · basisness

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1 Introduction

The Faber polynomials were introduced in 1903 by G. Faber in connection with applications of approximation on complex plane. The detailed information about these problems one may consult the monograph by V. I. Smirnov and N.A. Lebedev [32], also D. Gaier [15]. They replace polynomials of the variable z in a circle with respect to simply-connected domains. These polynomials play an important role in the problems of approximation on the complex plane and in the theory of conformal mappings. Series of classical Faber polynomials have been investigated enough well, and the results obtained here were comprehensively covered in [32]. The L_p topology case of the classical results was initiated to be studied by V. Kokilashvili in [20], where generalized Faber polynomials were introduced. The connection between generalized Faber and classical Faber polynomials was studied by J.E. Andersson in [1]. For some other aspects of approximation by Faber polynomials, see [15] and [33]. Note that the basisness problem of the system of generalized Faber polynomials in the Lebesgue spaces of functions defined on rectifiable closed Jordan curves was studied by B.T. Bilalov and T.I. Najafov in [9]. The degree of the approximation by generalized Faber polynomials in Smirnov's spaces was investigated by [19].

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Note that if the considered domain is a unit disk, then the generalized polynomials become parts of the system of exponentials $\left\{e^{-int}\right\}_{n\in N}$ (N is the set of positive integers) and $\left\{e^{int}\right\}_{n\in Z_+}$ ($Z_+ = \{0\} \cup N$). In this case the weighted Smirnov classes coincide with the weighted Hardy classes. To study the basisness of generalized Faber polynomials we will use the method of Riemann-Hilbert boundary value problems for analytic functions. This idea takes its origins from the note [11] of A. V. Bitsadze. The method was successfully used by S. Ponomarev [29] and E. I. Moiseev [21,22] to solve mixed type PDEs on special regions by the method of Fourier, and also to prove the criteria which guarantees the basisness of a trigonometric system with linear phase in Lebesgue spaces. The further development of this method to study the basisness (completeness, minimality and basisness) problems of special system of functions can be found in the work of B.T.Bilalov [2–7]. This method is still intensively developing, the boundary value problems and basisness problems are studied in various function spaces (see, e.g. [30,28,10,18,31,16,27]).

The present paper considers generalized Faber polynomials defined inside and outside a regular curve on the complex plane. The weighted Smirnov spaces corresponding to bounded and unbounded regions are defined. It is proved that the generalized Faber polynomials forms a basis in weighted Smirnov spaces, if the weight function satisfies the Muckenhoupt condition on the regular curve. Let us note that the Muckenhoupt condition plays a special role in basicity of the trigonometric systems in weighted Lebesgue spaces (see e.g. [17,33]).

2 Preliminaries

We give general notations and some definitions from the approximation theory and the theory of singular integral operators, which we will use through. By $O_r(z)$ we denote a disc with the radius r and the center z_0 in the complex plane: $O_r(z_0) \equiv \{z \in C : |z - z_0| < r\}$ (C is the complex plane); $|M|$ denotes the arc measure of the set $M \subset \Gamma$, where $\Gamma \subset C$ is a rectifiable curve. Denote $\omega = O_1(0)$ and $T = \partial\omega$. B- space will be used for Banach spaces, H-space for will be used for Hilbert spaces. We will also use the following standard notation. $Z_+ = \{0\} \bigcup N$; Z is the set of all integers.

Definition 2.1 *[19,13,14] A rectifiable Jordan curve Γ in the complex plane is called Carleson or regular if*

$$\sup_{z\in\Gamma} |\Gamma \cap O_r(z)| \le cr\ ,\ \ \forall r > 0\,,$$

where c is a constant, independent of r.

More details on this concept can be found in [19,14,12].

Let Γ be a rectifiable Jordan curve and $\rho(\cdot)$ be a positive function defined a.e. on Γ.

Definition 2.2 *The function $\rho(\cdot)$ is said to belong to the Muckenhoupt class $A_p(\Gamma)$ ($p > 1$) on the curve Γ if*

$$\sup_{z\in\Gamma}\sup_{r>0} \left(\frac{1}{r}\int_{\Gamma\cap O_r(z)} \rho(\xi)\ |d\xi|\right) \left(\frac{1}{r}\int_{\Gamma\cap O_r(z)} |\rho(\xi)|^{-\frac{1}{p-1}}\ |d\xi|\right)^{p-1} < +\infty.$$

Let us establish the definition of the generalized p-Faber polynomials $F^+_{p,n}$ and $F^-_{p,n}$ (see [20,19]). Let D^+ be a bounded region with the boundary Γ and the simple-connected complement $D^- = C\backslash\bar{D}^+$ ($\bar{D}^+$ is the closure of D^+). Let $w = \varphi(z)$ be a single-valued conformal mapping the region D^- on $C\backslash\overline{O_1(0)} \equiv O_1^-(0) : \varphi(\infty) = \infty\,,\ \varphi'(\infty) = \gamma > 0$. $\varphi(z)$ is the sum of its Laurent series at $z = \infty$:

$$\varphi(z) = \gamma\, z + \gamma_0 + \gamma_1 z^{-1} + \dots\ .$$

Let us take the analytic branch of $\sqrt[p]{\varphi'(z)}$ for which $\sqrt[p]{\varphi'(\infty)} > 0$. By $F_{p,n}^+$ we denote the principal part of the Laurent series of $[\varphi(z)]^n\ \sqrt[p]{\varphi'(z)}$ at $z = \infty$:

$$[\varphi(z)]^n\ \sqrt[p]{\varphi'(z)} \equiv F_{p,n}^+(z) + E_{p,n}^+(z)\,,\ z \in D^-,$$

where $E_{p,n}^+(\infty) = 0$. Here we take $F_{p,0}^+ \equiv 1$.

Similarly, the $F_{p,n}^-$ p-Faber polynomial corresponding to the mapping $\psi(\cdot)$ is defined. Now, let D^+ be a bounded simply-connected region, containing $z = 0$ and $w = \psi(z)$ conformal and single-valued function mapping D^+ on $O_1^-(0)\ :\ \psi(0) = \infty$, $\lim\limits_{z\to 0} z\psi(z) = \alpha > 0$. The function $\psi(z)$ has a Laurent expansion at $z = 0$:

$$\psi(z) = \alpha\, z^{-1} + \alpha_0 + \alpha_1 z + \dots\ .$$

It is clear that the point $z = 0$ is a pole of $[\psi(z)]^{n-\frac{2}{p}}\ \sqrt[p]{\psi'(z)}$ of order n and thus

$$[\psi(z)]^{n-\frac{2}{p}}\ \sqrt[p]{\psi'(z)} = F_{p,n}^-\left(z^{-1}\right) + E_{p,n}^-(z)\,,$$

where $F_{p,n}^-\left(z^{-1}\right)$ is the principal part of the series:

$$F_{p,n}^-\left(z^{-1}\right) = \alpha_n^{(n)} z^{-n} + \alpha_{n-1}^{(n)} z^{-n+1} + \dots + \alpha_1^{(n)} z^{-1}\ .$$

$L_{p,\rho}(\Gamma)$ is the usual weighted Lebesgue space equipped with the norm $\|\cdot\|_{p,\rho}$:

$$\|f\|_{L_{p,\rho}(\Gamma)} = \left(\int_\Gamma |f(\xi)|^p\, \rho(\xi)\, |d\xi|\right)^{\frac{1}{p}}.$$

Consider the Cauchy singular integral operator S_Γ:

$$S_\Gamma(f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - \tau} d\xi\,,\ \tau \in \Gamma.$$

The following theorem was proved by G.David.

Theorem 2.1 *[12] S_Γ is bounded in $L_p(\Gamma)$, $1 < p < +\infty$, if and only if Γ is a regular curve. Furthermore, if Γ is a regular curve then S_Γ is bounded in $L_{p,\rho}(\Gamma)$, $1 < p < +\infty$, if and only if $\rho \in A_p(\Gamma)$.*

For these and related results see, for example, [13,14].

We will need some facts about the basicity of classical exponential systems and its parts in Lebesgue and Hardy spaces, respectively. Recall some definitions.

Hardy Class H_p^+. H_p^+ $(p > 0)$ consists all analytic functions $f(z)$ on ω for which

$$\sup_{0\le r<1}\int_{-\pi}^{\pi} \left|f\left(re^{it}\right)\right|^p dt < +\infty.$$

The norm in H_p^+, $p \geq 1$ is given by

$$\|f\|_{H_p^+} = \sup_{0 \leq r < 1} \left(\int_{-\pi}^{\pi} \left| f\left(re^{it}\right) \right|^p dt \right)^{1/p} < +\infty.$$

With this norm H_p^+ is a Banach space (Hilbert space in the case of $p = 2$).

Define the class ${}_mH_p^-$ of functions analytic outside the unit disc. Let $f(z)$ be a function analytic outside ω for which $f(z) = \sum_{-\infty}^{m_0} a_k z^k$, $|z| > 1$, where $m_0 \leq m$. Thus, $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ and $f_2(z)$ are the principal and analytic parts of the Laurent expansion of $f(z)$, respectively. If the function $\varphi(z) \equiv \overline{f_2\left(\frac{1}{\bar{z}}\right)}$ belongs to H_p^+ $(p > 0)$, then we will say that $f(z) \in {}_mH_p^-$. The class ${}_{-1}H_p^-$ is usually denoted by H_p^-. Many properties of the functions of H_p^+ are easily proved to be true also for functions of ${}_mH_p^-$.

Now we define the weighted counterparts of the above spaces. Let

$$\tilde{H}^+ \equiv \left\{ f \in H_1^+ : \, f^+ \in L_{q,\,\nu^+} \right\},$$

where $H_1^{\pm}$ are Hardy classes of functions defined inside and outside of the unit disc, respectively, $\nu^+(\cdot)$ is a weight function defined on $[-\pi, \pi]$, $L_{q,\,\nu^+}$ is a weighted Lebesgue space on $(-\pi,\, \pi)$, $f^+\left(e^{it}\right)$ is the non-tangential boundary value of $f \in H_1^+$. Equip $\tilde{H}^+$ with the following norm:

$$\|f\|_{\tilde{H}^+} \equiv \left\| f^+\left(e^{it}\right) \right\|_{q,\,\nu^+}, \tag{2.1}$$

where $\|\cdot\|_{p,\,\nu^+}$ is the norm of $L_{p,\,\nu^+}$:

$$\|f\|_{p,\nu^+} = \left(\int_{-\pi}^{\pi} |f(t|^p \, \nu^+(t) \, dt \right)^{\frac{1}{p}}.$$

It is easy to prove the following

Proposition 2.1 *If* $|\nu^+|^{-\frac{p}{q}} \in L_1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq q < +\infty$ *then* $\tilde{H}^+$ *is a Banach space.*

Denote by $H_{q,\,\nu^+}^+$ the Banach space $\tilde{H}^+$ for which $|\nu^+|^{-\frac{p}{q}} \in L_1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq q < +\infty$.

Let ${}_mH_1^-$ be the Hardy class of the functions, which are analytic outside the unit disc and has a zero of order not greater than m at infinity. Let

$$\tilde{H}^- \equiv \left\{ f \in_m H_1^- : \, f^-\left(e^{it}\right) \in L_{q,\,\nu^-} \right\},$$

where ν^- is a weighted function on $[-\pi, \pi]$. The following assertion is the analog of the above one.

Proposition 2.2 *If* $|\nu^-|^{-\frac{p}{q}} \in L_1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq q < +\infty$ *then* $\tilde{H}^-$ *is a Banach space with respect to the norm*

$$\|f\|_{\tilde{H}^-} \equiv \left\| f^+\left(e^{it}\right) \right\|_{q,\,\nu^-}.$$

This space is denoted by ${}_mH_{q,\,\nu^-}^-$.

The restrictions of the functions belonging to $H_{p,\nu}^{+}$ and ${}_mH_{p,\nu}^{-}$ to the unit circle is denoted by $L_{p,\nu}^{+}$ and ${}_mL_{p,\nu}^{-}$, respectively: $H_{p,\nu}^{+}/_{\partial\omega} = L_{p,\nu}^{+}$; ${}_mH_{p,\nu}^{-}/_{\partial\omega} ={}_m L_{p,\nu}^{-}$.

We will say that the weight $\nu(\cdot)$ defined on $[-\pi,\pi]$ belongs to the Muckenhoupt class $A_p, 1 < p < +\infty$, if

$$\sup_{I\subset[-\pi;\pi]}\left(\frac{1}{|I|}\int_I \nu(t)\,dt\right)\left(\frac{1}{|I|}\int_I |\nu(t)|^{-\frac{1}{\rho-1}}\,dt\right)^{\rho-1} < +\infty,$$

where sup takes over all subintervals $I \subset [-\pi,\pi]$, $|I|$ is the Lebesgue measure of the interval I. Summarizing the results obtained earlier in [18], we reach the following.

Theorem 2.2 *The system of exponentials* $\left\{e^{int}\right\}_{n\in Z}$ *forms a basis in* $L_{p;\nu}(-\pi,\pi)$ *if and only if* $\nu \in A_p, \quad 1 < p < +\infty$

By using this theorem the following theorem is proved:

Theorem 2.3 *Let* $\nu \in A_p, \quad 1 < p < +\infty$. *Then: i) the system* $\{z^n\}_{n\in Z_+}$ *(i.e.* $\left\{e^{int}\right\}_{n\in Z_+}$*) forms a basis in* $H_{\rho,\nu}^{+}$ *(i.e. in* $L_{\rho,\nu}^{+}$*); ii) the system* $\{z^{-n}\}_{n\geq m}$ *(i.e.* $\left\{e^{-int}\right\}_{n\geq m}$*) forms a basis in* ${}_mH_{\rho,\nu}^{-}$ *(i.e. in* ${}_mL_{\rho,\nu}^{-}$*).*

For more comprehensive information about this and related results see, for example, [18].

3 Weighted Smirnov classes $E_{p,\rho}(D^+)$ and ${}_mE_{p,\rho}(D^-)$

Let $D^+ \subset C-$be a bounded region with the boundary $\Gamma = \partial D^+$which satisfies iii). By $E_p(D^+)$, $1 < p < \infty$, we denote the Smirnov space of analytic functions on D^+, which is also a Banach space with respect to the norm $\|\cdot\|_{E_p(D^+)}$:

$$\|f\|_{E_p(D^+)} =: \left\|f^+\right\|_{L_p(\Gamma)}, \forall f \in E_p\left(D^+\right), \tag{3.1}$$

where $f^+ = f/_{\Gamma}-$ is the non-tangential boundary values of f along Γ. Similarly, the Smirnov space $E_p(D^-)$ of functions defined on the region D^- with the boundary $\Gamma = \partial D^-$ is defined with by norm

$$\|f\|_{E_p(D^-)} =: \left\|f^-\right\|_{L_p(\Gamma)}, \forall f \in E_p\left(D^-\right),$$

where $f^- = f/_{\Gamma}-$ is the non-tangential boundary values of f along Γ.

Let $\rho \in L_1(\Gamma) -$ is a weight function. Denote

$$E_{p,\rho}\left(D^+\right) \equiv \left\{f \in E_1\left(D^+\right) : \left\|f^+\right\|_{L_{p,\rho}(\Gamma)} < +\infty\right\},$$

and equip it with the norm

$$\|f\|_{E_{p,\rho}(D^+)} = \left\|f^+\right\|_{L_{p,\rho}(\Gamma)}. \tag{3.2}$$

The Smirnov classes on unbounded regions are defined similarly. Let $D^- \subset C$ is a unbounded region containing infinity (∞). Denote by ${}_mE_1(D^-)$ the class of functions of $E_1(D^-)$, which have the Laurent expansion at $z = \infty$ of the form $f(z) = \sum_{k=-\infty}^{m} a_k z^k$, where m is an integer.

For the weight function $\rho \in L_1(\Gamma)$, the weighted class ${}_mE_{p,\rho}(D^-)$ is defined as

$$ {}_mE_{p,\rho}\left(D^-\right) \equiv \left\{ f \in_m E_1\left(D^-\right) : \left\|f^-\right\|_{L_{p,\rho}(\Gamma)} < +\infty \right\}, $$

and here the norm is given by

$$ \|f\|_{{}_mE_{p,\rho}(D^-)} = \left\|f^-\right\|_{L_{p,\rho}(\Gamma)}, $$

where $f^- = f/_{\Gamma}-$ is the non-tangential boundary values of f along Γ.

We have the following

Lemma 3.1 *If $\rho^{-\frac{q}{p}} \in L_1(\Gamma)$ then $E_{p,\rho}(D^+)$, $1 < p < +\infty$, is a Banach space.*

Proof. Let $\{f_n\}_{n\in N} \subset E_{p,\rho}(D^+)$ be any fundamental sequence, that is

$$ \|f_n - f_m\|_{E_{p,\rho}(D^+)} \to 0, \text{ as } n, m \to \infty. $$

Thus

$$ \left\|f_n^+(\xi) - f_m^+(\xi)\right\|_{L_{p,\rho}(\Gamma)} \to 0, \text{ as } n, m \to \infty. $$

As the space $L_{p,\rho}(\Gamma)$ is complete, then $\exists g \in L_{p,\rho}(\Gamma) : f_n^+(\xi) \to g(\xi)$, $n \to \infty$, in $L_{p,\rho}(\Gamma)$.

We have

$$ \|f_n - f_m\|_{E_1(D^+)} = \left\|f_n^+ - f_m^+\right\|_{L_1(\Gamma)} = \int_\Gamma \left|f_n^+(\xi) - f_m^+(\xi)\right| |d\xi| = $$

$$ = \int_\Gamma \left|f_n^+(\xi) - f_m^+(\xi)\right| \rho^{\frac{1}{p}}(\xi)\, \rho^{-\frac{1}{p}}(\xi)\, |d\xi| \le $$

$$ \le \left(\int_\Gamma \rho^{-\frac{q}{p}}(\xi)\, |d\xi| \right)^{\frac{1}{q}} \left(\int_\Gamma \left|f_n^+(\xi) - f_m^+(\xi)\right|^p \rho(\xi)\, |d\xi| \right)^{\frac{1}{p}}. $$

From $\rho^{-\frac{q}{p}} \in L_1(\Gamma)$ it follows that $\{f_n\}_{n\in N}$ is fundamental in $E_1(D^+)$ and hence, $\exists f \in E_1(D^+) : f_n \to f$, $n \to \infty$, in $E_1(D^+)$. Hence, $f_n^+(\xi) \to f^+(\xi)$, $n \to \infty$, in $L_1(\Gamma)$. Since

$$ \left\|f_n^+ - g\right\|_{L_1(\Gamma)} = \int_\Gamma \left|f_n^+(\xi) - g(\xi)\right| |d\xi| \le $$

$$ \le \left(\int_\Gamma \rho^{-\frac{q}{p}}(\xi)\, |d\xi| \right)^{\frac{1}{q}} \left(\int_\Gamma \left|f_n^+(\xi) - g(\xi)\right|^p \rho(\xi)\, |d\xi| \right)^{\frac{1}{p}}, $$

from $f_n^+ \to g$, $n \to \infty$, in $L_{p,\rho}(\Gamma)$ it follows that $f_n^+ \to g^+$, $n \to \infty$, in $L_1(\Gamma)$. Hence we get that $f^+(\xi) = g(\xi)$ a.e. on Γ, hereby $\|f_n - f\|_{E_{p,\rho}(D^+)}$, $n \to \infty$. That completes the proof.

The same reasoning proves the following

Lemma 3.2 *If $\rho^{-\frac{q}{p}} \in L_1(\Gamma)$ then ${}_mE_{p,\rho}(D^-)$ is a Banach space.*

4 Basisness of p-Faber polynomials in the weighted Smirnov spaces

In this section the basisness of generalized Faber polynomials in weighted Smirnov spaces is established under the condition that the weight and the boundary values of the conformal mapping performing an isomorphism between the unit circle and the simply-connected region under consideration satisfy the Muckenhoupt condition.

Let

$$f_+(w) = f[\varphi_{-1}(w)]\left(\varphi'_{-1}(w)\right)^{\frac{1}{p}}, f_-(w) = f[\psi_{-1}(w)]\left(\psi'_{-1}(w)\right)^{\frac{2}{p}}, \ w \in \omega.$$

Consider the following operator T_p^+:

$$\left(T_p^+ f\right)(z) = \frac{1}{2\pi i}\int_{\partial\omega} \frac{f(\xi)\left[\varphi'_{-1}(\xi)\right]^{\frac{1}{q}}}{\varphi_{-1}(\xi) - z}\, d\xi, \ z \in D,$$

where $f \in H^+_{p,\rho_+}$. In sequel we will use the following theorem proved in [19].

Theorem 4.1 *Let $\Gamma-$be a regular curve and $1 < p < +\infty$. If $\rho \in A_p(\Gamma)$ and $\rho_+ \in A_p(T)$, then $T_p^+ : H^+_{p,\rho_+} \leftrightarrow E_{p,\rho}(D^+)$ is a one-to-one operator onto $E_{p,\rho}(D^+)$.*

It is easy to see that

$$\left(T_p^+ f_n\right)(z) = F^+_{p,n}(z), \ \forall n \in Z_+,$$

where

$$f_n(z) = z^n.$$

Assume that the weight function $\rho_+(\cdot)$ satisfies $A_p(T)$, $1 < p < +\infty$ Muckenhoupt condition on the unit circle T. By Theorem 2.3, the system $\{z^n\}_{n\in Z_+}$ forms a basis in H^+_{p,ρ_+}. Theorem 2.1 implies that if $\rho \in A_p(\Gamma)$, then he system $\left\{F^+_{p,n}\right\}_{n\in Z_+}$ forms a basis in $E_{p,\rho}(D)$.

Denote by $L^+_{p,\rho}(\Gamma)$ the set of restrictions of the functions of $E_{p,\rho}(D)$ to Γ. We get that the system $\left\{F^+_{p,n}(\xi)\right\}_{n\in Z_+}$ forms a basis in $L^+_{p,\rho}(\Gamma)$, if Γ is a regular curve and $\rho_+ \in A_p(T)$ & $\rho \in A_p(\Gamma)$.

Show that analogous reasoning works also for the system $\left\{F^-_{p,n}\right\}_{n\in N}$. Indeed, denote by $L^-_p(\Gamma)$ the restriction of $E_p(D^-)$ to Γ. Prove that $\left\{F^-_{p,n}\right\}_{n\in N}$ forms a basis in $L^-_{p,\rho}(\Gamma)$ under some conditions on functions $\rho(\cdot)$ and $\rho_-(\cdot)$. Let us take any polynomial $P\left(z^{-1}\right) = a_1 z^{-1} + ... + a_r z^{-r}$ and define the operator T_p^- as follows:

$$T_p^- P\left(z^{-1}\right) = a_1 F^-_{p,1}\left(z^{-1}\right) + ... + a_r F^-_{p,r}\left(z^{-1}\right). \tag{4.1}$$

Assume that the weight function $\rho_-(\cdot)$ belongs to $A_p(T)$, $1 < p < +\infty$. Since the function $P\left(z^{-1}\right)$ is bounded in $C\backslash\omega$, from the definition of ${}_{-1}H^-_{p,\rho_-}$ it immediately follows that $P\left(z^{-1}\right) \in_{-1} H^-_{p,\rho_-}$. Since $\rho_-(\cdot) \in A_p(T)$, from the Theorem 2.3 it follows that *ii*) the system $\{z^{-n}\}_{n\in N}$ forms a basis in ${}_{-1}H^-_{p,\rho_-}$, which proves that the set of polynomials $P\left(z^{-1}\right)$ are dense in ${}_{-1}H^-_{p,\rho_-}$. Denote the set of polynomials of the form $P\left(z^{-1}\right)$ by Π^-. Thus, $\Pi^- \subset_{-1} H^-_{p,\rho_-}$ and $\overline{\Pi^-} =_{-1} H^-_{p,\rho_-}$ ($\bar{M}-$is the closure of the set in ${}_{-1}H^-_{p,\rho_-}$). Show that $T_p^- P\left(z^{-1}\right) \in_{-1} E_{p,\rho}(D^-)$. Due to (4.1) it is enough to show that $F^-_{p,n} \in_{-1} E_{p,\rho}(D^-)$, $\forall n \in N$. It is evident that there exists a number $M_n > 0$ for which

$$\left|F^-_{p,n}(z)\right| \le M_n, \ \forall z \in D^-.$$

Since, $F^-_{p,n}(z)$ is a polynomial from z^{-1} of order n. We have

$$\left\|F^-_{p,n}\right\|_{_{-1}E_{p,\rho}(D^-)} = \left\|F^-_{p,n}(\xi)\right\|_{L_{p,\rho}(\Gamma)} = \left(\int_\Gamma \left|F^-_{p,n}(\xi)\right|^p \rho(\xi)\,d\xi\right)^{\frac{1}{p}} \leq$$

$$\leq M_n \left(\int_\Gamma \rho(\xi)\,d\xi\right)^{\frac{1}{p}} < +\infty,$$

since, $\rho(\cdot) \in A_p(\Gamma)$. The inclusion $F^-_{p,n} \in_{-1} E_p(D^-)$ is clear. Then, from the above expression it follows that $F^-_{p,n} \in_{-1} E_{p,\rho}(D^-)$. Hence, $T^-_p : \Pi^- \to_{-1} E_{p,\rho}(D^-)$. Similarly to the case T^+_p, it is proved that the operator T^-_p is bounded in Π^-.

Consider the continuous extension of T^-_p to $_{-1}H^-_{p,\rho_-}$ and denote it again by T^-_p. Thus, Thus, we obtain that the operator T^-_p boundedly maps $_{-1}H^-_{p,\rho_-}$ onto $_{-1}E_{p,\rho}(D^-)$. T^-_p establishes an isomorphism between $_{-1}H^-_{p,\rho_-}$ and $_{-1}E_{p,\rho}(D^-)$. Therefore, by the Banach's theorem it follows that $\left(T^-_p\right)^{-1}$ is also bounded and as a result, T^-_p is an isomorphism between $_{-1}H^-_{p,\rho_-}$ and $_{-1}E_{p,\rho}(D^-)$. Furthermore, $\left[T^-_p\left(\xi^{-n}\right)\right](z) = F^-_{p,n}(z)$, $n \in N$. Since $\rho_-(\cdot) \in A_p(T)$, by Theorem 2.3, the system $\{z^{-n}\}_{n\in N}$ forms a basis in $_{-1}H^-_{p,\rho_-}$. As a result, the system $\left\{F^-_{p,n}(z)\right\}_{n\in N}$ forms a basis in $_{-1}E_{p,\rho}(D^-)$. Thus, $\forall F \in {}_{-1}E_{p,\rho}(D^-)$ there exists a unique expansion

$$F(z) = \sum_{n=1}^{\infty} F_n F^-_{p,n}(z) \ , \ \ z \in D^-.$$

Therefore we have proved the following.

Theorem 4.2 *Let be a regular curve and* $0 \in int\Gamma$. *If* $\rho_\pm(\cdot) \in A_p(T)$, $\rho(\cdot) \in A_p(\Gamma)$, $1 < p < +\infty$, *then the system of generalized p-Faber polynomials* $\left\{F^+_{p,n}\right\}_{n\in Z_+}$ *and* $\left\{F^-_{p,n}\right\}_{n\in N}$ *form a basis of the spaces* $E_{p,\rho}(D^+)$ *and* $_{-1}E_{p,\rho}(D^-)$, *respectively.*

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