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Diagonal lift in the semi-tangent bundle and its applications

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Abstract. *The main purpose of this paper is to investigate diagonal lift of tensor fields of type (1,1) from manifold M to its semi-tangent bundle tM. In this context cross-sections in semi-tangent (pull-back) bundle tM of tangent bundle TM by using projection (submersion) of the cotangent bundle T*M can be also defined. In addition, a new example for good square presented in this paper.*

Keywords. Vector field · complete lift · diagonal lift · pull-back bundle · cross-section · semi-tangent bundle

Mathematics Subject Classification (2010): 53A45, 55R10, 57R25

1 Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞, and let $\Big(T^*(M_n), \pi_1, M_n\Big)$ be a cotangent bundle over M_n. We use the notation $(x^i) = \left(x^{\overline{\alpha}}, x^\alpha\right)$, where the indices $i, j, ...$ run from 1 to $2n$, the indices $\overline{\alpha}, \overline{\beta}, ...$ from 1 to n and the indices $\alpha, \beta, ...$ from $n+1$ to $2n$, x^α are coordinates in M_n, $x^{\overline{\alpha}} = p_\alpha$ are fibre coordinates of the cotangent bundle $T^*(M_n)$ (For definition of the pull-back bundle, see for example [1–5]).

Let now $(T(M_n), \widetilde{\pi}, M_n)$ be a tangent bundle with base space M_n, and let $T^*(M_n)$ be cotangent bundle determined by a natural projection (submersion) $\pi_1 : T^*(M_n) \to M_n$.

The semi-tangent [1–5, 8] bundle (induced or pull-back) of the tangent bundle $\Big(T(M_n), \widetilde{\pi}, M_n\Big)$ is the bundle $\Big(t(M_n), \pi_2, T^*(M_n)\Big)$ over cotangent bundle $T^*(M_n)$ with a total space

$$t(M_n) = \left\{\left(\left(x^{\overline{\alpha}}, x^\alpha\right), x^{\overline{\overline{\alpha}}}\right) \in T^*(M_n) \times T_x(M_n) : \pi_1\left(x^{\overline{\alpha}}, x^\alpha\right) = \widetilde{\pi}\left(x^\alpha, x^{\overline{\overline{\alpha}}}\right) = (x^\alpha)\right\} \subset T^*(M_n) \times T_x(M_n)$$

and with the projection map $\pi_2 : t(M_n) \to T^*(M_n)$ defined by $\pi_2(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}}) = \left(x^{\overline{\alpha}}, x^\alpha\right)$, where $T_x(M_n)\left(x = \pi_1\left(\widetilde{x}\right), \widetilde{x} = \left(x^{\overline{\alpha}}, x^\alpha\right) \in T^*(M_n)\right)$ is the tangent space at a point x of

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M_n, where $x^{\overline{\overline{\alpha}}} = y^\alpha \left(\overline{\overline{\alpha}}, \overline{\overline{\beta}}, ... = 2n+1, ..., 3n\right)$ are fiber coordinates of the tangent bundle $T(M_n)$.

If $(x^{i'}) = (x^{\overline{\alpha}'}, x^{\alpha'}, x^{\overline{\overline{\alpha}}'})$ is another system of local adapted coordinates in the semi-tangent bundle $t(M_n)$, then we have

$$\begin{cases} x^{\overline{\alpha}'} = \frac{\partial x^\beta}{\partial x^{\alpha'}} p_\beta, \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\overline{\alpha}}'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} y^\beta. \end{cases} \tag{1.1}$$

The jacobian of (1.1) has components

$$\bar{A} = \left(A_J^{I'}\right) = \begin{pmatrix} A_{\alpha'}^\beta & p_\sigma A_\beta^{\beta'} A_{\beta'\alpha'}^\sigma & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & A_{\beta\varepsilon}^{\alpha'} y^\varepsilon & A_\beta^{\alpha'} \end{pmatrix}, \tag{1.2}$$

where

$$A_{\beta'\alpha'}^\alpha = \frac{\partial^2 x^\alpha}{\partial x^{\beta'} \partial x^{\alpha'}}, A_{\beta\varepsilon}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\varepsilon}. \tag{1.3}$$

We denote by $\Im_q^p(T^*(M_n))$ and $\Im_q^p(M_n)$ the modules over $F\left(T^*(M_n)\right)$ and $F\left(M_n\right)$ of all tensor fields of type (p,q) on $T^*(M_n)$ and M_n, respectively, where $F\left(T^*(M_n)\right)$ and $F\left(M_n\right)$ denote the rings of real-valued C^∞ $-$functions on $T^*(M_n)$ and M_n, respectively.

Let V be a vector field on $T^*(M_n)$. Then the transformation $P \to V_P$, V_P being the value of V at $P \in T^*(M_n)$, determines a cross-section β_V of semi-tangent bundle.

In addition, let θ be a $1-$form in M_n. Then the correspondence $P \to \theta^P$, θ^P being the value of θ at $P \in M_n$, determines a cross-section β_θ of cotangent bundle [[7], p. 301].

Thus if $\sigma : M_n \to T(M_n)$ is a cross-section of $(T(M_n), \widetilde{\pi}, M_n)$, such that $\widetilde{\pi} \circ \sigma = I_{(M_n)}$, an associated cross-section $\beta_V : T^*(M_n) \to t(M_n)$ of semi-tangent (pull-back) bundle $(t(M_n), \pi_2, T^*(M_n))$ of tangent bundle by using projection (submersion) of the cotangent bundle $T^*(M_n)$ defined by [[6], p. 217-218], [[7], p. 122-123]:

$$\beta_V\left(x^{\overline{\alpha}}, x^\alpha\right) = \left(x^{\overline{\alpha}}, x^\alpha, \sigma \circ \pi_1\left(x^{\overline{\alpha}}, x^\alpha\right)\right) = \left(x^{\overline{\alpha}}, x^\alpha, \sigma\left(x^\alpha\right)\right) = \left(x^{\overline{\alpha}}, x^\alpha, V^\alpha\left(x^\beta\right)\right).$$

If the vector field V has the local components $V^\alpha\left(x^\beta\right)$, the cross-section $\beta_V\left(T^*(M_n)\right)$ of $t(M_n)$ is locally expressed by

$$x^{\overline{\alpha}} = p_\alpha = \theta_\alpha\left(x^\beta\right), x^\alpha = x^\alpha, x^{\overline{\overline{\alpha}}} = y^\alpha = V^\alpha\left(x^\beta\right) \tag{1.4}$$

with respect to the coordinates $x^A = (x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ on $t(M_n)$. $x^{\overline{\alpha}} = p_\alpha$ being considered as parameters. Taking the derivative of (1.4) with respect to $x^{\overline{\alpha}} = p_\alpha$, we have vector fields $B_{(\overline{\beta})}$ $\left(\overline{\beta}, ... = 1, ..., n\right)$ with components

$$B_{(\overline{\beta})} = \frac{\partial x^A}{\partial x^{\overline{\beta}}} = \partial_{\overline{\beta}} x^A = \begin{pmatrix} \partial_{\overline{\beta}} p_\alpha \\ \partial_{\overline{\beta}} x^\alpha \\ \partial_{\overline{\beta}} y^\alpha \end{pmatrix}, \tag{1.5}$$

which are tangent to the cross-section $\beta_V\left(T^*(M_n)\right)$.

Thus $B_{(\overline{\beta})}$ have components

$$B_{(\overline{\beta})} : \left(B^A_{(\overline{\beta})}\right) = \begin{pmatrix} \delta^\beta_\alpha \\ 0 \\ 0 \end{pmatrix} \tag{1.6}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ in $t(M_n)$, where

$$\delta^\beta_\alpha = A^\beta_\alpha = \frac{\partial x^\beta}{\partial x^\alpha}.$$

Let $\omega \in \Im^0_1(M_n)$, i.e. $\omega = \omega_\alpha dx^\alpha$. We denote by $B\omega$ the vector field with local components

$$B\omega : \left(B^A_{(\overline{\beta})}\omega_\beta\right) = \begin{pmatrix} \delta^\beta_\alpha \omega_\beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \omega_\alpha \\ 0 \\ 0 \end{pmatrix} \tag{1.7}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ in $t(M_n)$, which is defined globally along $\beta_V\,(T^*(M_n))$. Then a mapping

$$B : \Im^0_1(M_n) \to \Im^1_0(t(M_n)) \tag{1.8}$$

is defined by (1.7) and so an isomorphism of $\Im^0_1(M_n)$ in to $\Im^1_0(t(M_n))$.

Since a cross-section is locally expressed by $x^{\overline{\alpha}} = p_\alpha = const.$, $x^{\overline{\overline{\alpha}}} = y^\alpha = const.$, $x^\alpha = x^\alpha$, x^α being considered as parameters. Taking the derivative of (1.4) with respect to x^α, we have vector fields $C_{(\beta)}$ $(\beta, ... = n+1, ..., 2n)$ with components

$$C_{(\beta)} = \frac{\partial x^A}{\partial x^\beta} = \partial_\beta x^A = \begin{pmatrix} \partial_\beta \theta_\alpha \\ \partial_\beta x^\alpha \\ \partial_\beta V^\alpha \end{pmatrix}, \tag{1.9}$$

which are tangent to the cross-section $\beta_V\,(T^*(M_n))$.

Thus $C_{(\beta)}$ have components

$$C_{(\beta)} : \left(C^A_{(\beta)}\right) = \begin{pmatrix} \partial_\beta \theta_\alpha \\ \delta^\alpha_\beta \\ \partial_\beta V^\alpha \end{pmatrix} \tag{1.10}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ in $t(M_n)$, where

$$\delta^\alpha_\beta = A^\alpha_\beta = \frac{\partial x^\alpha}{\partial x^\beta}.$$

Let $X \in \Im^1_0\,(T^*(M_n))$. Then we denote by CX the vector field with local components

$$CX : \left(C^A_{(\beta)} X^\beta\right) = \begin{pmatrix} X^\beta \partial_\beta \theta_\alpha \\ X^\alpha \\ X^\beta \partial_\beta V^\alpha \end{pmatrix} \tag{1.11}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ in $t(M_n)$, which is defined globally along $\beta_V\,(T^*(M_n))$. Then a mapping

$$C : \Im^1_0(T^*(M_n)) \to \Im^1_0(\beta_V\,(T^*(M_n))) \tag{1.12}$$

is defined by (1.11). The mapping C is the differential of $\beta_V : T^*(M_n) \to t(M_n)$ and so an isomorphism of $\Im^1_0(T^*(M_n))$ onto $\Im^1_0(\beta_V\,(T^*(M_n)))$.

Now, consider vector field $X \in \Im_0^1(T^*(M_n))$, then ${}^{vv}X$ (vertical lift), ${}^{cc}X$ (complete lift) and ${}^{HH}X$ (horizontal lift) have respectively, components on the semi-tangent bundle $t(M_n)$:

$$ {}^{vv}X = \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, {}^{cc}X = \begin{pmatrix} -p_\sigma(\partial_\alpha X^\sigma) \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix}, {}^{HH}X = \begin{pmatrix} X^\beta \Gamma_{\beta\alpha} \\ X^\alpha \\ -\Gamma_\beta^\alpha X^\beta \end{pmatrix} \tag{1.13} $$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$. Where

$$ \Gamma_{\beta\alpha} = \theta_\varepsilon \Gamma_{\beta\alpha}^\varepsilon, \Gamma_\beta^\alpha = V^\varepsilon \Gamma_{\varepsilon\ \beta}^{\ \alpha}. $$

On the other hand, the fibre is locally represented by

$$ x^{\overline{\alpha}} = p_\alpha = const., x^\alpha = const., x^{\overline{\alpha}} = y^\alpha = y^\alpha, $$

y^α being considered as parameters. Thus, on differentiating with respect to y^α, we easily see that the n local vector fields $E_{\left(\overline{\overline{\beta}}\right)} = {}^{vv}(\partial_\beta)\left(\overline{\overline{\beta}}, ... = 2n+1, ..., 3n\right)$ with components

$$ E_{\left(\overline{\overline{\beta}}\right)} : \left(E^A_{\left(\overline{\overline{\beta}}\right)}\right) = \partial_{\left(\overline{\overline{\beta}}\right)} x^A = \begin{pmatrix} \partial_{\overline{\overline{\beta}}} p_\alpha \\ \partial_{\overline{\overline{\beta}}} x^\alpha \\ \partial_{\overline{\overline{\beta}}} y^\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta_\beta^\alpha \end{pmatrix} \tag{1.14} $$

is tangent to the fibre, where

$$ \delta_\beta^\alpha = A_\beta^\alpha = \frac{\partial x^\alpha}{\partial x^\beta}. $$

Let $X \in \Im_0^1(T^*(M_n))$, i.e. $X = X^\alpha \partial_\alpha$. We denote by EX the vector field with local components

$$ EX : \left(E^A_{\left(\overline{\overline{\beta}}\right)} X^\beta\right) = \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, \tag{1.15} $$

which is tangent to the fibre. Then a mapping

$$ E : \Im_0^1(T^*(M_n)) \to \Im_0^1(\beta_V(T^*(M_n))) \tag{1.16} $$

is defined by (1.15) and so an isomorphism of $\Im_0^1(T^*(M_n))$ in to $\Im_0^1(\beta_V(T^*(M_n)))$.

From (1.7), (1.11) and (1.15), we obtain

Theorem 1.1 *Let X and Y be vector fields on $T^*(M_n)$. For the Lie product, we have*

$$ \begin{aligned} (i)\,[B\psi, B\omega] &= 0, \\ (ii)\,[CX, CY] &= C[X, Y], \\ (iii)\,[EX, EY] &= 0 \end{aligned} $$

for any ψ,$\omega \in \Im_1^0(M_n)$.

Proof. (i) If $\psi,\omega \in \Im_1^0(M_n)$ and $\begin{pmatrix} [B\psi, B\omega]^{\overline{\beta}} \\ [B\psi, B\omega]^{\beta} \\ [B\psi, B\omega]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[B\psi, B\omega]^J$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t^*(M_n)$, then we have

$$\begin{aligned} [B\psi, B\omega]^J &= (B\psi)^I \partial_I (B\omega)^J - (B\omega)^I \partial_I (B\psi)^J \\ &= (B\psi)^{\overline{\alpha}} \partial_{\overline{\alpha}} (B\omega)^J + (B\psi)^{\alpha} \partial_{\alpha} (B\omega)^J + (B\psi)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (B\omega)^J \\ &\quad - (B\omega)^{\overline{\alpha}} \partial_{\overline{\alpha}} (B\psi)^J - (B\omega)^{\alpha} \partial_{\alpha} (B\psi)^J - (B\omega)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (B\psi)^J \\ &= \psi_\alpha \partial_{\overline{\alpha}} (B\omega)^J - \omega_\alpha \partial_{\overline{\alpha}} (B\psi)^J . \end{aligned}$$

Firstly, if $J = \overline{\beta}$, we have

$$\begin{aligned} [B\psi, B\omega]^{\overline{\beta}} &= \psi_\alpha \partial_{\overline{\alpha}} (B\omega)^{\overline{\beta}} - \omega_\alpha \partial_{\overline{\alpha}} (B\psi)^{\overline{\beta}} \\ &= \psi_\alpha \partial_{\overline{\alpha}} \omega_\beta - \omega_\alpha \partial_{\overline{\alpha}} \psi_\beta \\ &= 0 \end{aligned}$$

by virtue of (1.7). Secondly, if $J = \beta$, we have

$$\begin{aligned} [B\psi, B\omega]^{\beta} &= \psi_\alpha \partial_{\overline{\alpha}} (B\omega)^{\beta} - \omega_\alpha \partial_{\overline{\alpha}} (B\psi)^{\beta} \\ &= 0 \end{aligned}$$

by virtue of (1.7). Thirdly, if $J = \overline{\overline{\beta}}$. Then we have

$$\begin{aligned} [B\psi, B\omega]^{\overline{\overline{\beta}}} &= \psi_\alpha \partial_{\overline{\alpha}} (B\omega)^{\overline{\overline{\beta}}} - \omega_\alpha \partial_{\overline{\alpha}} (B\psi)^{\overline{\overline{\beta}}} \\ &= 0 \end{aligned}$$

by virtue of (1.7). Thus, we have $[B\psi, B\omega] = 0$.

(ii) If X and Y are vector field on $T^*(M_n)$ and $\begin{pmatrix} [CX, CY]^{\overline{\beta}} \\ [CX, CY]^{\beta} \\ [CX, CY]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[CX, CY]^J$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t(M_n)$, then we have

$$[CX, CY]^J = (CX)^I \partial_I (CY)^J - (CY)^I \partial_I (CX)^J .$$

Firstly, if $J = \overline{\beta}$, we have

$$\begin{aligned} [CX, CY]^{\overline{\beta}} &= (CX)^I \partial_I (CY)^{\overline{\beta}} - (CY)^I \partial_I (CX)^{\overline{\beta}} \\ &= (CX)^{\overline{\alpha}} \partial_{\overline{\alpha}} (CY)^{\overline{\beta}} + (CX)^{\alpha} \partial_{\alpha} (CY)^{\overline{\beta}} + (CX)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (CY)^{\overline{\beta}} \\ &\quad - (CY)^{\overline{\alpha}} \partial_{\overline{\alpha}} (CX)^{\overline{\beta}} - (CY)^{\alpha} \partial_{\alpha} (CX)^{\overline{\beta}} - (CY)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (CX)^{\overline{\beta}} \\ &= X^{\varepsilon} \partial_{\varepsilon} \theta_\alpha \partial_{\overline{\alpha}} Y^{\gamma} \partial_\gamma \theta_\beta + X^{\alpha} \partial_\alpha Y^{\gamma} \partial_\gamma \theta_\beta + X^{\varepsilon} \partial_{\varepsilon} V^{\alpha} \partial_{\overline{\overline{\alpha}}} Y^{\gamma} \partial_\gamma \theta_\beta \\ &\quad - Y^{\varepsilon} \partial_{\varepsilon} \theta_\beta \partial_{\overline{\alpha}} X^{\gamma} \partial_\gamma \theta_\beta - Y^{\alpha} \partial_\alpha X^{\gamma} \partial_\gamma \theta_\beta - Y^{\varepsilon} \partial_{\varepsilon} V^{\alpha} \partial_{\overline{\overline{\alpha}}} X^{\gamma} \partial_\gamma \theta_\beta \\ &= X^{\alpha} \partial_\alpha Y^{\gamma} \partial_\gamma \theta_\beta - Y^{\alpha} \partial_\alpha X^{\gamma} \partial_\gamma \theta_\beta \\ &= (X^{\alpha} \partial_\alpha Y^{\gamma} - Y^{\alpha} \partial_\alpha X^{\gamma}) \partial_\gamma \theta_\beta \\ &= [X, Y]^{\gamma} \partial_\gamma \theta_\beta \end{aligned}$$

by virtue of (1.11). Secondly, if $J=\beta$, we have

$$\begin{aligned}[CX,CY]^{\beta}&=(CX)^{I}\,\partial_I(CY)^{\beta}-(CY)^{I}\partial_I(CX)^{\beta}\\&=(CX)^{\overline{\alpha}}\,\partial_{\overline{\alpha}}(CY)^{\beta}+(CX)^{\alpha}\,\partial_{\alpha}(CY)^{\beta}+(CX)^{\overline{\overline{\alpha}}}\,\partial_{\overline{\overline{\alpha}}}(CY)^{\beta}\\&\quad-(CY)^{\overline{\alpha}}\partial_{\overline{\alpha}}(CX)^{\beta}-(CY)^{\alpha}\partial_{\alpha}(CX)^{\beta}-(CY)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(CX)^{\beta}\\&=X^{\beta}\partial_{\beta}\theta_{\alpha}\partial_{\overline{\alpha}}Y^{\beta}+X^{\alpha}\partial_{\alpha}Y^{\beta}+X^{\beta}\partial_{\beta}V^{\alpha}\partial_{\overline{\overline{\alpha}}}Y^{\beta}\\&\quad-Y^{\beta}\partial_{\beta}\theta_{\alpha}\partial_{\overline{\alpha}}X^{\beta}-Y^{\alpha}\partial_{\alpha}X^{\beta}+Y^{\beta}\partial_{\beta}V^{\alpha}\partial_{\overline{\overline{\alpha}}}X^{\beta}\\&=X^{\alpha}\partial_{\alpha}Y^{\beta}-Y^{\alpha}\partial_{\alpha}X^{\beta}\\&=[X,Y]^{\beta}\end{aligned}$$

by virtue of (1.11). Thirdly, if $J=\overline{\overline{\beta}}$, then we have

$$\begin{aligned}[CX,CY]^{\overline{\overline{\beta}}}&=(CX)^{I}\,\partial_I(CY)^{\overline{\overline{\beta}}}-(CY)^{I}\partial_I(CX)^{\overline{\overline{\beta}}}\\&=(CX)^{\overline{\alpha}}\,\partial_{\overline{\alpha}}(CY)^{\overline{\overline{\beta}}}+(CX)^{\alpha}\,\partial_{\alpha}(CY)^{\overline{\overline{\beta}}}+(CX)^{\overline{\overline{\alpha}}}\,\partial_{\overline{\overline{\alpha}}}(CY)^{\overline{\overline{\beta}}}\\&\quad-(CY)^{\overline{\alpha}}\partial_{\overline{\alpha}}(CX)^{\overline{\overline{\beta}}}-(CY)^{\alpha}\partial_{\alpha}(CX)^{\overline{\overline{\beta}}}-(CY)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(CX)^{\overline{\overline{\beta}}}\\&=X^{\beta}\partial_{\beta}\theta_{\alpha}\partial_{\overline{\alpha}}Y^{\gamma}\partial_{\gamma}V^{\beta}+X^{\alpha}\partial_{\alpha}Y^{\gamma}\partial_{\gamma}V^{\beta}+X^{\beta}\partial_{\beta}V^{\alpha}\partial_{\overline{\overline{\alpha}}}Y^{\gamma}\partial_{\gamma}V^{\beta}\\&\quad-Y^{\beta}\partial_{\beta}\theta_{\alpha}\partial_{\overline{\alpha}}X^{\gamma}\partial_{\gamma}V^{\beta}-Y^{\alpha}\partial_{\alpha}X^{\gamma}\partial_{\gamma}V^{\beta}-Y^{\beta}\partial_{\beta}V^{\alpha}\partial_{\overline{\overline{\alpha}}}X^{\gamma}\partial_{\gamma}V^{\beta}\\&=(X^{\alpha}\partial_{\alpha}Y^{\gamma}-Y^{\alpha}\partial_{\alpha}X^{\gamma})\,\partial_{\gamma}V^{\beta}\\&=[X,Y]^{\gamma}\,\partial_{\gamma}V^{\beta}\end{aligned}$$

by virtue of (1.11). On the other hand, we know that $C\,[X,Y]$ have components

$$C\,[X,Y]=\begin{pmatrix}[X,Y]^{\gamma}\,\partial_{\gamma}\theta_{\beta}\\ [X,Y]^{\beta}\\ [X,Y]^{\gamma}\,\partial_{\gamma}V^{\beta}\end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\beta}},x^{\beta},x^{\overline{\overline{\beta}}})$ in $t(M_n)$. Thus, we have $[CX,CY]=C\,[X,Y]$.

(iii) $X,Y\in\Im_0^1(T^*(M_n))$ and $\begin{pmatrix}[EX,EY]^{\overline{\beta}}\\ [EX,EY]^{\beta}\\ [EX,EY]^{\overline{\overline{\beta}}}\end{pmatrix}$ are components of $[EX,EY]^{J}$ with respect to the coordinates $(x^{\overline{\beta}},x^{\beta},x^{\overline{\overline{\beta}}})$ in $t(M_n)$, then we have

$$[EX,EY]^{J}=(EX)^{I}\,\partial_I(EY)^{J}-(EY)^{I}\partial_I(EX)^{J}.$$

Firstly, if $J=\overline{\beta}$, we have

$$\begin{aligned}[EX,EY]^{\overline{\beta}}&=(EX)^{I}\,\partial_I(EY)^{\overline{\beta}}-(EY)^{I}\partial_I(EX)^{\overline{\beta}}\\&=(EX)^{\overline{\alpha}}\,\partial_{\overline{\alpha}}(EY)^{\overline{\beta}}+(EX)^{\alpha}\,\partial_{\alpha}(EY)^{\overline{\beta}}+(EX)^{\overline{\overline{\alpha}}}\,\partial_{\overline{\overline{\alpha}}}(EY)^{\overline{\beta}}\\&\quad-(EY)^{\overline{\alpha}}\partial_{\overline{\alpha}}(EX)^{\overline{\beta}}-(EY)^{\alpha}\partial_{\alpha}(EX)^{\overline{\beta}}-(EY)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(EX)^{\overline{\beta}}\\&=0\end{aligned}$$

by virtue of (1.15). Secondly, if $J=\beta$, we have

$$\begin{aligned}[EX,EY]^{\beta} &= (EX)^{I}\,\partial_{I}(EY)^{\beta}-(EY)^{I}\partial_{I}(EX)^{\beta}\\ &= (EX)^{\overline{\alpha}}\,\partial_{\overline{\alpha}}(EY)^{\beta}+(EX)^{\alpha}\,\partial_{\alpha}(EY)^{\beta}+(EX)^{\overline{\overline{\alpha}}}\,\partial_{\overline{\overline{\alpha}}}(EY)^{\beta}\\ &\quad -(EY)^{\overline{\alpha}}\partial_{\overline{\alpha}}(EX)^{\beta}-(EY)^{\alpha}\partial_{\alpha}(EX)^{\beta}-(EY)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(EX)^{\beta}\\ &= 0\end{aligned}$$

by virtue of (1.15). Thirdly, if $J=\overline{\overline{\beta}}$, then we have

$$\begin{aligned}[EX,EY]^{\overline{\overline{\beta}}} &= (EX)^{I}\,\partial_{I}(EY)^{\overline{\overline{\beta}}}-(EY)^{I}\partial_{I}(EX)^{\overline{\overline{\beta}}}\\ &= (EX)^{\overline{\alpha}}\,\partial_{\overline{\alpha}}(EY)^{\overline{\overline{\beta}}}+(EX)^{\alpha}\,\partial_{\alpha}(EY)^{\overline{\overline{\beta}}}+(EX)^{\overline{\overline{\alpha}}}\,\partial_{\overline{\overline{\alpha}}}(EY)^{\overline{\overline{\beta}}}\\ &\quad -(EY)^{\overline{\alpha}}\partial_{\overline{\alpha}}(EX)^{\overline{\overline{\beta}}}-(EY)^{\alpha}\partial_{\alpha}(EX)^{\overline{\overline{\beta}}}-(EY)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(EX)^{\overline{\overline{\beta}}}\\ &= X^{\alpha}\partial_{\overline{\overline{\alpha}}}Y^{\beta}-Y^{\alpha}\partial_{\overline{\overline{\alpha}}}X^{\beta}\\ &= 0\end{aligned}$$

by virtue of (1.15). Thus, we have $[EX,EY]=0$.

We consider in $\pi^{-1}(U)$ $3n$ local vector fields $B_{(\overline{\beta})}$, $C_{(\beta)}$ and $E_{(\overline{\overline{\beta}})}$ along $\beta_V(T^{*}(M_n))$, which are respectively represented by

$$B_{(\overline{\beta})}=Bdx^{\beta},C_{(\beta)}=C\frac{\partial}{\partial x^{\beta}},E_{(\overline{\overline{\beta}})}=E\frac{\partial}{\partial x^{\beta}}.$$

Theorem 1.2 *Let X be a vector field on $T^{*}(M_n)$. We have along $\beta_V(T^{*}(M_n))$ the formula*

$$^{cc}X=B(-L_X\theta)+CX+E(L_VX),$$

where L_VX denotes the Lie derivative of X with respect to V, and $L_X\theta$ denotes the Lie derivative of θ with respect to X.

Proof. Using (1.7), (1.11) and (1.15), we have

$$\begin{aligned}B(-L_X\theta)+CX+E(L_VX) &= \begin{pmatrix}-X^{\beta}\partial_{\beta}\theta_{\alpha}-\theta_{\beta}\partial_{\alpha}X^{\beta}\\ 0\\ 0\end{pmatrix}+\begin{pmatrix}X^{\beta}\partial_{\beta}\theta_{\alpha}\\ X^{\alpha}\\ X^{\beta}\partial_{\beta}V^{\alpha}\end{pmatrix}\\ &\quad+\begin{pmatrix}0\\ 0\\ V^{\beta}\partial_{\beta}X^{\alpha}-X^{\beta}\partial_{\beta}V^{\alpha}\end{pmatrix}\\ &= \begin{pmatrix}-\theta_{\beta}\partial_{\alpha}X^{\beta}\\ X^{\alpha}\\ V^{\beta}\partial_{\beta}X^{\alpha}\end{pmatrix}={}^{cc}X.\end{aligned}$$

Thus, we have Theorem 1.2.

On the other hand, on putting $C_{(\overline{\overline{\beta}})}=E_{(\overline{\overline{\beta}})}$, we write the adapted frame of $\beta_V(T^{*}(M_n))$ as $\left\{B_{(\overline{\beta})},C_{(\beta)},C_{(\overline{\overline{\beta}})}\right\}$. The adapted frame $\left\{B_{(\overline{\beta})},C_{(\beta)},C_{(\overline{\overline{\beta}})}\right\}$ of $\beta_V(T^{*}(M_n))$ is given by the matrix

$$\widetilde{A}=\left(\widetilde{A}_{B}^{A}\right)=\begin{pmatrix}\delta_{\alpha}^{\beta} & \partial_{\beta}\theta_{\alpha} & 0\\ 0 & \delta_{\beta}^{\alpha} & 0\\ 0 & \partial_{\beta}V^{\alpha} & \delta_{\beta}^{\alpha}\end{pmatrix}. \tag{1.17}$$

Since the matrix $\widetilde{A}$ in (1.17) is non-singular, it has the inverse. Denoting this inverse by $\left(\widetilde{A}\right)^{-1}$, we have

$$\left(\widetilde{A}\right)^{-1} = \left(\widetilde{A}_C^B\right)^{-1} = \begin{pmatrix} \delta_\beta^\theta & -\partial_\theta \theta_\beta & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & -\partial_\theta V^\beta & \delta_\theta^\beta \end{pmatrix}, \tag{1.18}$$

where $\widetilde{A}\left(\widetilde{A}\right)^{-1} = (\widetilde{A}_B^A)\left(\widetilde{A}_C^B\right)^{-1} = \delta_C^A = \widetilde{I}$, where $A = (\overline{\alpha}, \alpha, \overline{\overline{\alpha}})$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$, $C = \left(\overline{\theta}, \theta, \overline{\overline{\theta}}\right)$.

In fact, from (1.17) and (1.18), we easily see that

$$\begin{aligned} \widetilde{A}\left(\widetilde{A}\right)^{-1} &= (\widetilde{A}_B^A)\left(\widetilde{A}_C^B\right)^{-1} \\ &= \begin{pmatrix} \delta_\alpha^\beta & \partial_\beta \theta_\alpha & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & \partial_\beta V^\alpha & \delta_\beta^\alpha \end{pmatrix} \begin{pmatrix} \delta_\beta^\theta & -\partial_\theta \theta_\beta & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & -\partial_\theta V^\beta & \delta_\theta^\beta \end{pmatrix} \\ &= \begin{pmatrix} \delta_\alpha^\theta & \partial_\theta \theta_\alpha - \partial_\theta \theta_\alpha & 0 \\ 0 & \delta_\theta^\alpha & 0 \\ 0 & -\partial_\theta V^\alpha + \partial_\theta V^\alpha & \delta_\theta^\alpha \end{pmatrix} = \begin{pmatrix} \delta_\alpha^\theta & 0 & 0 \\ 0 & \delta_\theta^\alpha & 0 \\ 0 & 0 & \delta_\theta^\alpha \end{pmatrix} \\ &= \delta_C^A = \widetilde{I}. \end{aligned}$$

Then we see from Theorem 1.2 that the complete lift ${}^{cc}X$ of a vector field $X \in \Im_0^1(T^*(M_n))$ has along $\beta_V\left(T^*(M_n)\right)$ components of the form

$${}^{cc}X : \begin{pmatrix} -L_X \theta_\alpha \\ X^\alpha \\ L_V X^\alpha \end{pmatrix} \tag{1.19}$$

with respect to the adapted frame $\left\{B_{\left(\overline{\beta}\right)}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$.

$B\omega$, CX and EX also have components:

$$B\omega = \begin{pmatrix} \omega_\alpha \\ 0 \\ 0 \end{pmatrix}, CX = \begin{pmatrix} 0 \\ X^\alpha \\ 0 \end{pmatrix}, EX = \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix} \tag{1.20}$$

respectively, with respect to the adapted frame $\left\{B_{\left(\overline{\beta}\right)}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$ of the cross-section $\beta_V\left(T^*(M_n)\right)$ determined by a vector field V on $T^*(M_n)$.

2 Complete Lift of Tensor Fields of Type (1,1) on a Cross-Section in Semi-tangent Bundle

Suppose now that $F \in \Im_1^1(T^*(M_n))$ and F has local components F_β^α in a neighborhood U of M_n, $F = F_\beta^\alpha \partial_\alpha \otimes dx^\beta$. Then the semi-tangent (pull-back) bundle $t(M_n)$ of tangent bundle $T(M_n)$ by using projection of the cotangent bundle $T^*(M_n)$ admits the complete lift ${}^{cc}F$ of F with components:

$$ {}^{cc}F = ({}^{cc}F^I_J) = \begin{pmatrix} F^\beta_\alpha & p_\sigma(\partial_\beta F^\sigma_\alpha - \partial_\alpha F^\sigma_\beta) & 0 \\ 0 & F^\alpha_\beta & 0 \\ 0 & y^\varepsilon \partial_\varepsilon F^\alpha_\beta & F^\alpha_\beta \end{pmatrix} \tag{2.1} $$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ on $t(M_n)$. Then ${}^{cc}F$ has components F^A_B given by

$$ {}^{cc}F = ({}^{cc}F^A_B) = \begin{pmatrix} F^\beta_\alpha & \phi_F\theta & 0 \\ 0 & F^\alpha_\beta & 0 \\ 0 & L_V F^\alpha_\beta & F^\alpha_\beta \end{pmatrix} \tag{2.2} $$

with respect to the adapted frame $\left\{ B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)} \right\}$ of the cross-section $\beta_V\,(T^*(M_n))$ determined by a vector field V in $T^*(M_n)$, where $A = (\overline{\alpha}, \alpha, \overline{\overline{\alpha}})$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$.

Also, the component ${}^{cc}F^{\overline{\alpha}}_\beta$ of ${}^{cc}F^A_B$ is defined as Tachibana operator $\phi_F\theta$ of F, i.e.,

$$ {}^{cc}F^{\overline{\alpha}}_\beta = \phi_F\theta = (\partial_\beta F^\sigma_\alpha - \partial_\alpha F^\sigma_\beta)\theta_\sigma - F^\gamma_\beta \partial_\gamma \theta_\alpha + F^\gamma_\alpha \partial_\beta \theta_\gamma, \tag{2.3} $$

and $L_V F^\alpha_\beta$ denotes the Lie derivative of F^α_β with respect to V, i.e.,

$$ L_V F^\alpha_\beta = V^\gamma \partial_\gamma F^\alpha_\beta + F^\alpha_\gamma \partial_\beta V^\gamma - F^\gamma_\beta \partial_\gamma V^\alpha. \tag{2.4} $$

Let $F \in \Im^1_1(T^*(M_n))$. Then we have by (1.17), (1.18) and (2.1):

$$ \begin{aligned} {}^{cc}F &= \left(\widetilde{A}^B_A\right)^{-1} ({}^{cc}F^A_C) \left(\widetilde{A}^C_D\right) \\ &= \begin{pmatrix} \delta^\alpha_\beta & -\partial_\alpha\theta_\beta & 0 \\ 0 & \delta^\beta_\alpha & 0 \\ 0 & -\partial_\alpha V^\beta & \delta^\beta_\alpha \end{pmatrix} \begin{pmatrix} F^\gamma_\alpha & \theta_\sigma(\partial_\gamma F^\sigma_\alpha - \partial_\alpha F^\sigma_\gamma) & 0 \\ 0 & F^\alpha_\gamma & 0 \\ 0 & V^\varepsilon \partial_\varepsilon F^\alpha_\gamma & F^\alpha_\gamma \end{pmatrix} \begin{pmatrix} \delta^\psi_\gamma & \partial_\psi\theta_\gamma & 0 \\ 0 & \delta^\gamma_\psi & 0 \\ 0 & \partial_\psi V^\gamma & \delta^\gamma_\psi \end{pmatrix} \\ &= \begin{pmatrix} F^\gamma_\beta & -F^\alpha_\gamma \partial_\alpha\theta_\beta + \theta_\sigma\partial_\gamma F^\sigma_\beta - \theta_\sigma \partial_\beta F^\sigma_\gamma & 0 \\ 0 & F^\beta_\gamma & 0 \\ 0 & V^\varepsilon \partial_\varepsilon F^\beta_\gamma - F^\alpha_\gamma \partial_\alpha V^\beta & F^\beta_\gamma \end{pmatrix} \begin{pmatrix} \delta^\psi_\gamma & \partial_\psi\theta_\gamma & 0 \\ 0 & \delta^\gamma_\psi & 0 \\ 0 & \partial_\psi V^\gamma & \delta^\gamma_\psi \end{pmatrix} \\ &= \begin{pmatrix} F^\psi_\beta & -F^\alpha_\psi \partial_\alpha\theta_\beta + \theta_\sigma\partial_\psi F^\sigma_\beta - \theta_\sigma \partial_\beta F^\sigma_\psi + F^\gamma_\beta \partial_\psi \theta_\gamma & 0 \\ 0 & F^\beta_\psi & 0 \\ 0 & F^\beta_\gamma \partial_\psi V^\gamma + V^\varepsilon \partial_\varepsilon F^\beta_\psi - F^\alpha_\psi \partial_\alpha V^\beta & F^\beta_\psi \end{pmatrix} \\ &= \begin{pmatrix} F^\psi_\beta & \phi_F\theta & 0 \\ 0 & F^\beta_\psi & 0 \\ 0 & L_V F^\beta_\psi & F^\beta_\psi \end{pmatrix} \\ &= ({}^{cc}F^B_D), \end{aligned} $$

where $A = (\overline{\alpha}, \alpha, \overline{\overline{\alpha}})$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$, $C = (\overline{\gamma}, \gamma, \overline{\overline{\gamma}})$, $D = \left(\overline{\psi}, \psi, \overline{\overline{\psi}}\right)$.

Using (2.2), we have along $\beta_V\,(T^*(M_n))$:

Theorem 2.1 *If F and X are affinor and vector fields on $T^*(M_n)$, and $\omega \in \Im_1^0(M_n)$, then*

$$(i)^{cc}F\left(CX + EX\right) = B\left(P_X\right) + C\left(FX\right) + E\left(FX\right) + E\left(\left(L_V F\right)X\right),$$
$$(ii)^{cc}F\left(B\omega\right) = B\left(\omega \circ F\right),$$

where $P \in \Im_2^0(M_n)$ with local components

$$P_{\beta\alpha} = \phi_F\theta = (\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma)\theta_\sigma - F_\beta^\gamma \partial_\gamma \theta_\alpha + F_\alpha^\gamma \partial_\beta \theta_\gamma,$$

θ_β being local components of θ, and $P_X \in \Im_1^0(M_n)$ defined by $P_X\left(Y\right) = P\left(X,Y\right)$, for $Y \in \Im_0^1(T^*(M_n))$.

Proof. (i) If F and X are affinor and vector fields on $T^*(M_n)$, then by (1.20) and (2.2), we have

$$\begin{aligned}
{}^{cc}F\left(CX + EX\right) &= \begin{pmatrix} F_\alpha^\beta & \phi_F\theta & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & L_V F_\beta^\alpha & F_\beta^\alpha \end{pmatrix} \begin{pmatrix} 0 \\ X^\beta \\ X^\beta \end{pmatrix} \\
&= \begin{pmatrix} X^\beta \partial_\beta F_\alpha^\sigma \theta_\sigma - X^\beta \partial_\alpha F_\beta^\sigma \theta_\sigma - F_\beta^\gamma X^\beta \partial_\gamma \theta_\alpha + F_\alpha^\gamma X^\beta \partial_\beta \theta_\gamma \\ F_\beta^\alpha X^\beta \\ F_\beta^\alpha X^\beta + L_V F_\beta^\alpha X^\beta \end{pmatrix} \\
&= \begin{pmatrix} X^\beta \partial_\beta F_\alpha^\sigma \theta_\sigma - X^\beta \partial_\alpha F_\beta^\sigma \theta_\sigma - F_\beta^\gamma X^\beta \partial_\gamma \theta_\alpha + F_\alpha^\gamma X^\beta \partial_\beta \theta_\gamma \\ (FX)^\alpha \\ (FX)^\alpha + V^\gamma \partial_\gamma F_\beta^\alpha X^\beta + F_\gamma^\alpha \partial_\beta V^\gamma X^\beta - F_\beta^\gamma \partial_\gamma V^\alpha X^\beta \end{pmatrix} \\
&= \begin{pmatrix} P_X \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (FX)^\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ (FX)^\alpha \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \left(L_V F\right)X \end{pmatrix} \\
&= B\left(P_X\right) + C\left(FX\right) + E\left(FX\right) + E\left(\left(L_V F\right)X\right).
\end{aligned}$$

Thus, we have (i) of Theorem 2.1.

(ii) If $\omega \in \Im_1^0(M_n)$, F is an affinor fields on $T^*(M_n)$, then by (1.20) and (2.2), we have

$$\begin{aligned}
{}^{cc}F\left(B\omega\right) &= \begin{pmatrix} F_\alpha^\beta & \phi_F\theta & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & L_V F_\beta^\alpha & F_\beta^\alpha \end{pmatrix} \begin{pmatrix} \omega_\beta \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \omega_\beta F_\alpha^\beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \left(\omega \circ F\right)_\alpha \\ 0 \\ 0 \end{pmatrix} \\
&= B\left(\omega \circ F\right),
\end{aligned}$$

which gives equation (ii) of Theorem 2.1.

When ${}^{cc}F\left(CX + EX\right)$ is always tangent to $\beta_V\left(T^*(M_n)\right)$ for any vector field $X \in \Im_0^1\left(T^*(M_n)\right)$, ${}^{cc}F$ is said to leave the cross-section $\beta_V\left(T^*(M_n)\right)$ invariant.

Thus we have

Theorem 2.2 *The complete lift ${}^{cc}F$ of an element of $F \in \Im_1^1(T^*(M_n))$ leaves the cross-section $\beta_V\left(T^*(M_n)\right)$ invariant if and only if*
(i) $(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma)\theta_\sigma - F_\beta^\gamma \partial_\gamma \theta_\alpha + F_\alpha^\gamma \partial_\beta \theta_\gamma = 0$ (i.e. $\phi_F\theta = 0$),
(ii) $V^\gamma \partial_\gamma F_\beta^\alpha + F_\gamma^\alpha \partial_\beta V^\gamma - F_\beta^\gamma \partial_\gamma V^\alpha = 0$ (i.e. $L_V F = 0$),
where F_β^α,θ_β and V^α are local components of F,θ and V respectively.

3 Adapted Frames and Diagonal Lifts of Affinor Fields

Let ∇ be a symmetric affine connection in M_n. In each coordinate neighborhood $\{U, x^\alpha\}$ of M_n, we put

$$X_{(\alpha)} = \frac{\partial}{\partial x^\alpha}, \theta^{(\alpha)} = dx^\alpha. \tag{3.1}$$

Then $3n$ local vector fields $\theta^{(\alpha)}$, ${}^{HH}X_{(\alpha)}$ and ${}^{vv}X_{(\alpha)}$ have respectively components of the form

$$\theta^{(\alpha)}\colon \begin{pmatrix} \delta_\alpha^\beta \\ 0 \\ 0 \end{pmatrix}, {}^{HH}X_{(\alpha)}\colon \begin{pmatrix} -\Gamma_\beta^\alpha \\ \delta_\alpha^\beta \\ \Gamma_{\beta\alpha} \end{pmatrix}, {}^{vv}X_{(\alpha)}\colon \begin{pmatrix} 0 \\ 0 \\ \delta_\beta^\alpha \end{pmatrix} \tag{3.2}$$

with respect to the induced coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ in $\pi^{-1}(U)$, where we have used (1.13). We call the set $\left\{\theta^{(\alpha)}, {}^{HH}X_{(\alpha)}, {}^{vv}X_{(\alpha)}\right\}$ the frame adapted to the symmetric affine connection ∇ in $\pi^{-1}(U)$. On putting

$$\widehat{e}_{(\overline{\alpha})} = \theta^{(\alpha)}, \widehat{e}_{(\alpha)} = {}^{HH}X_{(\alpha)}, \widehat{e}_{(\overline{\overline{\alpha}})} = {}^{vv}X_{(\alpha)}, \tag{3.3}$$

we write the adapted frame as

$$\left\{\widehat{e}_{(B)}\right\} = \left\{\widehat{e}_{(\overline{\alpha})}, \widehat{e}_{(\alpha)}, \widehat{e}_{(\overline{\overline{\alpha}})}\right\}. \tag{3.4}$$

The adapted frame $\left\{\widehat{e}_{(B)}\right\} = \left\{\widehat{e}_{(\overline{\alpha})}, \widehat{e}_{(\alpha)}, \widehat{e}_{(\overline{\overline{\alpha}})}\right\}$ is given by the matrix

$$\widehat{A} = \left(\widehat{A}_B^A\right) = \begin{pmatrix} \delta_\alpha^\beta & \Gamma_{\beta\alpha} & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & -\Gamma_\beta^\alpha & \delta_\beta^\alpha \end{pmatrix}. \tag{3.5}$$

Since the matrix $\widehat{A}$ in (3.5) is non-singular, it has the inverse. Denoting this inverse by $\left(\widehat{A}\right)^{-1}$, we have

$$\left(\widehat{A}\right)^{-1} = \left(\widehat{A}_C^B\right)^{-1} = \begin{pmatrix} \delta_\beta^\theta & -\Gamma_{\theta\beta} & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & \Gamma_\theta^\beta & \delta_\theta^\beta \end{pmatrix}, \tag{3.6}$$

where $\widehat{A}\left(\widehat{A}\right)^{-1} = (\widehat{A}_B^A)\left(\widehat{A}_C^B\right)^{-1} = \delta_C^A = \widetilde{I}$, where $A = (\overline{\alpha}, \alpha, \overline{\overline{\alpha}})$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$, $C = \left(\overline{\theta}, \theta, \overline{\overline{\theta}}\right)$.

In fact, from (3.5) and (3.6), we easily see that

$$\begin{aligned} \widehat{A}\left(\widehat{A}\right)^{-1} = (\widehat{A}_B^A)\left(\widehat{A}_C^B\right)^{-1} &= \begin{pmatrix} \delta_\alpha^\beta & \Gamma_{\beta\alpha} & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & -\Gamma_\beta^\alpha & \delta_\beta^\alpha \end{pmatrix} \begin{pmatrix} \delta_\beta^\theta & -\Gamma_{\theta\beta} & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & \Gamma_\theta^\beta & \delta_\theta^\beta \end{pmatrix} \\ &= \begin{pmatrix} \delta_\alpha^\theta & \Gamma_{\theta\alpha} - \Gamma_{\theta\alpha} & 0 \\ 0 & \delta_\theta^\alpha & 0 \\ 0 & \Gamma_\theta^\alpha - \Gamma_\theta^\alpha & \delta_\theta^\alpha \end{pmatrix} = \begin{pmatrix} \delta_\alpha^\theta & 0 & 0 \\ 0 & \delta_\theta^\alpha & 0 \\ 0 & 0 & \delta_\theta^\alpha \end{pmatrix} \\ &= \delta_C^A = \widehat{I}. \end{aligned}$$

If we take account of (3.4), we see that the diagonal lift ${}^{DD}F$ of $F \in \Im_1^1(T^*(M_n))$ has components

$$ {}^{DD}F = ({}^{DD}F_J^I) = \begin{pmatrix} -F_\alpha^\beta & \Gamma_{\beta\sigma}F_\alpha^\sigma + \Gamma_{\alpha\sigma}F_\beta^\sigma & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & -\Gamma_\varepsilon^\alpha F_\beta^\varepsilon - \Gamma_\beta^\varepsilon F_\varepsilon^\alpha & -F_\beta^\alpha \end{pmatrix}, \tag{3.7} $$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ on $t(M_n)$, where

$$ \Gamma_{\alpha\sigma} = p_\gamma \Gamma_{\alpha\sigma}^\gamma, \Gamma_\varepsilon^\alpha = y^\gamma \Gamma_{\gamma\varepsilon}^\alpha. $$

Let $F \in \Im_1^1(T^*(M_n))$. Then we have by (3.5), (3.6) and (3.7):

$$ \begin{aligned} {}^{DD}F &= \left(\widehat{A}\right)\left({}^{DD}F\right)\left(\widehat{A}\right)^{-1} \\ &= \begin{pmatrix} \delta_\beta^\alpha & \Gamma_{\alpha\beta} & 0 \\ 0 & \delta_\alpha^\beta & 0 \\ 0 & -\Gamma_\alpha^\beta & \delta_\alpha^\beta \end{pmatrix} \begin{pmatrix} -F_\alpha^\gamma & \Gamma_{\gamma\sigma}F_\alpha^\sigma + \Gamma_{\alpha\sigma}F_\gamma^\sigma & 0 \\ 0 & F_\gamma^\alpha & 0 \\ 0 & -\Gamma_\varepsilon^\alpha F_\gamma^\varepsilon - \Gamma_\gamma^\varepsilon F_\varepsilon^\alpha & -F_\gamma^\alpha \end{pmatrix} \begin{pmatrix} \delta_\gamma^\psi & -\Gamma_{\psi\gamma} & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & \Gamma_\psi^\gamma & \delta_\psi^\gamma \end{pmatrix} \\ &= \begin{pmatrix} -F_\beta^\gamma & \Gamma_{\alpha\beta}F_\gamma^\alpha + \Gamma_{\gamma\sigma}F_\beta^\sigma + \Gamma_{\beta\sigma}F_\gamma^\sigma & 0 \\ 0 & F_\gamma^\beta & 0 \\ 0 & -\Gamma_\varepsilon^\beta F_\gamma^\varepsilon - \Gamma_\gamma^\varepsilon F_\varepsilon^\beta - \Gamma_\alpha^\beta F_\gamma^\alpha & -F_\gamma^\beta \end{pmatrix} \begin{pmatrix} \delta_\gamma^\psi & -\Gamma_{\psi\gamma} & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & \Gamma_\psi^\gamma & \delta_\psi^\gamma \end{pmatrix} \\ &= \begin{pmatrix} -F_\beta^\psi & \Gamma_{\alpha\beta}F_\psi^\alpha + \Gamma_{\psi\sigma}F_\beta^\sigma + \Gamma_{\beta\sigma}F_\psi^\sigma + \Gamma_{\psi\gamma}F_\beta^\gamma & 0 \\ 0 & F_\psi^\beta & 0 \\ 0 & -\Gamma_\psi^\gamma F_\gamma^\beta - \Gamma_\varepsilon^\beta F_\psi^\varepsilon - \Gamma_\psi^\varepsilon F_\varepsilon^\beta - \Gamma_\alpha^\beta F_\psi^\alpha & -F_\psi^\beta \end{pmatrix} \\ &= \begin{pmatrix} -F_\beta^\psi & \Gamma_{\psi\mu}F_\beta^\mu + \Gamma_{\beta\mu}F_\psi^\mu & 0 \\ 0 & F_\psi^\beta & 0 \\ 0 & -\Gamma_\rho^\beta F_\psi^\rho - \Gamma_\psi^\rho F_\rho^\beta & -F_\psi^\beta \end{pmatrix}, \end{aligned} $$

which proves (3.7).

We now see, from (3.4), that the diagonal lift ${}^{DD}F$ of $F \in \Im_1^1(T^*(M_n))$ has components of the form

$$ {}^{DD}F = ({}^{DD}F_B^A) = \begin{pmatrix} -F_\alpha^\beta & 0 & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & 0 & -F_\beta^\alpha \end{pmatrix} $$

with respect to the adapted frame $\left\{\widehat{e}_{(B)}\right\}$ in $t(M_n)$.

Let $F \in \Im_1^1(T^*(M_n))$. Then we have by (3.5), (3.6) and (3.7):

$$
\begin{aligned}
{}^{DD}F &= \left(\widehat{A}\right)^{-1}\left({}^{DD}F\right)\left(\widehat{A}\right) \\
&= \begin{pmatrix} \delta_\beta^\alpha & -\Gamma_{\alpha\beta} & 0 \\ 0 & \delta_\alpha^\beta & 0 \\ 0 & \Gamma_\alpha^\beta & \delta_\alpha^\beta \end{pmatrix}
\begin{pmatrix} -F_\alpha^\gamma & \Gamma_{\gamma\sigma}F_\alpha^\sigma + \Gamma_{\alpha\sigma}F_\gamma^\sigma & 0 \\ 0 & F_\gamma^\alpha & 0 \\ 0 & -\Gamma_\varepsilon^\alpha F_\gamma^\varepsilon - \Gamma_\gamma^\varepsilon F_\varepsilon^\alpha & -F_\gamma^\alpha \end{pmatrix}
\begin{pmatrix} \delta_\gamma^\psi & \Gamma_{\psi\gamma} & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & -\Gamma_\psi^\gamma & \delta_\psi^\gamma \end{pmatrix} \\
&= \begin{pmatrix} -F_\beta^\gamma & -\Gamma_{\alpha\beta}F_\gamma^\alpha + \Gamma_{\gamma\sigma}F_\beta^\sigma + \Gamma_{\beta\sigma}F_\gamma^\sigma & 0 \\ 0 & F_\gamma^\beta & 0 \\ 0 & -\Gamma_\varepsilon^\beta F_\gamma^\varepsilon - \Gamma_\gamma^\varepsilon F_\varepsilon^\beta + \Gamma_\alpha^\beta F_\gamma^\alpha & -F_\gamma^\beta \end{pmatrix}
\begin{pmatrix} \delta_\gamma^\psi & \Gamma_{\psi\gamma} & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & -\Gamma_\psi^\gamma & \delta_\psi^\gamma \end{pmatrix} \\
&= \begin{pmatrix} -F_\beta^\psi & -\Gamma_{\alpha\beta}F_\psi^\alpha + \Gamma_{\psi\sigma}F_\beta^\sigma + \Gamma_{\beta\sigma}F_\psi^\sigma - \Gamma_{\psi\gamma}F_\beta^\gamma & 0 \\ 0 & F_\psi^\beta & 0 \\ 0 & \Gamma_\psi^\gamma F_\gamma^\beta - \Gamma_\varepsilon^\beta F_\psi^\varepsilon - \Gamma_\psi^\varepsilon F_\varepsilon^\beta + \Gamma_\alpha^\beta F_\psi^\alpha & -F_\psi^\beta \end{pmatrix} \\
&= \begin{pmatrix} -F_\beta^\psi & 0 & 0 \\ 0 & F_\psi^\beta & 0 \\ 0 & 0 & -F_\psi^\beta \end{pmatrix}.
\end{aligned}
$$

We now obtain from (3.7) that the diagonal lift ${}^{DD}F$ of an affinor field $F \in \Im_1^1(T^*(M_n))$ has along $\beta_V\,(T^*(M_n))$ components of the form

$$
{}^{DD}F\colon \begin{pmatrix} -F_\alpha^\beta & -(\nabla_\beta\theta_\sigma)\,F_\alpha^\sigma - (\nabla_\alpha\theta_\sigma)\,F_\beta^\sigma & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & -(\nabla_\varepsilon V^\alpha)\,F_\beta^\varepsilon - (\nabla_\beta V^\varepsilon)\,F_\varepsilon^\alpha & -F_\beta^\alpha \end{pmatrix}, \tag{3.8}
$$

with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$.

Let $F \in \Im_1^1(T^*(M_n))$. Then we have by (1.17), (1.18) and (3.8):

$$
\begin{aligned}
{}^{DD}F &= \left(\widetilde{A}\right)^{-1}\left({}^{DD}F\right)\left(\widetilde{A}\right) \\
&= \begin{pmatrix} \delta_\beta^\alpha & -\partial_\alpha\theta_\beta & 0 \\ 0 & \delta_\alpha^\beta & 0 \\ 0 & -\partial_\alpha V^\beta & \delta_\alpha^\beta \end{pmatrix}
\begin{pmatrix} -F_\alpha^\gamma & \Gamma_{\gamma\sigma}F_\alpha^\sigma + \Gamma_{\alpha\sigma}F_\gamma^\sigma & 0 \\ 0 & F_\gamma^\alpha & 0 \\ 0 & -\Gamma_\varepsilon^\alpha F_\gamma^\varepsilon - \Gamma_\gamma^\varepsilon F_\varepsilon^\alpha & -F_\gamma^\alpha \end{pmatrix}
\begin{pmatrix} \delta_\gamma^\psi & \partial_\psi\theta_\gamma & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & \partial_\psi V^\gamma & \delta_\psi^\gamma \end{pmatrix} \\
&= \begin{pmatrix} -F_\beta^\gamma & -\partial_\alpha\theta_\beta F_\gamma^\alpha + \Gamma_{\gamma\sigma}F_\beta^\sigma + \Gamma_{\beta\sigma}F_\gamma^\sigma & 0 \\ 0 & F_\gamma^\beta & 0 \\ 0 & -\Gamma_\varepsilon^\beta F_\gamma^\varepsilon - \Gamma_\gamma^\varepsilon F_\varepsilon^\beta - \partial_\alpha V^\beta F_\gamma^\alpha & -F_\gamma^\beta \end{pmatrix}
\begin{pmatrix} \delta_\gamma^\psi & \partial_\psi\theta_\gamma & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & \partial_\psi V^\gamma & \delta_\psi^\gamma \end{pmatrix} \\
&= \begin{pmatrix} -F_\beta^\psi & -\partial_\alpha\theta_\beta F_\psi^\alpha + \Gamma_{\psi\sigma}F_\beta^\sigma + \Gamma_{\beta\sigma}F_\psi^\sigma - \partial_\psi\theta_\gamma F_\beta^\gamma & 0 \\ 0 & F_\psi^\beta & 0 \\ 0 & -\partial_\psi V^\gamma F_\gamma^\beta - \Gamma_\varepsilon^\beta F_\psi^\varepsilon - \Gamma_\psi^\varepsilon F_\varepsilon^\beta - \partial_\alpha V^\beta F_\psi^\alpha & -F_\psi^\beta \end{pmatrix} \\
&= \begin{pmatrix} -F_\beta^\psi & -(\nabla_\psi\theta_\gamma)\,F_\beta^\gamma - (\nabla_\beta\theta_\sigma)\,F_\psi^\sigma & 0 \\ 0 & F_\psi^\beta & 0 \\ 0 & -\left(\nabla_\gamma V^\beta\right)F_\psi^\gamma - (\nabla_\psi V^\gamma)\,F_\gamma^\beta & -F_\psi^\beta \end{pmatrix}.
\end{aligned}
$$

Then we see from (1.13) that the horizontal lift ${}^{HH}X$ of a vector field $X \in \Im_0^1(T^*(M_n))$ has along $\beta_V(T^*(M_n))$ components of the form

$$ {}^{HH}X \colon \begin{pmatrix} -(\nabla_\beta \theta_\alpha) X^\beta \\ X^\alpha \\ -X^\beta (\nabla_\beta V^\alpha) \end{pmatrix} \tag{3.9} $$

with respect to the adapted frame $\left\{ B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)} \right\}$.

Let $X \in \Im_0^1(T^*(M_n))$. Then we have by (1.13) and (1.18):

$$ \begin{aligned} {}^{HH}X &= \left(\widetilde{A}\right)^{-1} \left({}^{HH}X\right) \\ &= \begin{pmatrix} \delta_\alpha^\beta & -\partial_\beta \theta_\alpha & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & -\partial_\beta V^\alpha & \delta_\beta^\alpha \end{pmatrix} \begin{pmatrix} X^\alpha \theta_\varepsilon \Gamma_{\alpha\beta}^\varepsilon \\ X^\beta \\ -V^\varepsilon \Gamma_{\varepsilon\ \alpha}^{\beta} X^\alpha \end{pmatrix} \\ &= \begin{pmatrix} -\partial_\beta \theta_\alpha X^\beta + X^\theta \theta_\varepsilon \Gamma_{\theta\,\alpha}^\varepsilon \\ X^\alpha \\ -V^\varepsilon \Gamma_{\varepsilon\ \theta}^{\beta} X^\theta - \partial_\beta V^\alpha X^\beta \end{pmatrix} = \begin{pmatrix} -(\nabla_\beta \theta_\alpha) X^\beta \\ X^\alpha \\ -X^\beta (\nabla_\beta V^\alpha) \end{pmatrix}, \end{aligned} $$

which gives (3.9).

Using (1.13), (3.8) and (3.9), we have along $\beta_V(T^*(M_n))$:

Theorem 3.1 *If F and X are affinor and vector fields on $T^*(M_n)$, and $\omega \in \Im_1^0(M_n)$, then with respect to a symetric affine connection ∇ in M_n, we have*

$$ (i)\,{}^{DD}F\left({}^{HH}X\right) = {}^{HH}(FX), $$
$$ (ii)\,{}^{DD}F(B\omega) = -B(\omega \circ F). $$

Proof. (i) *If $F \in \Im_1^1(T^*(M_n))$ and $X \in \Im_0^1(T^*(M_n))$*, then by (3.8) and (3.9), we have

$$ \begin{aligned} {}^{DD}F\left({}^{HH}X\right) &= \begin{pmatrix} -F_\alpha^\beta & -(\nabla_\beta \theta_\sigma) F_\alpha^\sigma - (\nabla_\alpha \theta_\sigma) F_\beta^\sigma & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & -(\nabla_\varepsilon V^\alpha) F_\beta^\varepsilon - (\nabla_\beta V^\varepsilon) F_\varepsilon^\alpha & -F_\beta^\alpha \end{pmatrix} \begin{pmatrix} -(\nabla_\sigma \theta_\beta) X^\sigma \\ X^\beta \\ -X^\varepsilon \left(\nabla_\varepsilon V^\beta\right) \end{pmatrix} \\ &= \begin{pmatrix} -(\nabla_\alpha \theta_\sigma) F_\beta^\sigma X^\beta - (\nabla_\beta \theta_\sigma) F_\alpha^\sigma X^\beta + (\nabla_\sigma \theta_\beta) X^\sigma F_\alpha^\beta \\ (FX)^\alpha \\ F_\beta^\alpha X^\varepsilon \left(\nabla_\varepsilon V^\beta\right) - (\nabla_\varepsilon V^\alpha) F_\beta^\varepsilon X^\beta - (\nabla_\beta V^\varepsilon) F_\varepsilon^\alpha X^\beta \end{pmatrix} \\ &= \begin{pmatrix} -(\nabla_\sigma \theta_\alpha)(FX)^\sigma \\ (FX)^\alpha \\ -(\nabla_\varepsilon V^\alpha)(FX)^\varepsilon \end{pmatrix} \\ &= {}^{HH}(FX). \end{aligned} $$

Thus, we have ${}^{DD}F\left({}^{HH}X\right) = {}^{HH}(FX)$.

(ii) If $\omega \in \Im_1^0(M_n)$ and $F \in \Im_1^1(T^*(M_n))$, then by (1.20) and (3.8), we have

$$ \begin{aligned} {}^{DD}F(B\omega) &= \begin{pmatrix} -F_\alpha^\beta & -(\nabla_\beta \theta_\sigma) F_\alpha^\sigma - (\nabla_\alpha \theta_\sigma) F_\beta^\sigma & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & -(\nabla_\varepsilon V^\alpha) F_\beta^\varepsilon - (\nabla_\beta V^\varepsilon) F_\varepsilon^\alpha & -F_\beta^\alpha \end{pmatrix} \begin{pmatrix} \omega_\beta \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\omega_\beta F_\alpha^\beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -(\omega \circ F)_\alpha \\ 0 \\ 0 \end{pmatrix} \\ &= -B(\omega \circ F). \end{aligned} $$

Thus, we have (ii) of Theorem 3.1.

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