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On local componentwise equiconvergence for one-dimensional Dirac operator

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Abstract. *In the paper one-dimensional Dirac operator with a real-valued potential on the interval* $G = (0; 2\pi)$ *is considered. On any compact* $K \subset G$*, componentwise equiconvergence of orthogonal expansion in eigen-functions of this operator with trigonometric expansion of the function from the class* $L_2^2(G)$ *in the metric* $L_s,\ s \geq 1$*, is proved.*

Keywords. Dirac operator, eigen function, equiconvergence, orthogonal expansion.

Mathematics Subject Classification (2010): 34L10, 42A20

1 Introduction and formulation of results.

It is known that in [2-3], V.A. Il'in has developed a method for establishing uniform equiconvergence of spectral expansions responding to differential operators. This method was modified in [4] and allowed to establish componentwise uniform equiconvergence in the case of the Schrodinger operator with a matrix potential. Later componentwise uniform equiconvergence for an arbitrary order operator was established in [5]. The rate of componentwise equiconvergence was studied in [6] and the estimations for the velocity of componentwise equiconvergence of spectral expansions of the vector-function $f(x)$ in eigen and associated functions of an arbitrary order differential operator with expansion in trigonometric Fourier series of its component in the metric $L_s,\ s \geq 1$ on any compact of the main interval G were obtained. Componentwise uniform equiconvergence of spectral expansions in the case of the Dirac operator was studied in [7], [8]. In the given paper componentwise equiconvergence of orthogonal expansion for the Dirac operator in the metric $L_s,\ s \geq 1$, is studied on a compact and a sufficient condition providing equiconvergence in these metrics is established.

On the interval $G = (0, 2\pi)$ we consider one-dimensional Dirac operator

$$Dy = By' + P(x)y, \quad y(x) = (y_1(x),\ y_2(x))^T\,,$$

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where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $P(x) = diag\,(p(x),\ q(x))$, $p(x)$ and $q(x)$ are real-valued functions from the class $L_\alpha(G),\ \alpha \geq 1$.

Following V.A. Il'in [2] under the eigen vector-function of the operator D responding to the real eigen-value λ, we will understand any identically nonzero vector-function $y(x)$ that is absolutely continuous on $\overline{G} = [0, 2\pi]$ and almost everywhere in G satisfies the equation $Dy = \lambda y$.

Let $L_p^2(G),\ p \geq 1$, be a space of two-component vector-functions $f(x) = (f_1(x),\ f_2(x))^T$ with the norm $\|f\|_{p,2,G} \equiv \|f\|_{p,2} = \left(\int_G |f(x)|^p\, dx\right)^{1/p},\ \left(\|f\|_{\infty,2} = \sup_G vrai\,|f(x)|\right)$. Obviously, for $f(x) \in L_p^2(G),\ g(x) \in L_q^2(G),\ \ p^{-1} + q^{-1} = 1,\ p \geq 1$ there exists "scalar product"

$$(f, g) = \int_G \langle f, g\rangle\, dx = \int_G \sum_{j=1}^{2} f_j(x)\overline{g_j(x)} dx.$$

Let $\{u_n(x)\}_{n=1}^\infty$ be a complete, orthonormalized in $L_2^2(G)$ system of eigen vector-functions of the operator D, and $\{\lambda_n\}_{n=1}^\infty,\ \lambda_n \in R$, be appropriate system of eigenvalues.

Denote the partial sum of orthogonal expansion of the function $f(x) \in L_2^2(G)$ in the system of eigen vector-functions $\{u_n(x)\}_{n=1}^\infty$ by $\sigma_\nu(x, f)$:

$$\sigma_\nu(x, f) = \left(\sigma_\nu^1(x, f),\ \sigma_\nu^2(x, f)\right),\ \ \sigma_\nu^j(x, f) = \sum_{|\lambda_n| \leq \nu} (f, u_n) u_n^j(x), \quad (j = 1, 2)$$

$$u_n(x) = \left(u_n^1(x),\ \ u_n^2(x)\right)^T, \quad f(x) = (f_1(x),\ f_2(x))^T.$$

Along with partial sum $\sigma_\nu^j(x, f)$ we introduce a modified partial sum of trigonomteric series of the function $f_j(x)$, i.e.

$$S_\nu(x, f_j) - \frac{1}{\pi}\int_G \frac{\sin \nu(x-y)}{x-y} f_j(y) dy, \ \ j = 1, 2.$$

If on the compact $K \subset G$ the difference $\left\|\sigma_\nu^j(\cdot, f) - S_\nu(\cdot, f_j)\right\|_{L_s(K)}$ tends to zero as $\nu \to +\infty$, we say that the j-th component of the expansion of the function $f(x)$ in orthogonal series in the system $\{u_n(x)\}_{n=1}^\infty$ equinconverges on the compact $K \subset G$ with expansion in trigonometric series corresponding to the j-th component $f_j(x)$ of the vector-function $f(x)$ in the metric L_s.

The following theorem on componentwise equiconvergence on a compact is the main result of this paper.

Theorem 1.1 *Let $f(x) \in L_2^2(G)$, the coefficients $p(x)$ and $q(x)$ belong to $L_\alpha(G),\ \alpha \geq 1$ and the condition*

$$\frac{1}{\alpha} - \frac{1}{s} < \frac{1}{2} \tag{1.1}$$

be fulfilled.

Then on any compact $K \subset G$ the equality

$$\lim_{\nu \to +\infty} \left\|\sigma_\nu^j(\cdot, f) - S_\nu(\cdot, f_j)\right\|_{L_s(K)} = 0 \tag{1.2}$$

is valid. $j = 1, 2$, *i.e. the* j*-th component of expansion of the arbitrary function* $f(x) \in L_2^2(G)$ *in orthogonal series in the system* $\{u_n(x)\}_{n=1}^{\infty}$ *equiconverges on any compact* $K \subset G$ *with trigonometric series of the* j*-th component* $f_j(x)$ *of the vector-function* $f(x)$ *in the metric* L_s.

Remark 1.1. Relation (1.2) remains valid in the case $\alpha = 1,\ s = 2$ as well, because the system $\{u_n(x)\}_{n=1}^{\infty}$ is a basis in $L_2^2(G)$, and the trigonometric system forms a basis in $L_2(G)$.

Remark 1.2. For $s = \infty$, equality (1.2) means that

$$\lim_{\nu\to\infty} \left\|\sigma_\nu^j(\cdot, f) - S_\nu(\cdot, f_j)\right\|_{C(K)} = 0 \tag{1.3}$$

because the eigenfunctions $u_n(x)$ are continuous on $K \subset G$. Therefore, for $\alpha > 2$ we have uniform equiconvergent on any compact $K \subset G$ (see [8]).

Corollary 1.1. *For* $s = \infty$*, componentwise principle of localization is valid for orthogonal expansion of arbitrary* $f(x) \in L_2^2(G)$ *in the system* $\{u_n(x)\}_{n=1}^{\infty}$ *: convergence or divergence of the* j*-th component of orthogonal expansion at the point* $x_0 \in G$ *depends on the behavior in small vicinity of this point* x_0 *only of the appropriate* j*-th component* $f_j(x)$ *of the expandable vector-functions* $f(x)$ *(and does not depend on the behavior of another component).*

2 Some auxiliary statements.

The proof of theorem 1.1 is based on the following lemmas.

Lemma 2.1 *(Shift and mean-value formulas). If the functions* $p(x)$ *and* $q(x)$ *belong to the class* $L_1(G)$ *and the points* $x - t,\ x,\ x + t$ *are in the domain* $\overline{G}$*, the following formulas are valid:*

$$u_n(x \pm t) = (\cos\lambda_n t \cdot I \mp \sin\lambda_n t \cdot B)u_n(x) \pm \int_x^{x\pm t} \{\sin\lambda_n\,(t - |x-\xi|)$$

$$\times I + sgn(\xi - x)\cos\lambda_n\,(t - |x-\xi|)\cdot B\}\,P(\xi)u_n(\xi)d\xi; \tag{2.1}$$

$$\frac{u_n(x-t) + u_n(x-t)}{2} = u_n(x)\cos\lambda_n t + \frac{1}{2}\int_{x-t}^{x+t} \{\sin\lambda\,(t - |x-\xi|)$$

$$\times I + sign(\xi - x)\cos\lambda_n\,(t - |x-\xi|)\cdot B\}\,P(\xi)u_n(\xi)d\xi, \tag{2.2}$$

where I *is a unit operator in* E^2.

Lemma 2.1 was proved in [9].

Lemma 2.2 *Let* $p(x)$ *and* $q(x)$ *belong to the class* $L_1(G)$*. Then there exists such constants* C_1 *and* C_2 *that*

$$\|u_n\|_{\infty,2} \le C_1; \tag{2.3}$$

$$\sum_{|t-\lambda_n|\le 1} 1 \le C_2, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

are valid.

Estimation (2.3) follows from theorem 2 of [10], estimation (2.4) was established in [11] (see theorem 1.4).

Let

$$T_n^1(r,R,\nu)=\int_r^R \frac{\sin\nu t}{t}\sin\lambda_n(t-r)dt;$$

$$T_n^2(r,R,\nu)=\int_r^R \frac{\sin\nu t}{t}\cos\lambda_n(t-r)dt,$$

$$R_0/2\le R\le R_0,\ R_0>0,\ 0<r<R<\infty,\ \ \nu>0,\ \ n\in N;$$

$$\left\|T_n^j(\cdot,R,\nu)\right\|_{p,[0,R]}\equiv\left\{\int_0^R \left|T_n^j(r,R,\nu)\right|^p dr\right\}^{\frac{1}{p}}.$$

Lemma 2.3 *For any $\beta\in(0,1]$ for the integrals $T_n^j(r,R,\nu)$, $\ j=1,2;\ n\in N$ the following estimations are fulfilled:*

$$\left|T_n^j\right|\le C(\beta)\begin{cases}\left|\nu-|\lambda_n|\right|^{-\beta}r^{-\beta} & \text{for}\ \ \left|\nu-|\lambda_n|\right|\ge 1\\ \max\{|\ln r|,\ |\ln R|\} & \text{for}\ \ \left|\nu-|\lambda_n|\right|<1,\end{cases}\tag{2.5}$$

$$\left\|T_n^j(\cdot,R,\nu)\right\|_{\gamma,[0,R]}\le C(R_0)\begin{cases}\left|\nu-|\lambda_n|\right|^{-1/\gamma} & \text{for}\ \ \left|\nu-|\lambda_n|\right|\ge 1\\ 1 & \text{for}\ \ \left|\nu-|\lambda_n|\right|<1\end{cases},\gamma\in(1,\infty)\tag{2.6}$$

$$\left\|T_n^j(\cdot,R,\nu)\right\|_{1,[0,R]}\le C(R_0)\begin{cases}\left|\nu-|\lambda_n|\right|^{-1/\gamma} & \text{for}\ \ \left|\nu-|\lambda_n|\right|\ge 1,\ \forall\gamma>1,\\ 1 & \text{for}\ \ \left|\nu-|\lambda_n|\right|<1.\end{cases}\tag{2.7}$$

Proof. We consider the case $j=2$, because for $j=1$ the proof of estimation (2.5) is carried out quite similar. By the identity $\cos(\alpha-\beta)\cos\alpha\cos\beta+\sin\alpha\sin\beta$ we can represent the integral $T_n^2(r,R,\nu)$ in the form

$$T_n^2(r,R,\nu)=\cos\lambda_n r\int_r^R\frac{\sin\nu t}{t}\cos\lambda_n t\,dt+\sin\lambda_n r\int_r^R\frac{\sin\nu t}{t}$$
$$\times\sin\lambda_n t\,dt=F_1(r,R,\nu,\lambda_n)\cos\lambda_n r+F_2(r,R,\nu,\lambda_n)\sin\lambda_n t.\tag{2.8}$$

Transform the integral F_1 in the following way:

$$F_1=\int_r^R\frac{\sin\nu t}{t}\cos\lambda_n t\,dt=\int_r^R\frac{\sin\nu t}{t}\cos|\lambda_n|\,t\,dt=\int_r^\infty-\int_R^\infty$$
$$=\frac12\int_r^\infty\frac{\sin(\nu-|\lambda_n|)t}{t}dt+\frac12\int_r^\infty\frac{\sin(\nu+|\lambda_n|)t}{t}dt$$
$$-\frac12\int_R^\infty\frac{\sin(\nu-|\lambda_n|)t}{t}dt-\frac12\int_R^\infty\frac{\sin(\nu+|\lambda_n|)t}{t}dt$$
$$=\frac12 sign\,(\nu-|\lambda_n|)\,si\,(\left|\nu-|\lambda_n|\right|r)+\frac12 si\,((\nu+|\lambda_n|)r)$$
$$-\frac12 sign\,(\nu-|\lambda_n|)\,si\,(\left|\nu-|\lambda_n|\right|R)-\frac12 si\,((\nu+|\lambda_n|)R)\,,$$

where $si\ r=\int_r^\infty\frac{\sin t}{t}dt$.

Applying the inequality $|\sin x|\le C(\varepsilon)/x^\varepsilon$, $x>0,\ \varepsilon\in[0,1]$, we get

$$\|F_1\|\le\frac{C_1(\beta)}{\left|\nu-|\lambda_n|\right|^\beta r^\beta}+\frac{C_1(\beta)}{\left|\nu-|\lambda_n|\right|^\beta R^\beta}\le\frac{C(\beta)}{\left|\nu-|\lambda_n|\right|^\beta r^\beta}.$$

We estimate the integral F_2 and represent it in the form:

$$F_2 = sign\lambda_n \int_r^R \frac{\sin \nu t}{t} \sin |\lambda_n|\, t\, dt$$

$$= \frac{sign\lambda_n}{2} \int_r^R \frac{\cos(\nu - |\lambda_n|)\, t}{t} dt - \frac{sign\lambda_n}{2} \int_r^R \frac{\cos(\nu + |\lambda_n|)\, t}{t} dt.$$

Integrating by parts the first integral, in the right side of the last equality we get

$$\int_r^R \frac{\cos(\nu - |\lambda_n|)\, t}{t} dt = \frac{\sin(\nu - |\lambda_n|)\, t}{(\nu - |\lambda_n|)\, t} \int_r^R + \frac{1}{\nu - |\lambda_n|} \int_r^R \frac{\sin(\nu - |\lambda_n|)\, t}{t^2} dt.$$

Hence, by the inequality $|\sin x| \le |x|^{1-\beta}$, $\beta \in [0,1]$ we get

$$\left| \int_r^R \frac{\cos(\nu - |\lambda_n|)\, t}{t} dt \right| \le \frac{1}{|\nu - |\lambda_n||^{\beta} R^{\beta}} + \frac{1}{|\nu - |\lambda_n||^{\beta} r^{\beta}}$$

$$+ \frac{1}{|\nu - |\lambda_n||^{\beta}} \int_r^R \frac{dt}{t^{1+\beta}} \le \frac{2}{|\nu - |\lambda_n||^{\beta} r^{\beta}}$$

$$+ \frac{\beta^{-1}}{|\nu - |\lambda_n||^{\beta}} \left(r^{-\beta} - R^{-\beta} \right) \le \frac{C_2(\beta)}{|\nu - |\lambda_n||^{\beta} r^{\beta}},$$

where $C_2(\beta) = 2(1 + \beta^{-1}),\ \beta > 0.$

In the same way we prove

$$\left| \int_r^R \frac{\cos(\nu + |\lambda_n|)\, t}{t} dt \right| \le \frac{C_2(\beta)}{|\nu + |\lambda_n||^{\beta} r^{\beta}}.$$

Consequently, the following estimation is fulfilled for the integral F_2

$$|F_2| \le \frac{C(\beta)}{|\nu - |\lambda_n||^{\beta} r^{\beta}}.$$

Allowing for the estimations obtained for the integrals F_1 and F_2, from relation (2.8) we get

$$\left| T_n^2(r, R, \nu) \right| \le |F_1| + |F_2| \le \frac{C(\beta)}{|\nu - |\lambda_n||^{\beta} r^{\beta}}.$$

The second part of estimation (2.5) for the integral $T_n^2(r, R, \nu)$ follows from the inequality

$$\left| \int_r^R \frac{\sin \nu t}{t} \cos \lambda_n (t - r) dt \right| \le \int_r^R \frac{dt}{t} \le 2 \max \{ |\ln r|,\ |\ln R| \}.$$

We now prove estimation (2.6). Fix the number $R_0 > 0$ and consider the case $|\nu - |\lambda_n|| \ge \frac{2}{R_0} \ge 1$. Then $|\nu - |\lambda_n||^{-1} \le \frac{R_0}{2} \le R$. By the triangle inequality

$$\left\| T_n^j(\cdot, R, \nu) \right\|_{\gamma, [0,R]} \le \left\| T_n^j(\cdot, R, \nu) \right\|_{\gamma, \left[0, |\nu - |\lambda_n||^{-1}\right]}$$

$$+ \left\| T_n^j(\cdot, R, \nu) \right\|_{\gamma, \left[|\nu - |\lambda_n||^{-1}, R\right]}.$$

Apply to the first addend of estimation (2.5) for $|\nu - |\lambda_n|| \geq 1,\ \beta \in (0,1),\ \beta\gamma < 1$, and to the second addend for $\beta = 1$. As a result we get

$$\left\|T_n^j(\cdot, R, \nu)\right\|_{\gamma,\left[0,|\nu-|\lambda_n||^{-1}\right]} = O\left(\left(\int_0^{|\nu-|\lambda_n||^{-1}} |\nu-|\lambda_n||^{-\beta\gamma} r^{-\beta\gamma} dr\right)^{1/\gamma}\right)$$

$$= O\left(|\nu-|\lambda_n||^{-\beta}\right)\left(|\nu-|\lambda_n||^{\gamma\beta-1}\right) = O\left(|\nu-|\lambda_n||^{-1/\gamma}\right);$$

$$\left\|T_n^j(\cdot, R, \nu)\right\|_{\gamma,\left[|\nu-|\lambda_n||^{-1},R\right]} = O\left(\left(\int_{|\nu-|\lambda_n||^{-1}}^{R} |\nu-|\lambda_n||^{-\gamma} r^{-\gamma} dr\right)^{1/\gamma}\right)$$

$$= O\left(|\nu-|\lambda_n||^{-1}\right)\left(\int_{|\nu-|\lambda_n||^{-1}}^{R} r^{-\gamma} dr\right)^{1/\gamma}$$

$$= O\left(|\nu-|\lambda_n||^{-1}\right)\left[\left(R^{-1}\right)^{\gamma-1} - |\nu-|\lambda_n||^{\gamma-1}\right]^{1/\gamma}$$

$$= O\left(|\nu-|\lambda_n||^{-1}\right)\left(2\,|\nu-|\lambda_n||^{\gamma-1}\right)^{1/\gamma} = O\left(|\nu-|\lambda_n||^{-1/\gamma}\right).$$

If $1 \leq |\nu - |\lambda_n|| < \frac{2}{R_0}$, then applying inequality (2.5) for $\beta = \beta_0 < \frac{1}{\gamma}$, we get

$$\left\|T_n^j(\cdot, r, \nu)\right\|_{\gamma,[0,R]} = O\left(|\nu-|\lambda_n||^{-\beta_0}\right)\left(\int_0^R r^{-\beta_0\gamma} dr\right)^{1/\gamma}$$

$$= O\left(|\nu-|\lambda_n||^{-\beta_0}\right) R_0^{\frac{1}{\gamma}-\beta_0} = O\left(|\nu-|\lambda_n||^{-\beta_0}\right)|\nu-|\lambda_n||^{\beta_0-1/\gamma}$$

$$= O\left(|\nu-|\lambda_n||^{-1/\gamma}\right).$$

For $|\nu - |\lambda_n|| < 1$, estimation (2.6) directly follows from estimation (2.5) allowing for integrability of the function $|\ln r|^{\gamma}$.

Lemma 2.3. is proved.

3 Proof of theorem 1.1.

Fix an arbitrary connected compact $K \subset G$ and a number R_0, satisfying the condition $0 < 2R_0 < dist(K, \partial G)$. Let $f(x) = (f_1(x),\ f_2(x))^T$ be an arbitrary function from the space $L_2^2(G)$ and $R \in [R_0,\ 2R_0]$. Let us introduce the vector $\widetilde{S}_\nu(x, f) = \left(\widetilde{S}_\nu(x, f_1),\ \widetilde{S}_\nu(x, f_2)\right)^T$, where $\widetilde{S}_\nu(x, f_j),\ j = 1, 2$ is a modified partial sum of order ν of ordinary trigonometric Fourier series of the function $f_j(x)$, i.e.

$$\widetilde{S}_\nu(x, f_j) = \frac{1}{\pi}\int_{|x-y|\leq R} \frac{\sin\nu(x-y)}{x-y} f_j(y)dy,\ \ x \in K,\ \ j = 1, 2.$$

As the difference $S_\nu(x, f_j) - \widetilde{S}_\nu(x, f_j)$ tends to zero uniformly with respect to $x \in K$ as $\nu \to +\infty$ (including in the metrics $L_s(K),\ s \geq 1$), then for proving theorem 1.1 if suffices to establish equality (1.2) for the partial sum $\widetilde{S}_\nu(x, f)$.

Because of the basicity of the system $\{u_n(x)\}$ in the space $L_2^2(G)$ we can represent every function $f(x) \in L_2^2(G)$ in the form

$$f(x) = \sum_{n=1}^{\infty} f_n u_n(x), \quad f_n = (f, u_n),$$

converging in the norm of this space. Consequently, we can represent the partial sum $\widetilde{S}_\nu(x,f)$ in the form

$$\widetilde{S}_\nu(x,f) = \frac{2}{\pi}\sum_{n=1}^{\infty}(f,u_n)\int_0^R \frac{\sin\nu t}{t}\frac{u_n(x+t)+u_n(x-t)}{2}dt. \tag{3.1}$$

Transform the integral $\frac{1}{2}\int_0^R t^{-1}\sin\nu t\,(u_n(x+t)+u_n(x-t))\,dt$. For that we apply the mean value formula (2.2)

$$\frac{2}{\pi}\int_0^R \frac{\sin\nu t}{t}\frac{u_n(x-t)+u_n(x+t)}{2}dt = \frac{2}{\pi}u_n(x)\int_0^R \frac{\sin\nu t}{t}$$

$$\times\cos\lambda_n t\;dt + \frac{1}{\pi}\int_0^R \frac{\sin\nu t}{t}\int_{x-t}^{x+t}\{\sin\lambda_n(t-|x-\xi|)\cdot I$$

$$+\,sign(\xi-x)\cos\lambda_n(t-|x-\xi|)\cdot B\}\,P(\xi)u_n(\xi)d\xi$$

$$= \frac{2}{\pi}u_n(x)\int_0^R \frac{\sin\nu t}{t}\cos\lambda_n t\;dt + \frac{1}{\pi}\int_{|x-\xi|}^{x+R}\left(\int_{|x-\xi|}^R \frac{\sin\nu t}{t}\right.$$

$$\times\{\sin\lambda_n\,(t-|x-\xi|)\cdot I + sign(\xi-x)\cos\lambda_n\,(t-|x-\xi|)\cdot B\}\,dt)$$

$$\times P(\xi)u_n(\xi)d\xi = \frac{2}{\pi}u_n(x)\int_0^R \frac{\sin\nu t}{t}\cos\lambda_n t\;dt$$

$$+\frac{1}{\pi}\int_{x-R}^{x+R}\left\{T_n^1\left(|x-\xi|,R,\nu\right)\cdot I\right.$$

$$\left.+sign\,(\xi-x)\,T_n^2\left(|x-\xi|,R,\nu\right)\cdot B\right\}P(\xi)u_n(\xi)d\xi.$$

Let $\delta(\nu,\lambda_n)$ be Dirichlet's discontinuous multiplier, i.e.

$$\delta(\nu,\lambda_n) = \begin{cases} 1 & \text{for } \nu > |\lambda_n| \\ \frac{1}{2} & \text{for } \nu = |\lambda_n| \\ 0 & \text{for } \nu < |\lambda_n|. \end{cases}$$

Then for the previous integral we get the following representation:

$$\frac{2}{\pi}\int_0^R \frac{\sin\nu t}{t}\frac{u_n(x-t)+u_n(x+t)}{2}dt = u_n(x)\delta(\nu,\lambda_n)$$

$$+u_n(x)\left\{\frac{2}{\pi}\int_0^R \frac{\sin\nu t}{t}\cos\lambda_n t\;dt - \delta(\nu,\lambda_n)\right\} + \frac{1}{\pi}\int_{x-R}^{x+R}\left\{T_n^1\left(|x-\xi|,R,\nu\right)I\right.$$

$$\left.+sign(\xi-x)T_n^2\left(|x-\xi|,R,\nu\right)\cdot B\right\}P(\xi)u_n(\xi)d\xi$$

$$= u_n(x)\delta(\nu,\lambda_n) + u_n(x)I(\nu,\lambda_n) + \frac{1}{\pi}\int_{x-R}^{x+R}\left\{T_n^1\left(|x-\xi|,R,\nu\right)I\right.$$

$$+sign(\xi - x)T_n^2\left(|x-\xi|, R, \nu\right) \cdot B\Big\} P(\xi)u_n(\xi)d\xi,$$

where for $I(\nu, \lambda_n)$ we have the following estimation (see [3], [8])

$$|I(\nu, \lambda_n)| \leq \frac{C(R)}{1 + |\nu - |\lambda_n||}. \tag{3.2}$$

By virtue of the given representation, from equality (3.1) it follows that

$$\tilde{S}_\nu(x,f) = \sum_{n=1}^{\infty}(f,u_n)\delta(\nu,\lambda_n)u_n(x) + \sum_{n=1}^{\infty}(f,u_n)u_n(x)\cdot I(\nu,\lambda_n)$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}(f,u_n)\int_0^R \{P(x+r)u_n(x+r) + P(x-r)u_n(x-r)\}\, T_n^1(r,R,\nu)dr$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}(f,u_n)\int_0^R \{P(x+r)u_n(x+r) + P(x-r)u_n(x-r)\}\, T_n^2(r,R,\nu)dr.$$

Since

$$\sum_{n=1}^{\infty}(f,u_n)\delta(\nu,\lambda_n)u_n(x) = \sigma_\nu(x,f) - \frac{1}{2}\sum_{\nu=|\lambda_n|}(f,u_n)u_n(x),$$

then for the difference $\tilde{S}_\nu(x,f) - \sigma_\nu(x,f)$ we get

$$\tilde{S}_\nu(x,f) - \sigma_\nu(x,f) = -\frac{1}{2}\sum_{\nu=|\lambda_n|}(f,u_n)u_n(x) + \sum_{n=1}^{\infty}(f,u_n)u_n(x)I(\nu,\lambda_n)$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}(f,u_n)\int_0^R \{P(x+r)u_n(x+r) + P(x-r)u_n(x-r)\}\, T_n^1(r,R,\nu)dr$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}(f,u_n)\int_0^R \{P(x+r)u_n(x+r) - P(x-r)u_n(x-r)\}\, T_n^2(r,R,\nu)dr$$

$$= \Phi_1(x) + \Phi_2(x) + \Phi_3(x) + \Phi_4(x), \;\; x \in K. \tag{3.3}$$

We should estimate the series $\Phi_i(x), \;\; i = \overline{1,4}$, in the metric $L_s^2(K)$. To estimate $\Phi_1(x)$ we use the Bessel inequality and estimations (2.3), (2.4):

$$\|\Phi_1\|_{s,2,K} = \frac{1}{2}\left\|\sum_{|\lambda_n|=\nu}(f,u_n)u_n(x)\right\|_{s,2,K} \leq \frac{1}{2}\sum_{|\lambda_n|=\nu}|(f,u_n)|$$

$$\times \|u_n\|_{s,2} \leq C\sum_{|\lambda_n|=\nu}|(f,u_n)| \leq C\left(\sum_{|\lambda_n|=\nu}|(f,u_n)|^2\right)^{1/2}\left(\sum_{|\lambda_n|=\nu}1\right)^{1/2} \leq C\,\|f\|_{2,2}\,.$$

By virtue of estimations (2.3), (2.4), (3.2) and the Bessel inequality for the series $\Phi_2(x)$ we get

$$\|\Phi_2\|_{s,2,K} = \left\|\sum_{n=1}^{\infty}(f,u_n)u_n(x)I(\nu,\lambda_n)\right\|_{s,2,K}$$

$$\leq C(R)\sum_{n=1}^{\infty}|(f,u_n)|\,|I(\nu,\lambda_n)|\,\|u_n\|_{s,2,K}$$

$$\leq C\sum_{n=1}^{\infty}|(f,u_n)|\,|I(\nu,\lambda_n)|\leq C\left(\sum_{n=1}^{\infty}|(f,u_n)|^2\right)^{1/2}$$

$$\times\left(\sum_{n=1}^{\infty}\frac{1}{(1+||\lambda_n|-\nu|)^2}\right)^{1/2}\leq C\,\|f\|_{2,2}\left(\sum_{k=1}^{\infty}\left(\frac{1}{1+k}\right)^2\sum_{k\leq|\nu-|\lambda_n||\leq k+1}1\right)^{1/2}$$

$$\leq C\,\|f\|_{2,2}\left(\sum_{l=1}^{\infty}l^{-2}\right)^{1/2}\leq C\,\|f\|_{2,2}\,.$$

We now estimate the series $\Phi_3(x)$. We apply estimation (2.3) from lemma 2.2. As a result we get

$$|\Phi_3(x)|=\frac{1}{\pi}\left|\sum_{n=1}^{\infty}(f,u_n)\int_0^R\{P(x+r)u_n(x+r)+P(x-r)\}\,T_n^1(r,R,\nu)dr\right|$$

$$\leq\frac{1}{\pi}\sum_{n=1}^{\infty}|(f,u_n)|\int_0^R\left\{\left(\left|p(x+r)u_n^1(x+r)\right|^2+\left|q(x+r)u_n^2(x+r)\right|^2\right)^{1/2}\right.$$

$$\left.+\left(\left|p(x-r)u_n^1(x-r)\right|^2+\left|q(x-r)u_n^2(x-r)\right|^2\right)^{1/2}\right\}\left|T_n^1(r,R,\nu)\right|dr$$

$$\leq C\sum_{n=1}^{\infty}|(f,u_n)|\int_0^R Q(x,r)\left|T_n^1(r,R,\nu)\right|dr, \tag{3.4}$$

where

$$Q(x,r)=|p(x+r)|+|p(x-r)|+|q(x-r)|+|q(x+r)|\,.$$

Let us consider the following integrals

$$L^{\pm}(x)=\int_0^R|p(x\pm r)|\left|T_n^1(r,R,\nu)\right|dr;\quad M^{\pm}(x)=\int_0^R|q(x\pm r)|\left|T_n^1(r,R,\nu)\right|dr.$$

We estimate these integrals in the norm $L_s(K)$. For that we use the Young inequality (see [1], p. 25)

$$\left\|L^{\pm}\right\|_{s,K}=\left\|\int_0^R|p(\cdot\pm r)|\left|T_n^1(r,R,\nu)\right|dr\right\|_{s,K}\leq\|p\|_{\alpha}\left\|T_n^1(\cdot,R,\nu)\right\|_{\gamma,[0,R]}\,,$$

where $\gamma^{-1}=1+s^{-1}-\alpha^{-1}$ for $s>\alpha;\ \gamma=1,$ for $s\leq\alpha$.

Hence by lemma 2.3 (see estimation (2.6), (2.7)) it follows that

$$\left\|L^{\pm}\right\|_{s,K}\leq C(R_0)\,\|p\|_{\alpha}\begin{cases}|\nu-|\lambda_n||^{\frac{1}{\alpha}-\frac{1}{s}-1} & \text{for } |\nu-|\lambda_n||\geq1,\ s>\alpha\\ |\nu-|\lambda_n||^{-\frac{2}{3}} & \text{for } |\nu-|\lambda_n||\geq1,\ s\leq\alpha\\ 1 & \text{for } |\nu-|\lambda_n||<1.\end{cases} \tag{3.5}$$

The same estimation is fulfilled for the integrals $M^{\pm}(x)$, i.e.

$$\left\|M^{\pm}\right\|_{s,K} \leq C(R_0)\left\|q\right\|_{\alpha} \begin{cases} \left|\nu-|\lambda_n|\right|^{\frac{1}{\alpha}-\frac{1}{s}-1} & \text{for } \left|\nu-|\lambda_n|\right| \geq 1,\ s>\alpha, \\ \left|\nu-|\lambda_n|\right|^{-\frac{2}{3}} & \text{for } \left|\nu-|\lambda_n|\right| \geq 1,\ s\leq\alpha \\ 1 & \text{for } \left|\nu-|\lambda_n|\right| < 1. \end{cases} \tag{3.6}$$

For $s>\alpha$ form inequalities (3.4) allowing for estimations (3.5), (3.6) it follows that

$$\left\|\Phi_3\right\|_{s,2,K} \leq C\sum_{n=1}^{\infty}\left|(f,u_n)\right|\left\|\int_0^R Q(x,r)\left|T_n^1(r,R,\nu)\right|dr\right\|_{s,2,K}$$

$$\leq C\left(\sum_{\left|\nu-|\lambda_n|\right|<1}\left|(f,u_n)\right|\left\{\left\|L^+\right\|_{s,K}+\left\|L^-\right\|_{s,K}+\left\|M^+\right\|_{s,K}\right.\right.$$

$$\left.+\left\|M^-\right\|_{s,K}\right\}+\sum_{\left|\nu-|\lambda_n|\right|\geq 1}\left|(f,u_n)\right|\left\{\left\|L^+\right\|_{s,K}+\left\|L^-\right\|_{s,K}\right.$$

$$\left.\left.+\left\|M^+\right\|_{s,K}+\left\|M^-\right\|_{s,K}\right\}\right) \leq C\left(\left\|p\right\|_{\alpha}+\left\|q\right\|_{\alpha}\right)$$

$$\times\left\{\sum_{\left|\nu-|\lambda_n|\right|<1}\left|(f,u_n)\right|+\sum_{\left|\nu-|\lambda_n|\right|\geq 1}\left|(f,u_n)\right|\left|\nu-|\lambda_n|\right|^{\frac{1}{\alpha}-\frac{1}{s}-1}\right\}.$$

Hence by the Bessel inequality, condition $\frac{1}{\alpha}-\frac{1}{s}<\frac{1}{2}$ and inequality (2.4) we get

$$\left\|\Phi_3\right\|_{s,2,K} \leq C\left\|P\right\|_{\alpha}\left\|f\right\|_{2,2},$$

where $\left\|P\right\|_{\alpha}=\left\|p\right\|_{\alpha}+\left\|q\right\|_{\alpha}$.

For $s\leq\alpha$ by the above scheme we find

$$\left\|\Phi_3\right\|_{s,2,K} \leq C\left\|P\right\|_{\alpha}\left\{\sum_{\left|\nu-|\lambda_n|\right|<1}\left|(f,u_n)\right|+\sum_{\left|\nu-|\lambda_n|\right|\geq 1}\left|(f,u_n)\right|\right.$$

$$\left.\times\left|\nu-|\lambda_n|\right|^{-\frac{2}{3}}\right\} \leq C\left\|P\right\|_{\alpha}\left\|f\right\|_{2,2}.$$

Consequently, for $\Phi_3(x)$ at $\frac{1}{\alpha}-\frac{1}{s}<\frac{1}{2}$ the following estimation is fulfilled:

$$\left\|\Phi_3\right\|_{s,2,K} \leq C(K)\left\|f\right\|_{2,2}. \tag{3.7}$$

Obviously, estimation (3.7) is fulfilled for the vector-function $\Phi_4(x)$ as well.

Taking into account estimations for $\left\|\Phi_i\right\|_{s,2,K}$, $i=\overline{1,4}$, from equality (3.3) we obtain the inequality

$$\left\|\sigma_\nu(\cdot,f)-\widetilde{S}_\nu(\cdot,f)\right\|_{s,2,K} \leq C_0(K)\left\|f\right\|_{2,2}, \tag{3.8}$$

where $f(x)$ is an arbitrary vector-function from $L_2^2(G)$.

Now from this estimation we derive relation (1.2) for the partial sum $\widetilde{S}_\nu(x,f)$. By the basicity of the system $\{u_n(x)\}_{n=1}^{\alpha}$ in the space $L_2^2(G)$ for an arbitrary $f(x)\in L_2^2(G)$ and for any $\varepsilon>0$ there exists such $n_0(\varepsilon,f)\in N$ that

$$\left\|f-\sum_{n=1}^{n_0(\varepsilon,f)}(f,u_n)u_n(x)\right\|_{2,2} < \varepsilon/(2C_0(K)) \tag{3.9}$$

where $C_0(K)$ is a constant from inequality (3.8).

Denote $g(x) = \sum\limits_{n=1}^{n_0(\varepsilon,f)} (f, u_n)u_n(x)$. By the triangle inequality

$$\left\|\sigma_\nu(\cdot, f) - \widetilde{S}_\nu(\cdot, f)\right\|_{s,2,K} = \|\sigma_\nu(\cdot, f-g) + \sigma_\nu(\cdot, g)$$

$$-\widetilde{S}_\nu(\cdot, f-g) - \widetilde{S}_\nu(\cdot, g)\right\|_{s,2,K} \leq \left\|\sigma_\nu(\cdot, f-g) - \widetilde{S}_\nu(\cdot, f-g)\right\|_{s,2,K}$$

$$+\left\|\sigma_\nu(\cdot, g) - \widetilde{S}_\nu(\cdot, g)\right\|_{s,2,K}.$$

By virtue of inequality (3.8) and equality $\sigma_\nu(x, g) = g$, for sufficiently large ν, we get:

$$\left\|\sigma_\nu(\cdot, f) - \widetilde{S}_\nu(\cdot, f)\right\|_{s,2,K} \leq C_0(K)\, \|f-g\|_{2,2}$$

$$+\left\|g(\cdot) - \widetilde{S}_\nu(\cdot, g)\right\|_{s,2,K} < \varepsilon/2 + \left\|g(\cdot) - \widetilde{S}_\nu(\cdot, g)\right\|_{s,2,K}. \tag{3.10}$$

Show that

$$\lim_{\nu\to\infty} \left\|g(\cdot) - \widetilde{S}_\nu(\cdot, g)\right\|_{s,2,K} = 0. \tag{3.11}$$

From the definition of the eigen-function $u_n(x)$ it follows that it belongs to the class $W_\alpha^1(G)$. Consequently, the function $g(x)$ also belongs to $W_\alpha^1(G)$. If $s \leq 2$, then equality (3.11) is a corollary of basicity trigonometric system in $L_2(G)$. If $s > 2$, then by the condition $\frac{1}{\alpha} - \frac{1}{s} < \frac{1}{2}$ we find $\alpha > \frac{2s}{s-2} > 1$. This shows that the function $g(x)$ belongs to $W_\alpha^1(G)$, $\alpha > 1$. Consequently, every component $g_i(x)$ of the vector-function $g(x) = (g_1(x),\, g_2(x))^T$ satisfies the Holder condition. Therefore, on any fixed compact $K \subset (0, 2\pi)$ the difference $\widetilde{S}_\nu(x, g_i) - g_i(x)$, $i = 1, 2$, tends to zero as $\nu \to +\infty$ uniformly with respect to $x \in K$, i.e.

$$\lim_{\nu\to\infty} \left\|g(\cdot) - \widetilde{S}_\nu(\cdot, g)\right\|_{C(K)} = 0.$$

Thus, the validity of equality (3.11) is justified.

From inequality (3.10), allowing for equality (3.11) we get that for $\nu \geq \nu_0$ (ν_0 is a sufficiently large number) the following inequality is fulfilled:

$$\left\|\widetilde{S}_\nu(\cdot, f) - \sigma_\nu(\cdot, f)\right\|_{s,2,K} < \varepsilon.$$

Theorem 1.1 is completely proved.

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