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Approximation by trigonometric polynomials in Morrey spaces

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Abstract. *In this paper, we investigate the best approximation by trigonometric polynomials in Morrey space $L_{p,\lambda}(I_0)$ with $1 < p < \infty$ and $I_0 = [0, 2\pi]$. We prove the direct and inverse theorems of approximation by trigonometric polynomials in the spaces $\widetilde{L}_{p,\lambda}(I_0)$ the closure of $C^{\infty}(I_0)$ in $L_{p,\lambda}(I_0)$. To prove these theorems we get the characterization of $K-$functionals in terms of the modulus of smoothness and give the Bernstein type inequality for trigonometric polynomials in the spaces $L_{p,\lambda}(I_0)$.*

Keywords. Morrey space, best approximation, modulus of smoothness.

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1 Introduction and main results

The classical Morrey spaces, were introduced by C.B. Morrey in 1938, have been studied intensively by various authors and play an important role in theory of partial differential equations; they appeared to be quite useful in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than L_p spaces.

We consistently write

$$I_0 = [0, 2\pi], \quad I(x, r) = (x - r, x + r) \subset \mathbb{R}, \quad I_0(x, r) = I(x, r) \cap I_0$$

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for intervals in this paper. Let $0 \leq \lambda \leq 1$ and $1 \leq p < \infty$. The classical Morrey space $L_{p,\lambda}(I_0)$ is defined as the set of all functions $f \in L_p(I_0)$ such that

$$\|f\|_{L_{p,\lambda}(I_0)} = \sup\{r^{-\frac{\lambda}{p}}\|f\|_{L_p(I_0(x,r))} : x \in I_0, 0 < r < 2\pi\} < \infty.$$

Under this definition $L_{p,\lambda}(I_0)$ is a Banach space; furthermore, for $\lambda = 0$ it coincides with $L_p(I_0)$ and for $\lambda = 1$ with $L_\infty(I_0)$. If $\lambda < 0$ or $\lambda > 1$, then $L_{p,\lambda}(I_0) = \Theta(I_0)$, where $\Theta(I_0)$ denotes the set of all functions equivalent to 0 on I_0. If $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, then $L_{p,\lambda_2}(I_0) \subset L_{p,\lambda_1}(I_0)$. If $f \in L_{p,\lambda}(I_0)$, then $f \in L_p(I_0)$ and hence $f \in L_1(I_0)$.

Compared to Lebesgue spaces, Morrey spaces have the following remarkable features: Let $1 < p < \infty$ and $0 < \lambda \leq 1$.

1 The function $f(x) = x^{-(1-\lambda)/p}$ is in $L_{p,\lambda}(I_0)$.
2 The Morrey space $L_{p,\lambda}(I_0)$ is not reflexive; see [11, Example 5.2] and [16, Theorem 1.3].
3 Denote by $C^\infty(I_0)$ the set of all functions that are realized as the restriction to I_0 of elements in $C^\infty(\mathbb{R})$. The Morrey space $L_{p,\lambda}(I_0)$ does not have $C^\infty(I_0)$ as a dense closed subspace; see [15, Proposition 2.16].
4 The Morrey space $L_{p,\lambda}(I_0)$ is not separable; see [15, Proposition 2.16].

If $\lambda = 0$, all of these properties above fail to hold, since $L_{p,0}(I_0) = L_p(I_0)$ with norm coincidence. Based on these properties, we define $\widetilde{L}_{p,\lambda}(I_0)$ to be the closure of $C^\infty(I_0)$ in $L_{p,\lambda}(I_0)$.

Hardy-Littlewood maximal function Mf of f on I_0 is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|I(x,r)|} \int_{I_0(x,r)} |f(t)|dt, \ x \in I_0.$$

Using the suitable result from [2] , for every $f \in L_{p,\lambda}(I_0)$, $0 \leq \lambda < 1$, $1 < p < \infty$ we obtain

$$\|Mf\|_{L_{p,\lambda}(I_0)} \leq C\|f\|_{L_{p,\lambda}(I_0)} \tag{1.1}$$

with a positive constant C does not depend of f.

Denote by $\mathcal{P}_n$ the set of trigonometric polynomials having degree not exceeding n and $C(I_0)$ the set of $2\pi-$periodic continuous functions. Let $f \in C(I_0)$ and $E_n(f)$ be the best approximation of f by the trigonometric polynomials, i.e.,

$$E_n(f) = \inf_{T_n \in \mathcal{P}_n} \|f - T_n\|_{C(I_0)},$$

and $\omega(f, \delta)$ be the modulus of continuity of f. The Weierstrass well-known theorem on the approximation of the continuous function by the trigonometric polynomials and its quantitative refinement represented by Jackson inequality

$$E_n(f) \leq C\omega(f, \frac{1}{n})$$

are one of the basics of the Approximation Theory. The analog of Jackson inequality is correct for the mean approximation and higher order modulus of continuity as well (see [13]). S. Bernstein [1] obtained the reversed estimations of Jackson's inequality in the space of continuous functions for some specific cases. Later E.S. Quade [10], S.B. Stechkin [12], A.F. Timan [13], A.F. and M.F. Timan [14] etc. proved the reversed type inequalities of Jackson's inequality, including in $L_p, 1 < p < \infty$, spaces. These type inequalities played an important role in the investigation of properties of the conjugate functions, in the study of absolute convergent Fourier series [12], and in the related problems. For the approximation in weighted and nonweighted Lebesgue spaces the sufficiently wide presentation can be found in the works [4], [6], [7] and [8].

Let $\mathbb{N} := \{1, 2, ...\}$ and $\triangle_t^r f(x) := \sum\limits_{s=0}^{r} \binom{r}{s} (-1)^{r+s+1} f(x + st),\ r \in \mathbb{N}$ for $f \in L_{p,\lambda}(I_0),\quad 0 \leq \lambda \leq 1, p \geq 1$ and

$$\sigma_\delta^r(f)(x) := \frac{1}{\delta} \int\limits_0^\delta |\triangle_t^r f(x)| dt. \tag{1.2}$$

One of the main problems observed in the investigations on the approximation theory is the correct definition of the modulus of smoothness that will provide us with a better tool to deal with the rate of the best approximation, inverse theorems and also some other similar problems. Now we define the modulus of smoothness in Morrey spaces $L_{p,\lambda}(I_0)$.

Definition 1.1 *If $g \in L_{p,\lambda}(I_0)$, $0 \leq \lambda \leq 1$, $1 < p < \infty$, then the function $\Omega^r(g, \cdot, L_{p,\lambda}(I_0)) : [0, \infty] \to [0, +\infty)$ defined by*

$$\Omega^r(g, h, L_{p,\lambda}(I_0)) := \sup_{0<\delta\leq\min\{2\pi,h\}} \|\sigma_\delta^r(g)\|_{L_{p,\lambda}(I_0)},\quad r \in \mathbb{N}$$

is called the r th modulus of smoothness of g in $L_{p,\lambda}(I_0)$.

From Corollary 2.1 we get $\Omega^r(g, h, L_{p,\lambda}(I_0)) \leq C\|g\|_{L_{p,\lambda}(I_0)}$ for every $g \in L_{p,\lambda}(I_0)$, $0 \leq \lambda \leq 1$, $1 < p < \infty$ and

$$\Omega^r(g_1 + g_2, \cdot, L_{p,\lambda}(I_0)) \leq \Omega^r(g_1, \cdot, L_{p,\lambda}(I_0)) + \Omega^r(g_2, \cdot, L_{p,\lambda}(I_0))$$

for $g_1, g_2 \in L_{p,\lambda}(I_0)$. Also for $f \in \widetilde{L}_{p,\lambda}(I_0)$, $\lim\limits_{\delta\to0} \Omega^r(g, h, \widetilde{L}_{p,\lambda}(I_0)) = 0$ (see [4]).

For $f \in L_{p,\lambda}(I_0)$, $0 \leq \lambda \leq 1$ and $p \geq 1$, we denote

$$E_n(f)_{L_{p,\lambda}(I_0)} := \inf_{T_n\in\mathcal{P}_n} \|f - T_n\|_{L_{p,\lambda}(I_0)}$$

the minimal error of approximation of f in the class $\mathcal{P}_n$ of trigonometric polynomials of degree not exceeding n.

The homogeneous Sobolev-Morrey space $\dot{W}_{p,\lambda}^r(I_0)$ is defined as the set of all functions $f \in L_1^{loc}(I_0)$ for which the weak derivative $f^{(r)}$ exists on I_0 and

$$\|f\|_{\dot{W}_{p,\lambda}^r(I_0)} = \|f^{(r)}\|_{\mathcal{M}_{p,\lambda}(I_0)} < \infty.$$

We define the non-homogeneous Sobolev–Morrey space $W_{p,\lambda}^r(I_0)$ as the subset of $\dot{W}_{p,\lambda}^r(I_0)$, consisting of all functions $f \in \dot{W}_{p,\lambda}^r(I_0)$ for which

$$\|f\|_{W_{p,\lambda}^r(I_0)} := \|f\|_{L_{p,\lambda}(I_0)} + \|f^{(r)}\|_{L_{p,\lambda}(I_0)} < \infty.$$

For $f \in L_{p,\lambda}(I_0)$, $0 \leq \lambda \leq 1$, $1 < p < \infty$ and $r \geq 1$ the K- functional is defined as follows

$$K_r(f, t)_{L_{p,\lambda}(I_0)} = \inf_{g\in W^r L_{p,\lambda}(I_0)} \left\{ \|f - g\|_{L_{p,\lambda}(I_0)} + t^r \|g^{(r)}\|_{L_{p,\lambda}(I_0)} \right\}, t > 0.$$

In this paper we study the direct and inverse problems of approximation theory in the $\widetilde{L}_{p,\lambda}(I_0)$ modified Morrey spaces which contains the set of trigonometric polynomials as a dense subset with $1 < p < \infty$. We give a characterization of K-functionals in terms of the modulus of smoothness in Morrey spaces $L_{p,\lambda}(I_0)$.

The direct result can be formulated as follows:

Theorem 1.1 *Let $f \in \widetilde{L}_{p,\lambda}(I_0)$, $0 \le \lambda \le 1$, $1 < p < \infty$. Then for every $r \in \mathbb{N}$ we have*

$$E_n(f)_{\widetilde{L}_{p,\lambda}(I_0)} \le C\ \Omega^r(f, \frac{1}{n}, \widetilde{L}_{p,\lambda}(I_0)), \quad n \ge r$$

with a constant $C > 0$ independent of n.

Similar result in Lebesgue spaces $L_p(I_0)$, in term of usually modulus of smoothness, defined as

$$\sup_{|t|\le h} \|\triangle_t^r f(x)\|_{L_p(I_0)}, \quad h > 0,\ r \in \mathbb{N}$$

was proved by S.B. Stechkin in [12].

The inverse result can be formulated as follows:

Theorem 1.2 *Let $f \in \widetilde{L}_{p,\lambda}(I_0)$, $0 \le \lambda < 1$, $1 < p < \infty$. Then for every $r \in \mathbb{N}$ we have*

$$\Omega^r(f, \frac{1}{n}, \widetilde{L}_{p,\lambda}(I_0)) \le \frac{C}{n^r}\left\{E_0(f)_{\widetilde{L}_{p,\lambda}(I_0)} + \sum_{m=1}^{n} m^{r-1} E_m(f)_{\widetilde{L}_{p,\lambda}(I_0)}\right\},\ n \in \mathbb{N}$$

with a constant $C > 0$ independent of n.

In Lebesgue spaces $L_p(I_0)$ similar result was proved in [14] (see also [13], pp. 208-211).

The letter C is used for various constants, and may change from one occurrence to another.

2 Some auxiliary results

In this section we give some lemmas which we will need while proving our main results.

Lemma 2.1 *Let $f \in \dot{W}^r L_{p,\lambda}(I_0)$, $r \in \mathbb{N} \cup \{0\}$, $0 \le \lambda < 1$, $1 < p < \infty$. Then for every $r \in \mathbb{N}$*

$$\|\sigma_\delta^r(f)\|_{L_{p,\lambda}(I_0)} \le C\delta^r\ \|f^{(r)}\|_{L_{p,\lambda}(I_0)}$$

with a constant $C > 0$.

Proof. Let $f \in \dot{W}^r L_{p,\lambda}(I_0)$. Then $f^{(r)} \in L_{p,\lambda}(I_0)$ and

$$\triangle_t^r f(x) = \int_0^t \cdots \int_0^t f^{(r)}(x + t_1 + t_2 + ... + t_r) dt_1 ... dt_r.$$

Then

$$\|\sigma_\delta^r(f)\|_{L_{p,\lambda}(I_0)} = \left\| \frac{1}{\delta}\int_0^\delta |\triangle_t^r f(\cdot)| dt \right\|_{L_{p,\lambda}(I_0)}$$

$$\le \left\| \frac{1}{\delta}\int_0^\delta \int_0^t \cdots \int_0^t |f^{(r)}(\cdot + t_1 + t_2 + ... + t_r)| dt_1 ... dt_r dt \right\|_{L_{p,\lambda}(I_0)}$$

$$\leq \delta^r \left\| \frac{1}{\delta^r} \int\limits_0^{\delta} \cdots \int\limits_0^{\delta} |f^{(r)}(\cdot + t_1 + t_2 + ... + t_r)| dt_1 ... dt_r \right\|_{L_{p,\lambda}(I_0)}$$

$$\leq \delta^r \left\| \frac{1}{\delta^{r-1}} \int\limits_0^{\delta} \cdots \int\limits_0^{\delta} \left\{ \frac{1}{\delta} \int\limits_{t_1+...+t_{r-1}}^{t_1+...+t_{r-1}+\delta} |f^{(r)}(\cdot + t)| dt \right\} dt_1 ... dt_{r-1} \right\|_{L_{p,\lambda}(I_0)}$$

$$\leq \delta^r \frac{1}{\delta^{r-1}} \int\limits_0^{\delta} \cdots \int\limits_0^{\delta} \left\| \frac{1}{\delta} \int\limits_0^{\delta+(r-1)\delta} |f^{(r)}(\cdot + t)| dt \right\|_{L_{p,\lambda}(I_0)} dt_1 ... dt_{r-1}$$

$$= \delta^r \left\| \frac{1}{\delta} \int\limits_0^{r\delta} |f^{(r)}(\cdot + t)| dt \right\|_{L_{p,\lambda}(I_0)}$$

$$\leq C\delta^r \, \|Mf^{(r)}\|_{L_{p,\lambda}(I_0)}$$

$$\leq C\delta^r \, \|f^{(r)}\|_{L_{p,\lambda}(I_0)}.$$

Corollary 2.1 *Let $f \in L_{p,\lambda}(I_0)$, $0 \leq \lambda < 1$, $1 < p < \infty$. Then for every $r \in \mathbb{N}$*

$$\|\sigma_\delta^r(f)\|_{L_{p,\lambda}(I_0)} \leq C \, \|f\|_{L_{p,\lambda}(I_0)}$$

with a constant $C > 0$.

Proof. Using the triangle inequality we have

$$\|\sigma_\delta^r(f)\|_{L_{p,\lambda}(I_0)} = \left\| \frac{1}{\delta} \int\limits_0^{\delta} |\triangle_t^r f(\cdot)| dt \right\|_{L_{p,\lambda}(I_0)}$$

$$= \left\| \frac{1}{\delta} \int_0^{\delta} \left| \sum_{s=0}^{r} \binom{r}{s} (-1)^{r+s+1} f(\cdot + st) \right| dt \right\|_{L_{p,\lambda}(I_0)}$$

$$\leq \sum_{s=0}^{r} \binom{r}{s} \left\| \frac{1}{\delta} \int_0^{\delta} |f(\cdot + st)| \, dt \right\|_{L_{p,\lambda}(I_0)}$$

$$\leq \left\| \frac{1}{\delta} \int_0^{\delta} |f(\cdot)| dt \right\|_{L_{p,\lambda}(I_0)} + \sum_{s=1}^{r} \binom{r}{s} \left\| \frac{1}{\delta} \int_0^{\delta} |f(\cdot + st)| \, dt \right\|_{L_{p,\lambda}(I_0)}$$

$$= \|f\|_{L_{p,\lambda}(I_0)} + \sum_{s=1}^{r} \binom{r}{s} \left\| \frac{1}{s\delta} \int_0^{s\delta} |f(\cdot + u)| du \right\|_{L_{p,\lambda}(I_0)}$$

$$\leq \|f\|_{L_{p,\lambda}(I_0)} + \sum_{s=1}^{r} \binom{r}{s} \left\| \frac{1}{\delta} \int_0^{r\delta} |f(\cdot + u)| du \right\|_{L_{p,\lambda}(I_0)}$$

$$\leq \|f\|_{L_{p,\lambda}(I_0)} + r2^r \left\| \frac{1}{r\delta} \int_0^{r\delta} |f(\cdot + u)| du \right\|_{L_{p,\lambda}(I_0)}.$$

Since function f on $\mathbb{R}$ is 2π-periodic , without loss of generality, we can assume $r\delta < 2\pi$ and by boundedness of maximal operator in Morrey spaces [2], we get

$$\|\sigma_\delta^r(f)\|_{L_{p,\lambda}(I_0)} \le \|f(\cdot)\|_{L_{p,\lambda}(I_0)} + r2^r C(p)\,\|f(\cdot)\|_{L_{p,\lambda}(I_0)}$$
$$= C(p,r)\,\|f(\cdot)\|_{L_{p,\lambda}(I_0)}\,.$$

In the following lemma we give a characterization of $K-$functionals in terms of the modulus of smoothness in Morrey spaces $L_{p,\lambda}(I_0)$.

Lemma 2.2 *Let $f \in L_{p,\lambda}(I_0)$, $0 \le \lambda < 1$ and $1 < p < \infty$. Then for every $r \in \mathbb{N}$ we have*

$$c\,\Omega^r(f,h,L_{p,\lambda}(I_0)) \le K_r(f,h)_{L_{p,\lambda}(I_0)} \le C\Omega^r(f,h,L_{p,\lambda}(I_0)), \quad 0 < h \le c(r,\lambda)$$

with constants $c, C > 0$.

Proof. Let $g \in W^r L_{p,\lambda}(I_0)$. Then $g^{(r)} \in L_{p,\lambda}(I_0)$ and hence $g^{(r)} \in L_1(I_0)$. Therefore we write

$$\triangle_t^r g(x) = \int\limits_0^t \cdots \int\limits_0^t g^{(r)}(x+t_1+t_2+...+t_r)dt_1...dt_r.$$

Using generalized Minkowski inequality and (1.1), we have

$$\Omega^r(g,h,L_{p,\lambda}(I_0)) := \sup_{0<\delta\le\min\{2\pi,h\}} \left\| \frac{1}{\delta}\int\limits_0^\delta |\triangle_t^r g(\cdot)|dt \right\|_{L_{p,\lambda}(I_0)}$$
$$\le \sup_{0<\delta\le\min\{2\pi,h\}} \frac{1}{\delta}\int\limits_0^\delta \left\| \int\limits_0^t \cdots \int\limits_0^t |g^{(r)}(\cdot+t_1+t_2+...+t_r)|dt_1...dt_r \right\|_{L_{p,\lambda}(I_0)} dt$$
$$\le h^r \left\| \frac{1}{h^r}\int\limits_0^h \cdots \int\limits_0^h |g^{(r)}(\cdot+t_1+t_2+...+t_r)|dt_1...dt_r \right\|_{L_{p,\lambda}(I_0)}$$
$$= h^r \left\| \frac{1}{h^{r-1}}\int\limits_0^h \cdots \int\limits_0^h \left\{ \frac{1}{h} \int\limits_{t_1+...+t_{r-1}}^{t_1+...+t_{r-1}+h} |g^{(r)}(\cdot+t)|dt \right\} dt_1...dt_{r-1} \right\|_{L_{p,\lambda}(I_0)}$$
$$\le h^r \frac{1}{h^{r-1}}\int\limits_0^h \cdots \int\limits_0^h \left\| \frac{1}{h}\int\limits_0^{h+(r-1)h} |g^{(r)}(\cdot+t)|dt \right\|_{L_{p,\lambda}(I_0)} dt_1...dt_{r-1}$$
$$= h^r \left\| \frac{1}{h}\int\limits_0^{rh} |g^{(r)}(\cdot+t)|dt \right\|_{L_{p,\lambda}(I_0)}$$
$$\le c_r h^r \|Mg^{(r)}\|_{L_{p,\lambda}(I_0)}$$
$$\le c_r h^r \|g^{(r)}\|_{L_{p,\lambda}(I_0)}.$$

Hence, from the definition of $K_r(f,h)_{L_{p,\lambda}(I_0)}$, we obtain

$$\begin{aligned}\Omega^r(f,h,L_{p,\lambda}(I_0)) &\le \Omega^r(f-g,h,L_{p,\lambda}(I_0)) + \Omega^r(g,h,L_{p,\lambda}(I_0)) \\ &\le c\left(\|f-g\|_{L_{p,\lambda}(I_0)} + h^r\|g^{(r)}\|_{L_{p,\lambda}(I_0)}\right) \\ &\le cK_r(f,h)_{L_{p,\lambda}(I_0)}\end{aligned}$$

for any $f \in L_{p,\lambda}(I_0)$. In order to prove the converse inequality, we introduce a Steklov-type transform for $f \in L_{p,\lambda}(I_0)$, $r \ge 1$, $h > 0$:

$$f_{r,h}(x) := \frac{2}{h}\int_{\frac{h}{2}}^{h}\left(\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\sum_{s=0}^{r-1}\binom{r}{s}(-1)^{r+s+1}f\left(x+\frac{r-s}{r}(t_1+...+t_r)\right)dt_1...dt_r\right)d\delta.$$

By simple calculations we have

$$\begin{aligned}&|f_{r,h}(x)-f(x)| \\ &= \left|\frac{2}{h}\int_{\frac{h}{2}}^{h}\left(\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\sum_{s=0}^{r}\binom{r}{s}(-1)^{r+s+1}f\left(x+\frac{r-s}{r}(t_1+...+t_r)\right)dt_1...dt_r\right)d\delta\right| \\ &= \left|\frac{2}{h}\int_{\frac{h}{2}}^{h}\left(\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\sum_{s=0}^{r}\binom{r}{s}(-1)^{r+s+1}f\left(x+\frac{s}{r}(t_1+...+t_r)\right)dt_1...dt_r\right)d\delta\right| \\ &= \left|\frac{2}{h}\int_{\frac{h}{2}}^{h}\left(\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\triangle^r_{\frac{t_1+...+t_r}{r}}f(x)dt_1...dt_r\right)d\delta\right|.\end{aligned}$$

Taking the norm and applying generalized Minkowski inequality, we get

$$\begin{aligned}\|f_{r,h}-f\|_{L_{p,\lambda}(I_0)} &\le \frac{2}{h}\int_{\frac{h}{2}}^{h}\left\|\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\triangle^r_{\frac{t_1+...+t_r}{r}}f(x)dt_1...dt_r\right\|_{L_{p,\lambda}(I_0)}d\delta \\ &\le \sup_{\frac{h}{2}\le\delta\le h}\left\|\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\triangle^r_{\frac{t_1+...+t_r}{r}}f(x)dt_1...dt_r\right\|_{L_{p,\lambda}(I_0)} \\ &= \sup_{\frac{h}{2}\le\delta\le h}\left\|\frac{1}{\delta^{r-1}}\int_0^{\delta}\cdots\int_0^{\delta}\left(\frac{1}{\delta}\int_{t_1+...+t_{r-1}}^{\delta+t_1+...+t_{r-1}}\triangle^r_{\frac{t}{r}}f(x)dt\right)dt_1...dt_{r-1}\right\|_{L_{p,\lambda}(I_0)}\end{aligned}$$

$$\le \sup_{\frac{h}{2}\le\delta\le h} \frac{1}{\delta^{r-1}} \int_0^\delta \cdots \int_0^\delta \left\| \frac{1}{\delta} \int_0^{\delta+(r-1)\delta} |\triangle^r_{\frac{t}{r}} f(x)| dt \right\|_{L_{p,\lambda}(I_0)} dt_1 ... dt_{r-1}$$

$$\le \sup_{\frac{h}{2}\le\delta\le h} \left\| \frac{1}{\delta} \int_0^{r\delta} |\triangle^r_{\frac{t}{r}} f(x)| dt \right\|_{L_{p,\lambda}(I_0)}$$

$$\le r \sup_{0\le\delta\le h} \left\| \frac{1}{\delta} \int_0^{\delta} |\triangle^r_{t} f(x)| dt \right\|_{L_{p,\lambda}(I_0)} \le r\, \Omega^r(f,h,L_{p,\lambda}(I_0)).$$

Differentiating $f_{r,h}$, we have

$$f^{(r)}_{r,h}(x) = \frac{2}{h} \int_{\frac{h}{2}}^{h} \frac{1}{\delta^r} \sum_{s=0}^{r-1} \binom{r}{s} (-1)^{r+s+1} \left(\frac{r}{r-s}\right)^r \triangle^r_{\frac{r-s}{r}} f(x) d\delta.$$

Therefore,

$$|f^{(r)}_{r,h}(x)| \le \frac{2}{h} \int_{\frac{h}{2}}^{h} \frac{1}{\delta^r} \sum_{s=0}^{r-1} \binom{r}{s} \left(\frac{r}{r-s}\right)^r |\triangle^r_{\frac{r-s}{r}} f(x)| d\delta$$

$$\le \frac{c_r}{h^r} \sum_{s=0}^{r-1} \binom{r}{s} \left(\frac{r}{r-s}\right)^r \frac{1}{h} \int_0^h |\triangle^r_{\frac{r-s}{r}} f(x)| d\delta$$

$$= \frac{c_r}{h^r} \sum_{s=0}^{r-1} \binom{r}{s} \left(\frac{r}{r-s}\right)^r \frac{1}{\frac{r-s}{r}h} \int_0^{\frac{r-s}{r}h} |\triangle^r_{u} f(x)| du.$$

Hence,

$$\|f^{(r)}_{r,h}\|_{L_{p,\lambda}(I_0)} \le c_r h^{-r} \|\sigma^r_{\frac{r-s}{r}h} f\|_{L_{p,\lambda}(I_0)}$$
$$\le c_r h^{-r} \Omega^r(f,h,L_{p,\lambda}(I_0)).$$

Thus,

$$K_r(f,h)_{L_{p,\lambda}(I_0)} \le \|f - f_{r,h}\|_{L_{p,\lambda}(I_0)} + h^r \|f^{(r)}_{r,h}\|_{L_{p,\lambda}(I_0)}$$
$$\le C\, \Omega^r(f,h,L_{p,\lambda}(I_0)).$$

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k coskx + b_k sinkx \tag{2.1}$$

be the Fourier series of $f \in \widetilde{L}_{p,\lambda}(I_0)$, $0 \le \lambda \le 1$, $1 < p < \infty$ and $S_n(x,f)$ be its n th partial sum. Using the method of the proof of Lemma 2.1 we see that

$$\begin{aligned} \|f - S_n(\cdot,f)\|_{\widetilde{L}_{p,\lambda}(I_0)} &\le C E_n(f)_{\widetilde{L}_{p,\lambda}(I_0)}, \\ E_n(\tilde{f})_{\widetilde{L}_{p,\lambda}(I_0)} &\le C E_n(f)_{\widetilde{L}_{p,\lambda}(I_0)}, \end{aligned} \tag{2.2}$$

where $\tilde{f}$ is the conjugate function of f.

Lemma 2.3 *Let $r \geq 1$. Then for any $f \in \dot{W}^r \widetilde{L}_{p,\lambda}(I_0)$, $0 \leq \lambda < 1$, $1 < p < \infty$, we have*

$$E_n(f)_{\widetilde{L}_{p,\lambda}(I_0)} \leq \frac{C}{n^r} \|f^{(r)}\|_{\widetilde{L}_{p,\lambda}(I_0)}, \quad n \in \mathbb{N}$$

with a constant $C = C(p, \lambda, r)$.

Proof. Let

$$f(x) \sim \sum_{k=0}^{\infty} a_k coskx + b_k sinkx$$

be the Fourier series of $f \in \widetilde{L}_{p,\lambda}(I_0)$, $0 \leq \lambda < 1$, $1 < p < \infty$ and $S_n(x, f)$ be its n th partial sum. Then

$$\tilde{f}(x) \sim \sum_{k=0}^{\infty} b_k coskx - a_k sinkx.$$

Setting

$$A_k(x, f) := a_k coskx + b_k sinkx, \quad k \in \mathbb{N}$$

we have $f(x) = \sum\limits_{k=0}^{\infty} A_k(x, f)$ in the norm of $\widetilde{L}_{p,\lambda}(I_0)$. Since

$$\begin{aligned}
A_k(x, f) &= a_k coskx + b_k sinkx \\
&= a_k cos\left(kx + \frac{r\pi}{2} - \frac{r\pi}{2}\right) + b_k sin\left(kx + \frac{r\pi}{2} - \frac{r\pi}{2}\right) \\
&= cos\frac{r\pi}{2}\left[a_k cosk\left(x + \frac{r\pi}{2k}\right) + b_k sink\left(x + \frac{r\pi}{2k}\right)\right] \\
&+ sin\frac{r\pi}{2}\left[a_k sink\left(x + \frac{r\pi}{2k}\right) - b_k cosk\left(x + \frac{r\pi}{2k}\right)\right] \\
&= A_k\left(x + \frac{r\pi}{2k}, f\right) cos\frac{r\pi}{2} + A_k\left(x + \frac{r\pi}{2k}, \tilde{f}\right) sin\frac{r\pi}{2}
\end{aligned}$$

and

$$A_k\left(x, f^{(r)}\right) = k^r A_k\left(x + \frac{r\pi}{2k}, f\right),$$

we get

$$\sum_{k=0}^{\infty} A_k(x, f) = A_0(x, f) + cos\frac{r\pi}{2} \sum_{k=1}^{\infty} A_k(x + \frac{r\pi}{2k}, f)$$

$$+sin\frac{r\pi}{2} \sum_{k=1}^{\infty} A_k(x + \frac{r\pi}{2k}, \tilde{f})$$

$$= A_0(x, f) + cos\frac{r\pi}{2} \sum_{k=1}^{\infty} \frac{1}{k^r} A_k(x, f^{(r)}) + sin\frac{r\pi}{2} \sum_{k=1}^{\infty} \frac{1}{k^r} A_k(x, \tilde{f}^{(r)}).$$

Then

$$\begin{aligned}
f(x) - S_n(x, f) &= \sum_{k=n+1}^{\infty} A_k(x, f) \\
&= cos\frac{r\pi}{2} \sum_{k=n+1}^{\infty} \frac{1}{k^r} A_k(x, f^{(r)}) + sin\frac{r\pi}{2} \sum_{k=n+1}^{\infty} \frac{1}{k^r} A_k(x, \tilde{f}^{(r)}).
\end{aligned}$$

Taking into account

$$\sum_{k=n+1}^{\infty} \frac{1}{k^r} A_k(x, f^{(r)}) = \sum_{k=n+1}^{\infty} \frac{1}{k^r} \left[S_k(x, f^{(r)}) - S_{k-1}(x, f^{(r)}) \right]$$

$$= \sum_{k=n+1}^{\infty} \frac{1}{k^r} \left\{ \left[S_k(x, f^{(r)}) - f^{(r)}(x) \right] - \left[S_{k-1}(x, f^{(r)}) - f^{(r)}(x) \right] \right\}$$

$$= \sum_{k=n+1}^{\infty} \left(\frac{1}{k^r} - \frac{1}{(k+1)^r} \right) \left[S_k(x, f^{(r)}) - f^{(r)}(x) \right] - \frac{1}{(n+1)^r} \left[S_n(x, f^{(r)}) - f^{(r)}(x) \right]$$

and

$$\sum_{k=n+1}^{\infty} \frac{1}{k^r} A_k(x, \tilde{f}^{(r)}) = \sum_{k=n+1}^{\infty} \left(\frac{1}{k^r} - \frac{1}{(k+1)^r} \right) \left[S_k(x, \tilde{f}^{(r)}) - \tilde{f}^{(r)}(x) \right]$$

$$- \frac{1}{(n+1)^r} \left[S_n(x, \tilde{f}^{(r)}) - \tilde{f}^{(r)}(x) \right],$$

by (2.2), we have

$$\|f - S_n(\cdot, f)\|_{\widetilde{L}_{p,\lambda}(I_0)} \le \sum_{k=n+1}^{\infty} \left(\frac{1}{k^r} - \frac{1}{(k+1)^r} \right) \left\| S_k(\,\cdot, f^{(r)}) - f^{(r)} \right\|_{\widetilde{L}_{p,\lambda}(I_0)}$$

$$+ \frac{1}{(n+1)^r} \left\| S_n(\cdot, f^{(r)}) - f^{(r)} \right\|_{\widetilde{L}_{p,\lambda}(I_0)} + \sum_{k=n+1}^{\infty} \left(\frac{1}{k^r} - \frac{1}{(k+1)^r} \right) \left\| S_k(\,\cdot, \tilde{f}^{(r)}) - \tilde{f}^{(r)} \right\|_{\widetilde{L}_{p,\lambda}(I_0)}$$

$$+ \frac{1}{(n+1)^r} \left\| S_n(\,\cdot, \tilde{f}^{(r)}) - \tilde{f}^{(r)} \right\|_{\widetilde{L}_{p,\lambda}(I_0)}$$

$$\le C \left\{ \sum_{k=n+1}^{\infty} \left(\frac{1}{k^r} - \frac{1}{(k+1)^r} \right) E_k(f^{(r)})_{\widetilde{L}_{p,\lambda}(I_0)} + \frac{1}{(n+1)^r} E_n(f^{(r)})_{\widetilde{L}_{p,\lambda}(I_0)} \right\}$$

$$+ C \left\{ \sum_{k=n+1}^{\infty} \left(\frac{1}{k^r} - \frac{1}{(k+1)^r} \right) E_k(\tilde{f}^{(r)})_{\widetilde{L}_{p,\lambda}(I_0)} + \frac{1}{(n+1)^r} E_n(\tilde{f}^{(r)})_{\widetilde{L}_{p,\lambda}(I_0)} \right\}.$$

After simple calculations and using second relation of (2.2), we get

$$\|f - S_n(\cdot, f)\|_{\widetilde{L}_{p,\lambda}(I_0)} \le C E_n(f^{(r)})_{\widetilde{L}_{p,\lambda}(I_0)} \left\{ \sum_{k=n+1}^{\infty} \left(\frac{1}{k^r} - \frac{1}{(k+1)^r} \right) + \frac{1}{(n+1)^r} \right\}$$

$$+ C E_n(\tilde{f}^{(r)})_{\widetilde{L}_{p,\lambda}(I_0)} \left\{ \sum_{k=n+1}^{\infty} \left(\frac{1}{k^r} - \frac{1}{(k+1)^r} \right) + \frac{1}{(n+1)^r} \right\}$$

$$\le \frac{C}{(n+1)^r} E_n(f^{(r)})_{\widetilde{L}_{p,\lambda}(I_0)}.$$

Hence,

$$E_n(f)_{\widetilde{L}_{p,\lambda}(I_0)} \le \|f - S_n(\,\cdot, f)\|_{\widetilde{L}_{p,\lambda}(I_0)}$$

$$\le \frac{C}{n^r} E_n(f^{(r)})_{\widetilde{L}_{p,\lambda}(I_0)}$$

$$\le \frac{C}{n^r} \|f^{(r)}\|_{\widetilde{L}_{p,\lambda}(I_0)}.$$

Now we will give the Bernstein inequality in Morrey spaces. Bernstein inequalities date back to 1912 when S.N. Bernstein proved the first inequality of this type for L_∞ norms of trigonometric polynomials. A generalization can be found in [3]; this result, which is credited to Zygmund, states that any trigonometric polynomial T of degree $n \in \mathbb{N} \cup \{0\}$ satisfies

$$\|T_n^{(k)}\|_{L_p(I_0)} \leq Cn^k \|T_n\|_{L_p(I_0)}$$

for $1 < p < \infty$. Therefore we have the following:

Lemma 2.4 *(Bernstein inequality in Morrey spaces) Let* $f \in L_{p,\lambda}(I_0)$, $0 \leq \lambda < 1$ *and* $1 < p < \infty$. *Then for every trigonometric polynomial* T_n *and* $k \in \mathbb{N}$

$$\|T_n^{(k)}\|_{L_{p,\lambda}(I_0)} \leq Cn^k \|T_n\|_{L_{p,\lambda}(I_0)}, \quad n \in \mathbb{N} \cup \{0\}$$

with a constant C independent of n.

Proof. The desired result is obtained by iteration to the inequality in [4, Lemma 5.2].

3 Proofs of the main results

Proof of Theorem 1.1 Let $g \in \dot{W}^r \widetilde{L}_{p,\lambda}(I_0)$. By Lemma 2.3

$$E_n(f)_{\widetilde{L}_{p,\lambda}(I_0)} \leq E_n(f-g)_{\widetilde{L}_{p,\lambda}(I_0)} + E_n(g)_{\widetilde{L}_{p,\lambda}(I_0)}$$

$$\leq \|f-g\|_{\widetilde{L}_{p,\lambda}(I_0)} + \frac{C}{n^r}\|g^{(r)}\|_{\widetilde{L}_{p,\lambda}(I_0)}.$$

Since this inequality holds for every $g \in \dot{W}^r \widetilde{L}_{p,\lambda}(I_0)$, by the definition of the $K-$functional and by Lemma 2.2, we get

$$E_n(f)_{\widetilde{L}_{p,\lambda}(I_0)} \leq C\, K_r\left(f, \frac{1}{n}\right)_{\widetilde{L}_{p,\lambda}(I_0)} \leq C\, \Omega^r(f, \frac{1}{n}, \widetilde{L}_{p,\lambda}(I_0)).$$

Thus the proof is completed.

Proof of Theorem 1.2 Let $T_n \in \mathcal{P}_n$ be the polynomial of best approximation to f in $\widetilde{L}_{p,\lambda}(I_0)$. For any integer $j = 1, 2, ...,$

$$K_r\left(f, \frac{1}{n}\right)_{\widetilde{L}_{p,\lambda}(I_0)} = \inf_{g \in W^r \widetilde{L}_{p,\lambda}(I_0)} \left\{\|f-g\|_{\widetilde{L}_{p,\lambda}(I_0)} + \frac{1}{n^r}\|g^{(r)}\|_{\widetilde{L}_{p,\lambda}(I_0)}\right\}$$
$$\leq \|f - T_{2^{j+1}}\|_{\widetilde{L}_{p,\lambda}(I_0)} + \frac{1}{n^r}\|T^{(r)}_{2^{j+1}}\|_{\widetilde{L}_{p,\lambda}(I_0)}.$$

Using Lemma 2.4, we get

$$\begin{aligned}
\|T^{(r)}_{2^{j+1}}\|_{\widetilde{L}_{p,\lambda}(I_0)} &\le \|T^{(r)}_1 - T^{(r)}_0\|_{\widetilde{L}_{p,\lambda}(I_0)} + \sum_{i=0}^{j} \|T^{(r)}_{2^{j+1}} - T^{(r)}_{2^i}\|_{\widetilde{L}_{p,\lambda}(I_0)} \\
&\le C\left\{\|T_1 - T_0\|_{\widetilde{L}_{p,\lambda}(I_0)} + \sum_{i=0}^{j} 2^{(i+1)r}\|T_{2^{i+1}} - T_{2^i}\|_{\widetilde{L}_{p,\lambda}(I_0)}\right\} \\
&\le C\Big\{E_1(f)_{\widetilde{L}_{p,\lambda}(I_0)} + E_0(f)_{\widetilde{L}_{p,\lambda}(I_0)} \\
&+ \sum_{i=0}^{j} 2^{(i+1)r}\left\{E_{2^{i+1}}(f)_{\widetilde{L}_{p,\lambda}(I_0)} + E_{2^i}(f)_{\widetilde{L}_{p,\lambda}(I_0)}\right\}\Big\} \\
&\le C\left\{E_0(f)_{\widetilde{L}_{p,\lambda}(I_0)} + \sum_{i=0}^{j} 2^{(i+1)r} E_{2^i}(f)_{\widetilde{L}_{p,\lambda}(I_0)}\right\} \\
&= C\left\{E_0(f)_{\widetilde{L}_{p,\lambda}(I_0)} + 2^r E_1(f)_{\widetilde{L}_{p,\lambda}(I_0)} + \sum_{i=1}^{j} 2^{(i+1)r} E_{2^i}(f)_{\widetilde{L}_{p,\lambda}(I_0)}\right\}.
\end{aligned}$$

Since

$$2^{(i+1)r} E_{2^i}(f)_{\widetilde{L}_{p,\lambda}(I_0)} \le 2^{2r} \sum_{m=2^{i-1}+1}^{2^i} m^{r-1} E_m(f)_{\widetilde{L}_{p,\lambda}(I_0)} \tag{3.1}$$

for $i \ge 1$, we have

$$\begin{aligned}
&\|T^{(r)}_{2^{j+1}}\|_{\widetilde{L}_{p,\lambda}(I_0)} \\
&\le C\left\{E_0(f)_{\widetilde{L}_{p,\lambda}(I_0)} + 2^r E_1(f)_{\widetilde{L}_{p,\lambda}(I_0)} + 2^{2r}\sum_{m=2}^{2^j} m^{r-1} E_m(f)_{\widetilde{L}_{p,\lambda}(I_0)}\right\} \\
&\le C\left\{E_0(f)_{\widetilde{L}_{p,\lambda}(I_0)} + \sum_{m=1}^{2^j} m^{r-1} E_m(f)_{\widetilde{L}_{p,\lambda}(I_0)}\right\}.
\end{aligned}$$

Selecting j such that $2^j \le n < 2^{j+1}$, from (3.1) we get

$$\begin{aligned}
E_{2^{j+1}}(f)_{\widetilde{L}_{p,\lambda}(I_0)} &= \frac{2^{(j+1)r} E_{2^{j+1}}(f)_{\widetilde{L}_{p,\lambda}(I_0)}}{2^{(j+1)r}} \\
&\le \frac{1}{n^r} 2^{(j+1)r} E_{2^{j+1}}(f)_{\widetilde{L}_{p,\lambda}(I_0)} \\
&\le \frac{1}{n^r} \sum_{m=2^{j-1}+1}^{2j} m^{r-1} E_m(f)_{\widetilde{L}_{p,\lambda}(I_0)}.
\end{aligned}$$

Now by Lemma 2.2, we conclude that

$$\begin{aligned}\Omega^r(f,\frac{1}{n},\widetilde{L}_{p,\lambda}(I_0)) &\le CK_r\left(f,\frac{1}{n}\right)_{\widetilde{L}_{p,\lambda}(I_0)} \\ &\le CE_{2^{j+1}}(f)_{\widetilde{L}_{p,\lambda}(I_0)} + \frac{1}{n^r}\|T_{2^{j+1}}\|_{\widetilde{L}_{p,\lambda}(I_0)} \\ &\le \frac{C}{n^r}\sum_{m=2^{j-1}+1}^{2^j} m^{r-1}E_m(f)_{\widetilde{L}_{p,\lambda}(I_0)} \\ &+ \frac{C}{n^r}\left\{E_0(f)_{\widetilde{L}_{p,\lambda}(I_0)} + \sum_{m=1}^{2^j} m^{r-1}E_m(f)_{\widetilde{L}_{p,\lambda}(I_0)}\right\} \\ &\le \frac{C}{n^r}\left\{E_0(f)_{\widetilde{L}_{p,\lambda}(I_0)} + \sum_{m=1}^{n} m^{r-1}E_m(f)_{\widetilde{L}_{p,\lambda}(I_0)}\right\}.\end{aligned}$$

Thus the proof is completed.

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