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Existence for delay fractional differential equations with mixed fractional derivatives on an infinite interval

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Abstract. *Using the Krasnoselskii's fixed point theorem in a weighted Banach space, we give sufficient conditions for the existence of solutions for initial value problems for delay fractional differential equations with the mixed Riemann-Liouville and Caputo fractional derivatives on an infinite interval. At the end, an example is given to illustrate our main results.*

Keywords. Mixed derivatives, Delay fractional differential equations, Existence, Fixed point theorems.

Mathematics Subject Classification (2010): 34A08, 34K05.

1 Introduction

The great interest recently in fractional differential equations, have been the principle raison of both the intensive development of the theory of fractional calculus itself and its various applications of such contributions in many scientific disciplines such as physics, chemistry, biology, engineering, viscoelasticity, signal processing, electrotechnical, electrochemistry and controllability. For more details about applicabilities internal and external of fractional calculus, see [4–10] and the references therein.

On the other hand, to the best of our knowledge, the use of mixed fractional derivative in neutral fractional differential equations is still not sufficiently generalized as a important kind of fractional differential equations, where we will interest in this work to study this type of fractional differential equations. Beside, neutral fractional differential equations have been studied extensively in the last decades and by different methods as fixed point theorems, upper and lower solution method, spectral theory. For some recent contributions in fractional boundary value problems, we can see the papers [1–3] and the references therein.

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Benchohra et al. [3], investigated the existence of solutions for the following Riemann-Liouville fractional order functional differential equation with infinite delay

$$\begin{cases} {}^{RL}D^{\alpha}[u(t)-g(t,u_t)] = f(t,u_t),\ t\in[0,T],\ 0<\alpha<1, \\ u\left(t\right)=\phi\left(t\right),\ t\in\left(-\infty,0\right]. \end{cases}$$

Agarwal et al. [1], studied the initial value problem of fractional neutral Caputo fractional derivative

$$\begin{cases} {}^{C}D^{\alpha}[u(t)-g(t,u_t)] = f(t,u_t),\ t\in\left(t_0,\infty\right),\ t_0\geq 0,\ 0<\alpha<1, \\ u_{t_0}=\phi, \end{cases}$$

and established the existence results of solutions of this problem by using Krasnoselskii's fixed point theorem. In [2], Ahmad et al. studied the existence and uniqueness of solutions to the following boundary value problem

$$\begin{cases} D^{\alpha}\left(D^{\beta}u(t)-g\left(t,u_t\right)\right) = f(t,u_t),\ t\in[1,b], \\ u(t)=\phi(t),\ t\in[1-r,1], \\ D^{\beta}u(1)=\eta\in\mathbb{R}, \end{cases}$$

where D^{α} and D^{β} are the Caputo-Hadamard fractional derivatives, $0<\alpha,\beta<1$.

Motivated and inspired by the works mentioned above and the references therein, in this paper we investigate the existence of solutions for the following initial value problem of the mixed Riemann-Liouville and Caputo fractional functional differential equation with delay on an infinite interval

$$\begin{cases} {}^{RL}D^{\alpha}[{}^{C}D^{\beta}u(t)-g(t,u\left(t-r\right))] = f(t,u\left(t-r\right)),\ t\in I=\left[0,\infty\right), \\ u\left(t\right)=\phi\left(t\right),\ t\in\left[-r,0\right], \\ \lim\limits_{t\to 0}t^{1-\alpha}\,{}^{C}D^{\beta}u(t)=0,\ u'(0)=u_0, \end{cases} \tag{1.1}$$

where ${}^{RL}D^{\alpha}$ and ${}^{C}D^{\beta}$ are the Riemann Liouville and the Caputo fractional derivatives respectively, $0<\alpha<1$, $1<\beta<2$, $f,g\in I\times\mathbb{R}\to\mathbb{R}$ are given continuous functions and $\phi\in C([-r,0],\mathbb{R})$. To show the existence of solutions, we transform (1.1) into an integral equation and then use Krasnoselskii's fixed point theorem in a weighted Banach space. The obtained integral equation splits in the sum of two mappings, one is a contraction and the other is compact.

The organization of this paper is as follows. In Section 2 we recall some useful preliminaries, also we present the equivalent fixed point problem corresponding to (1.1). In Section 3 we discuss the existence of solutions for (1.1) via fixed point theory in a weighted Banach space. An example is constructed for illustrating the obtained results.

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts that we need in the sequel. By C_λ we denote the Banach space of all continuous functions from $[-r,+\infty)$ into $\mathbb{R}$ with the norm

$$\|u\|_\lambda=\sup_{t\geq -r}\left\{e^{-\lambda t}\left|u\left(t\right)\right|\right\},$$

where $\lambda>1$. Also $C_r=C([-r,0],\mathbb{R})$ is endowed with norm

$$\|\phi\|_C:=\sup\{|\phi(t)|:t\in[-r,0]\}.$$

Definition 2.1 **([5,7,10])** *The Riemann-Liouville fractional integral of the function u of order $\alpha > 0$ is defined by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$.

Definition 2.2 **([5,7,10])** *The Riemann-Liouville fractional derivative of the function u of order $\alpha \in (n-1, n]$ is defined by*

$$^{RL}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds.$$

Definition 2.3 **([5,7,10])** *The Caputo fractional derivative of the function u of order $\alpha \in (n-1, n]$ is defined by*

$$^{C}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds.$$

Let $\alpha > 0$ be a real number, we have two following lemmas.

Lemma 2.1 **([7])** *The unique solution of the linear fractional differential equation*

$$^{RL}D^{\alpha}u(t) = 0,$$

is given by

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + ... + c_n t^{\alpha-n},\ c_i \in \mathbb{R},\ i = 1, 2, ..., n.$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Lemma 2.2 **([7])** *The unique solution of the linear fractional differential equation*

$$^{C}D^{\alpha}u(t) = 0,$$

is given by

$$u(t) = c_1 + c_2 t + ... + c_n t^{n-1},\ c_i \in \mathbb{R},\ i = 1, 2, ..., n,$$

where $n = [\alpha] + 1$.

Lemma 2.3 *Problem (1.1) is equivalent to the following Caputo type fractional differential equation with delay*

$$\begin{cases} ^{C}D^{\beta}u(t) = I^{\alpha} f(t, u(t-r)) + g(t, u(t-r)),\ t \geq 0, \\ u(t) = \phi(t),\ t \in [-r, 0], \\ u'(0) = u_0. \end{cases} \tag{2.1}$$

Proof. Using Lemma 2.1, equation one of (1.1) can be written as

$$^{C}D^{\beta}u(t) = I^{\alpha} f(t, u(t-r)) + g(t, u(t-r)) + c_0 t^{\alpha-1},$$

using condition $\lim_{t\to 0} t^{1-\alpha}\, {}^{C}D^{\beta}u(t) = 0$, we get $c_0 = 0$. Then we obtain the desired result.

Lemma 2.4 *Let f and g are continuous functions. Then $u \in C([-r,+\infty))$ is a solution of the problem (2.1) if and only if u is a solution of the delay Cauchy type problem*

$$\begin{cases} u'(t) = I^{\alpha+\beta-1} f(t,u(t-r)) + I^{\beta-1} g(t,u(t-r)) + u_0,\ t \geq 0, \\ u(t) = \phi(t),\ t \in [-r,0]. \end{cases} \tag{2.2}$$

Proof. Let $u \in C([-r,+\infty))$ be a solution of the problem (2.1), for any $t \in \mathbb{R}_+$, we have

$${}^C D^\beta u(t) = \left({}^C D^{\beta-1} D^1 u\right)(t) = I^\alpha f(t,u(t-r)) + g(t,u(t-r)).$$

According to Lemma 2.2 and according to the condition $u'(0) = u_0$, one gets

$$u'(t) = I^{\beta-1}\left[I^\alpha f(t,u(t-r)) + g(t,u(t-r))\right] + u_0,$$

which means that u is a solution of the problem (2.2).

Conversely, let u be a solution of the problem (2.2). Also, for any $t \in \mathbb{R}_+$, it is easy to see that

$$\begin{aligned} {}^C D^\beta u(t) &= {}^C D^{\beta-1} u'(t) \\ &= {}^C D^{\beta-1}\left(I^{\alpha+\beta-1} f(t,u(t-r)) + I^{\beta-1} g(t,u(t-r))\right) + {}^C D^{\beta-1} u_0 \\ &= I^\alpha f(t,u(t-r)) + g(t,u(t-r)). \end{aligned}$$

Besides, we have $u'(0) = u_0$.

Lemma 2.5 *Let $k \in \mathbb{R}^*$ satisfies that $|k| \leq \frac{\lambda-1}{2}$, clearly $\lambda + k > 0$. Then (2.2) can be equivalently written as*

$$\begin{aligned} u(t) &= \phi(0)e^{-kt} + \frac{1-e^{-kt}}{k}u_0 + k\int_0^t e^{-k(t-s)}u(s)ds \\ &+ \frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t \int_\tau^t e^{-k(t-s)}(s-\tau)^{\alpha+\beta-2} ds f(\tau,u(\tau-r))d\tau \\ &+ \frac{1}{\Gamma(\beta-1)}\int_0^t \int_\tau^t e^{-k(t-s)}(s-\tau)^{\beta-2} ds g(\tau,u(\tau-r))d\tau. \end{aligned} \tag{2.3}$$

Proof. It is clear that (2.2) can be written as follow

$$\begin{cases} u'(t) + ku(t) = ku(t) + \frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t (t-s)^{\alpha+\beta-2} f(s,u(s-r))ds \\ + \frac{1}{\Gamma(\beta-1)}\int_0^t (t-s)^{\beta-2} g(s,u(s-r))ds + u_0, \\ u(t) = \phi(t),\ t \in [-r,0]. \end{cases}$$

By the variation of constants formula, we have

$$\begin{aligned} u(t) &= \phi(0)e^{-kt} \\ &+ e^{-kt}\int_0^t \Bigg[ku(s) + \frac{1}{\Gamma(\alpha+\beta-1)}\int_0^s (s-\tau)^{\alpha+\beta-2} f(\tau,u(\tau-r))d\tau \\ &+ \frac{1}{\Gamma(\beta-1)}\int_0^s (s-\tau)^{\beta-2} g(\tau,u(\tau-r))d\tau + u_0\Bigg] e^{ks}ds, \end{aligned}$$

by simplifications, one gets

$$\begin{aligned} u(t) &= \phi(0) e^{-kt} + k\int_0^t e^{-k(t-s)}u(s)ds \\ &+\frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t \int_0^s e^{-k(t-s)}(s-\tau)^{\alpha+\beta-2} f(\tau, u(\tau-r))d\tau ds \\ &+\frac{1}{\Gamma(\beta-1)}\int_0^t \int_0^s e^{-k(t-s)}(s-\tau)^{\beta-2} g(\tau, u(\tau-r))d\tau ds \\ &+u_0 e^{-kt}\int_0^t e^{ks}ds, \end{aligned}$$

some computations and simplifications give us

$$\begin{aligned} u(t) &= \phi(0) e^{-kt} + k\int_0^t e^{-k(t-s)}u(s)ds \\ &+\frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t \int_\tau^t e^{-k(t-s)}(s-\tau)^{\alpha+\beta-2} dsf(\tau, u(\tau-r))d\tau \\ &+\frac{1}{\Gamma(\beta-1)}\int_0^t \int_\tau^t e^{-k(t-s)}(s-\tau)^{\beta-2} dsg(\tau, u(\tau-r))d\tau \\ &+\frac{1-e^{-kt}}{k}u_0. \end{aligned}$$

Furthermore, it's clear that

$$\left(e^{kt}u(t)\right)' = \left(u'(t)+ku(t)\right)e^{kt},$$

using this fact, we get

$$\begin{aligned} &\left(u'(t)+ku(t)\right)e^{kt} \\ &= \Bigg[\phi(0) + k\int_0^t e^{ks}u(s)ds \\ &+\frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t \int_\tau^t e^{ks}(s-\tau)^{\alpha-2}dsf(\tau, u(\tau-r))d\tau \\ &+\frac{1}{\Gamma(\beta-1)}\int_0^t \int_\tau^t e^{ks}(s-\tau)^{\beta-2}dsg(\tau, u(\tau-r))d\tau + \frac{e^{kt}-1}{k}u_0\Bigg]' \\ &= e^{kt}u_0 + ke^{kt}u(t) \\ &+\left[\int_0^t e^{k\tau}I^{\alpha+\beta-1}f(\tau, u(\tau-r))d\tau + \int_0^t e^{k\tau}I^{\beta-1}g(\tau, u(\tau))d\tau\right]' \\ &= e^{kt}\left(u_0 + I^{\alpha+\beta-1}f(t, u(t-r)) + I^{\beta-1}g(t, u(t-r)) + ku(t)\right). \end{aligned}$$

This means that

$$u'(t) = I^{\alpha+\beta-1}f(t, u(t-r)) + I^{\beta-1}g(t, u(t-r)) + u_0.$$

On the other hand, if (2.3) holds we have $u(0) = \phi(0)$. From the argument above, we get that the system (2.2) can be equivalently written as (2.3). Then our following study will focus on the integral equation (2.3). This is complete the proof.

Our main results is based on the Krasnoselskii fixed point theorem.

Theorem 2.1 (Krasnoselskii fixed point theorem [11,12]) *If $\mathcal{M}$ is a nonempty bounded, closed and convex subset of a Banach space E, $\mathcal{A}$ and $\mathcal{B}$ two operators defined on $\mathcal{M}$ with values in E such that*
i) $\mathcal{A}u+\mathcal{B}v \in \mathcal{M}$, for all $u, v \in \mathcal{M}$,
ii) $\mathcal{A}$ is continuous and compact,
iii) $\mathcal{B}$ is a contraction.
Then there exists $w \in \mathcal{M}$ such that $w = \mathcal{A}w + \mathcal{B}w$.

In order to prove our main result, we give the following modified compactness criterion.

Lemma 2.6 ([8]) *Let $\mathcal{M}$ be a subset of the Banach space C_λ. Then $\mathcal{M}$ is relatively compact in C_λ if the following conditions are satisfied:*
i) $\left\{e^{-\lambda t}u(t) : u \in \mathcal{M}\right\}$ is uniformly bounded;
ii) $\left\{e^{-\lambda t}u(t) : u \in \mathcal{M}\right\}$ is equicontinuous on any compact interval of $\mathbb{R}$;
iii) $\left\{e^{-\lambda t}u(t) : u \in \mathcal{M}\right\}$ is equiconvergent at infinity i.e. for any given $\epsilon > 0$, there exists a $T_0 > 0$ such that for all $u \in \mathcal{M}$and $t_1, t_2 > T_0$, it holds

$$\left|e^{-\lambda t_2}u(t_2) - e^{-\lambda t_1}u(t_1)\right| < \epsilon.$$

3 Main results

This section devoted to presenting and proving our main results. Consider the following hypothesis.

$(H1)$ $f, g : I \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

$(H2)$ There exists $\eta \in C\left(I, \mathbb{R}_+^*\right)$ such that

$$|g(t,u) - g(t,v)| \leq \eta\,(t)\,|u - v|\,,\ g(t,0) = 0,$$

with η is bounded.

$(H3)$ There exist $\zeta, \Psi \in C\left(I, \mathbb{R}_+^*\right)$ such that

$$\left|f(t, e^{\lambda(t-r)}u)\right| \leq e^{\lambda t}\zeta\,(t)\,\Psi\,(|u|)\,,$$

with Ψ is nondecreasing and $\zeta \in L^1\left([0,\infty)\right)$.

Theorem 3.1 *Assume that $(H1)-(H3)$ hold. Then problem (1.1) has at least one solution on $[-r, +\infty)$, provided that there exist constants $M_1, M_2 > 0$ such that*

$$\sup_{t\geq 0}\int_0^t e^{-\lambda(t-\tau)}\mathcal{H}\,(t-\tau)\,\eta\,(\tau)\,d\tau \leq M_1 < 1 - \frac{|k|}{\lambda + k} < 1, \tag{3.1}$$

and

$$\sup_{t\geq 0}\int_0^t e^{-\lambda(t-\tau)}\mathcal{K}\,(t-\tau)\,\zeta\,(\tau)\,d\tau \leq M_2, \tag{3.2}$$

where

$$\mathcal{K}\,(t-\tau) = \begin{cases} \frac{1}{\Gamma(\alpha+\beta-1)}\int_\tau^t e^{-k(t-s)}(s-\tau)^{\alpha+\beta-2}ds, & if\, t-\tau \geq 0,\\ 0, & if\, t-\tau \leq 0,\end{cases}$$

and

$$\mathcal{H}\,(t-\tau) = \begin{cases} \frac{1}{\Gamma(\beta-1)}\int_\tau^t e^{-k(t-s)}(s-\tau)^{\beta-2}ds, & if\, t-\tau \geq 0,\\ 0, & if\, t-\tau \leq 0,\end{cases}$$

Proof. Choosing

$$R \geq \frac{\Psi\left(R\right)M_2 + \left|\phi\left(0\right)\right| + 2\frac{|u_0|}{|k|}}{1 - M_1 - \frac{|k|}{\lambda+k}},$$

and define

$$B_R = \{u \in C_\lambda([-r, +\infty), \mathbb{R}) : \|u\| \leq R \text{ and } u\left(t\right) = \phi(t) \text{ if } t \in [-r, 0]\},$$

for any $R > 0$. We define two mapping $\mathcal{A}, \mathcal{B} : B_R \to C_\lambda$ by

$$\left(\mathcal{A}u\right)\left(t\right) = \begin{cases} 0, & \text{if } t \in [-r, 0], \\ k\int_0^t e^{-k(t-s)}u(s)ds + \int_0^t \mathcal{K}\left(t-\tau\right)f(\tau, u(\tau - r))d\tau, & \text{if } t \in I, \end{cases} \tag{3.3}$$

and

$$\left(\mathcal{B}u\right)\left(t\right) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi\left(0\right)e^{-kt} + \dfrac{1 - e^{-kt}}{k}u_0 + \int_0^t \mathcal{H}\left(t-\tau\right)g(\tau, u(\tau - r))d\tau, & \text{if } t \in I. \end{cases} \tag{3.4}$$

Clearly, for $u \in B_R$, both $\mathcal{A}u$ and $\mathcal{B}u$ are continuous functions on $[-r, +\infty)$. Also, for $u \in B_R$, for any $t \geq 0$, we have

$$\begin{aligned} e^{-\lambda t}\left|\left(\mathcal{A}u\right)\left(t\right)\right| &\leq |k|\,e^{-\lambda t}\int_0^t e^{-k(t-s)}\left|u(s)\right|ds \\ &+ \int_0^t e^{-\lambda t}\mathcal{K}\left(t-\tau\right)\left|f(\tau, u(\tau - r))\right|d\tau \\ &\leq |k|\int_0^t e^{-\lambda(t-s)}e^{-k(t-s)}\left|e^{-\lambda s}u(s)\right|ds \\ &+ \int_0^t e^{-\lambda t}\mathcal{K}\left(t-\tau\right)\zeta\left(\tau\right)\Psi\left(\left|u(\tau - r)\right|\right)d\tau \\ &\leq |k|\,\|u\|_\lambda\int_0^\infty e^{-(k+\lambda)s}ds \\ &+ \int_0^t e^{-\lambda(t-\tau)}\mathcal{K}\left(t-\tau\right)\zeta\left(\tau\right)\Psi\left(e^{-\lambda(\tau-r)}\left|u(\tau - r)\right|\right)d\tau \\ &\leq \frac{|k|}{\lambda+k}\|u\|_\lambda + \Psi\left(\|u\|_\lambda\right)\int_0^t e^{-\lambda(t-\tau)}\mathcal{K}\left(t-\tau\right)\zeta\left(\tau\right)d\tau \\ &\leq \frac{|k|}{\lambda+k}R + \Psi\left(R\right)M_2 < \infty, \end{aligned} \tag{3.5}$$

also

$$
\begin{aligned}
e^{-\lambda t}\left|(\mathcal{B}u)(t)\right| &\leq \left|\Phi(0)\right| e^{-(\lambda+k)t} + \frac{e^{-\lambda t}+e^{-(\lambda+k)t}}{|k|}\left|u_0\right| \\
&+ \int_0^t e^{-\lambda t}\mathcal{H}(t-\tau)\, g(\tau,u(\tau-r))d\tau \\
&\leq \left|\Phi(0)\right| + 2\frac{|u_0|}{|k|} + \int_0^t e^{-\lambda(t-\tau)}\mathcal{H}(t-\tau)\, e^{-\lambda\tau} g(\tau,u(\tau-r))d\tau \\
&\leq \left|\Phi(0)\right| + 2\frac{|u_0|}{|k|} + \int_0^t e^{-\lambda(t-\tau)}\mathcal{H}(t-\tau)\,\eta(\tau)\left|e^{-\lambda\tau}u(\tau)\right| d\tau \\
&\leq \left|\Phi(0)\right| + 2\frac{|u_0|}{|k|} + \left\{\int_0^t e^{-\lambda(t-\tau)}\mathcal{H}(t-\tau)\,\eta(\tau)\, d\tau\right\}\|u\|_\lambda \\
&\leq \left|\Phi(0)\right| + 2\frac{|u_0|}{|k|} + M_1 R < \infty. \qquad (3.6)
\end{aligned}
$$

Then $\mathcal{A}B_R \subset C_\lambda$ and $\mathcal{B}B_R \subset C_\lambda$. Now we shall to prove that there exists at least one fixed point of the operator $\mathcal{A}+\mathcal{B}$. To this end, we divide the proof into three claims.

Claim 1. We show that $\mathcal{A}u+\mathcal{B}v \in B_R$ for all $u, v \in B_R$, from (3.5) and (3.6), we get

$$
\|\mathcal{A}u+\mathcal{B}v\|_\lambda \leq \left(\frac{|k|}{\lambda+k}+M_1\right)R + \Psi(R)\,M_2 + |\phi(0)| + 2\frac{|u_0|}{|k|} \leq R, \qquad (3.7)
$$

this means that $\mathcal{A}u+\mathcal{B}v \in B_R$, for all $u, v \in B_R$.

Claim 2. Obviously, $\mathcal{A}$ is continuous operator, it remains to prove that $\mathcal{A}B_R$ is relatively compact in C_λ. In fact, from (3.7), we get that $\left\{e^{-\lambda t}u(t) : u \in B_R\right\}$ is uniformly bounded in C_λ. Moreover, a classical theorem states the fact that the convolution of an L^1-function with a function tending to zero, does also tend to zero. Then we conclude that for $t \geq \tau$, we have

$$
\begin{aligned}
&\lim_{t\to\infty} e^{-\lambda(t-\tau)}\mathcal{K}(t-\tau) \\
&\leq \lim_{t\to\infty}\frac{1}{\Gamma(\alpha+\beta-1)}\int_\tau^t \left[e^{-\lambda(t-\tau)}e^{-k(t-s)}\right]\left[e^{-\lambda(s-\tau)}(s-\tau)^{\alpha+\beta-2}\right] ds \\
&= \lim_{t\to\infty}\frac{1}{\Gamma(\alpha+\beta-1)}\int_\tau^t \left[e^{-\lambda(t-\tau)}e^{-k(t-\tau-s)}\right]\left[e^{-\lambda s}s^{\alpha+\beta-2}\right] ds = 0, \qquad (3.8)
\end{aligned}
$$

due to the fact $\lim_{t\to\infty} e^{-\lambda t}t^{\alpha+\beta-2} = 0$. Together with the continuity of functions $\mathcal{K}$ and $t \longmapsto e^{-\lambda t}$, we get that there exists a constant $M_3 > 0$ such that

$$
e^{-\lambda(t-\tau)}\left|\mathcal{K}(t-\tau)\right| \leq M_3.
$$

Also, for any fixed $T_0 \geq 0$ and any $t_1, t_2 \in [0, T_0]$, $t_1 < t_2$, we have

$$\begin{aligned}
&\left| e^{-\lambda t_2} (\mathcal{A}u)(t_2) - e^{-\lambda t_1} (\mathcal{A}u)(t_1) \right| \\
&= \left| k \int_0^{t_2} e^{-\lambda t_2} e^{-k(t_2-s)} u(s) ds - k \int_0^{t_1} e^{-\lambda t_1} e^{-k(t_1-s)} u(s) ds \right. \\
&+ \int_0^{t_2} e^{-\lambda t_2} \mathcal{K}(t_2 - \tau) f(\tau, u(\tau - r)) d\tau \\
&\left. - \int_0^{t_1} e^{-\lambda t_1} \mathcal{K}(t_1 - \tau) f(\tau, u(\tau - r)) d\tau \right| \\
&\leq |k| \int_0^{t_1} \left| e^{-\lambda t_2} e^{-k(t_2-s)} - e^{-\lambda t_1} e^{-k(t_1-s)} \right| |u(s)| ds \\
&+ \int_0^{t_1} \left| e^{-\lambda t_2} \mathcal{K}(t_2 - \tau) - e^{-\lambda t_1} \mathcal{K}(t_1 - \tau) \right| |f(\tau, u(\tau - r))| d\tau \\
&+ |k| \int_{t_1}^{t_2} e^{-\lambda t_2} e^{-k(t_2-s)} |u(s)| ds + \int_{t_1}^{t_2} e^{-\lambda t_2} \mathcal{K}(t_2 - \tau) |f(\tau, u(\tau - r))| d\tau \\
&\leq |k| \int_0^{t_1} \left| e^{-(\lambda+k)(t_2-s)} - e^{-(\lambda+k)(t_1-s)} \right| \left| e^{-\lambda s} u(s) \right| ds \\
&+ \int_0^{t_1} \left| e^{-\lambda t_2} \mathcal{K}(t_2 - \tau) - e^{-\lambda t_1} \mathcal{K}(t_1 - \tau) \right| \zeta(\tau) \Psi(|u(\tau - r)|) d\tau \\
&+ \int_{t_1}^{t_2} e^{-\lambda(t_2-\tau)} \mathcal{K}(t_2 - \tau) \zeta(\tau) e^{-\lambda\tau} \Psi(|u(\tau - r)|) d\tau \\
&+ |k| \int_{t_1}^{t_2} e^{-(\lambda+k)(t_2-s)} \left| e^{-\lambda s} u(s) \right| ds \\
&\leq \left\{ |k| \int_0^{t_1} \left| e^{-(\lambda+k)(t_2-s)} - e^{-(\lambda+k)(t_1-s)} \right| ds + |k| \int_{t_1}^{t_2} e^{-(\lambda+k)(t_2-s)} ds \right\} R \\
&+ \left\{ \int_0^{t_1} \left| e^{-\lambda t_2} \mathcal{K}(t_2 - \tau) - e^{-\lambda t_1} \mathcal{K}(t_1 - \tau) \right| \zeta(\tau) d\tau \right. \\
&\left. + \|\zeta\|_\infty M_3 (t_2 - t_1) \right\} \Psi(R) \\
&\to 0 \text{ as } t_2 \to t_1,
\end{aligned}$$

this means that $\left\{ e^{-\lambda t} u(t) : u \in B_R \right\}$ is equicontinuous on any compact interval of $\mathbb{R}_+$, it remains also to show that the set $\left\{ e^{-\lambda t} u(t) : u \in B_R \right\}$ is equiconvergent at infinity. In fact, for any $\epsilon \geq \frac{6R|k|}{\lambda+k} > 0$, there exists a $L > 0$ such that

$$M_3 \int_L^\infty \zeta(\tau) d\tau \leq \frac{\epsilon}{6}.$$

According to (3.8), we get that

$$\lim_{t\to\infty} \sup_{\tau\in[0,L]} e^{-\lambda(t-\tau)} \mathcal{K}(t-\tau) \leq \max\left\{ \lim_{t\to\infty} e^{-\lambda(t-L)} \mathcal{K}(t-L), \lim_{t\to\infty} e^{-\lambda t} \mathcal{K}(t) \right\} = 0.$$

Then, there exists $T > L$ such that for $t_1, t_2 \geq T$, we have

$$\begin{aligned}
&\sup_{\tau\in[0,L]} \left| e^{-\lambda t_2} \mathcal{K}\left(t_2-\tau\right) e^{\lambda\tau} - e^{-\lambda t_1} \mathcal{K}\left(t_1-\tau\right) e^{\lambda\tau} \right| \\
&\leq \sup_{\tau\in[0,L]} \left| e^{-\lambda(t_2-\tau)} \mathcal{K}\left(t_2-\tau\right) \right| + \sup_{\tau\in[0,L]} \left| e^{-\lambda(t_1-\tau)} \mathcal{K}\left(t_1-\tau\right) \right| \\
&\leq \frac{\epsilon}{6} \left(\Psi\left(R\right) \int_0^\infty \zeta\left(\tau\right) d\tau \right)^{-1}.
\end{aligned}$$

Furthermore, for $t \geq s$, one gets

$$\lim_{t\to\infty} e^{-(\lambda+k)(t-s)} = 0,$$

then for $t_1, t_2 \geq T$, we have

$$\begin{aligned}
&\sup_{\tau\in[0,L]} \left| e^{-(\lambda+k)(t_2-s)} - e^{-(\lambda+k)(t_1-s)} \right| \\
&\leq \sup_{\tau\in[0,L]} \left| e^{-(\lambda+k)(t_2-s)} \right| + \sup_{\tau\in[0,L]} \left| e^{-(\lambda+k)(t_1-s)} \right| \leq \frac{\epsilon}{6} \left(R\left|k\right|\right)^{-1}.
\end{aligned}$$

Therefore, for $t_1, t_2 \geq T$, we have

$$\begin{aligned}
&\left| e^{-\lambda t_2} \left(\mathcal{A}u\right)\left(t_2\right) - e^{-\lambda t_1} \left(\mathcal{A}u\right)\left(t_1\right) \right| \\
&= \left| k \int_0^{t_2} e^{-\lambda t_2} e^{-k(t_2-s)} u(s) ds - k \int_0^{t_1} e^{-\lambda t_1} e^{-k(t_1-s)} u(s) ds \right. \\
&+ \int_0^{t_2} e^{-\lambda t_2} \mathcal{K}\left(t_2-\tau\right) f\left(\tau, u(\tau-r)\right) d\tau \\
&\left. - \int_0^{t_1} e^{-\lambda t_1} \mathcal{K}\left(t_1-\tau\right) f\left(\tau, u(\tau-r)\right) d\tau \right| \\
&\leq R\left|k\right| \int_0^L \left| e^{-(\lambda+k)(t_2-s)} - e^{-(\lambda+k)(t_1-s)} \right| ds + R\left|k\right| \int_L^{t_2} e^{-(\lambda+k)(t_2-s)} ds \\
&+ R\left|k\right| \int_L^{t_1} e^{-(\lambda+k)(t_1-s)} ds + \Psi\left(R\right) M_3 \int_L^{t_2} \zeta\left(\tau\right) d\tau + \Psi\left(R\right) M_3 \int_L^{t_1} \zeta\left(\tau\right) d\tau \\
&+ \Psi\left(R\right) \int_0^L \left| e^{-\lambda t_2} \mathcal{K}\left(t_2-\tau\right) e^{\lambda\tau} - e^{-\lambda t_1} \mathcal{K}\left(t_1-\tau\right) e^{\lambda\tau} \right| \zeta\left(\tau\right) d\tau \\
&\leq \frac{\epsilon}{6} + \frac{2R\left|k\right|}{\lambda+k} + 2\Psi\left(R\right) M_3 \int_L^\infty \zeta\left(\tau\right) d\tau + \frac{\epsilon}{6} \leq \epsilon,
\end{aligned}$$

Claim 3. We show that $\mathcal{B} : B_R \to C_\lambda$ is a contraction mapping. In fact, for any $u, v \in B_R$, from $(H2)$, we have

$$\begin{aligned}
&\sup_{t\geq 0} e^{-\lambda t} \left|(\mathcal{B}u)(t) - (\mathcal{B}v)(t)\right| \\
&= \sup_{t\geq 0} \left\{ \int_0^t e^{-\lambda t} \mathcal{H}(t-\tau) \left|g(\tau, u(\tau - r)) - g(\tau, v(\tau - r))\right| d\tau \right\} \\
&\leq \sup_{t\geq 0} \left\{ \int_0^t e^{-\lambda t} \mathcal{H}(t-\tau)\, \eta(\tau) \left|u(\tau) - v(\tau)\right| d\tau \right\} \\
&\leq \sup_{t\geq 0} \left\{ \int_0^t e^{-\lambda (t-\tau)} \mathcal{H}(t-\tau)\, \eta(\tau) \left[e^{-\lambda\tau} \left|u(\tau) - v(\tau)\right| \right] d\tau \right\} \\
&\leq \left\{ \sup_{t\geq 0} \int_0^t e^{-\lambda (t-\tau)} \mathcal{H}(t-\tau)\, \eta(\tau)\, d\tau \right\} \|u - v\|_\lambda \leq M_1 \|u - v\|_\lambda ,
\end{aligned}$$

from (3.3) $\mathcal{B}$ is a contraction mapping.

Thus all the assumptions of Theorem 2.1 are satisfied. So the conclusion of Theorem 3.1 implies that the problem (1.1) has at least one continuous solution on $[-r, +\infty)$. This is complete the proof.

We give an example to illustrate the effectiveness of our main results.

Example 1 Let us consider the following nonlinear fractional initial value problems

$$\begin{cases}
{}^{RL}D^{\frac{1}{2}}[{}^{C}D^{\frac{3}{2}}u(t) - \frac{1}{\theta^2+t^2}\sin(u(t-r))] \\
= \frac{e^{\lambda\left(\left(1-\lambda^{-1}\right)t + 2e^{-\lambda(t-r)}u(t-r)\right)}\arctan(tu(t-r))}{1+e^{2\lambda e^{-\lambda(t-r)}u(t-r)}},\ t \geq 0, \\
u(t) = \phi(t),\ t \in [-r, 0], \\
\lim_{t\to 0} t^{1-\alpha}\, {}^{C}D^{\beta}u(t) = 0,\ u'(0) = u_0 \in \mathbb{R},
\end{cases} \tag{3.9}$$

where

$$\alpha = \frac{1}{2},\ \beta = \frac{3}{2},\ g(t,x) = \frac{1}{\theta^2 + t^2}\sin(x),\ \theta > 0,\ g(t,0) = 0,$$
$$f(t,x) = \frac{e^{\lambda\left(\left(1-\lambda^{-1}\right)t + 2xe^{-\lambda(t-r)}\right)}\arctan(tx)}{1 + e^{2\lambda x e^{-\lambda(t-r)}}},$$

then we have

$$|g(t,x) - g(t,y)| \leq \frac{1}{\theta^2 + t^2}|x - y| \text{ i.e. } \eta(t) = \frac{1}{\theta^2 + t^2},$$
$$\left| f(t, e^{\lambda(t-r)}x) \right| \leq e^{\lambda t} \frac{\pi}{2e^t} \frac{e^{2\lambda x}}{1 + e^{2\lambda x}} \text{ i.e. } \zeta(t) = \frac{\pi}{2e^t} \text{ and } \Psi(x) = \frac{e^{2\lambda x}}{1 + e^{2\lambda x}},$$

where Ψ is positive continuous nondecreasing function, ζ is a positive continuous integrable function on $[0, \infty)$ and $\int_0^\infty \zeta(t)\, dt = \frac{\pi}{2}$. Furthermore

$$\begin{aligned}
\left| e^{-\lambda(t-\tau)} \mathcal{K}(t-\tau) \right| &= e^{-\lambda(t-\tau)} \left| \int_\tau^t e^{-\lambda(t-s)} ds \right| \\
&= \frac{1}{k} \left| e^{-\lambda(t-\tau)} - e^{-(\lambda+k)(t-\tau)} \right| \leq \frac{2}{|k|} = M_3,
\end{aligned}$$

$$\int_0^t e^{-\lambda(t-\tau)}\mathcal{K}(t-\tau)\zeta(\tau)\,d\tau \leq \frac{\pi}{|k|}\int_0^t e^{-\tau}d\tau \leq \frac{\pi}{|k|} = M_2,$$

also

$$\begin{aligned} e^{\lambda(t-\tau)}\mathcal{H}(t-\tau) &= \frac{1}{\Gamma(1/2)}\int_\tau^t \frac{1}{e^{(\lambda+k)(t-s)}}\frac{(s-\tau)^{-1/2}}{e^{\lambda(s-\tau)}}ds \\ &\leq \frac{\int_\tau^t \frac{(s-\tau)^{-1/2}}{e^{\lambda(s-\tau)}}ds}{\Gamma(1/2)} = \frac{\int_0^{t-\tau}\frac{w^{-1/2}}{e^{\lambda w}}dw}{\Gamma(1/2)} \leq \lambda^{1/2}, \end{aligned}$$

for all $t \geq 0$. Also, if we choose $\theta \geq \lambda^{1/2}\pi(\lambda+k)$ then for all $t \geq 0$ we get

$$\begin{aligned} \int_0^t e^{-\lambda(t-\tau)}\mathcal{H}(t-\tau)\eta(\tau)\,d\tau &= \theta^{-1}\lambda^{1/2}\int_0^{t\theta^{-1}}\frac{d\tau}{1+\tau^2} \\ &\leq \frac{1}{2(\lambda+k)} = M_1 < 1 - \frac{|k|}{\lambda+k}. \end{aligned}$$

All conditions of theorem 3.1 are satisfied, so the system 3.9 has at least one solution in C_λ follows from Lemma 2.1.

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