Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **39** (1), 73-82 (2019).

On smoothness of solution for the higher-order partial differential equations

Leyla Sh. Kadimova ⋆ · **Rena E. Kerbalayeva**

Received: 02.11.2018 / Revised: 28.04.2019 / Accepted: 06.05.2019

Abstract. *In this paper a smoothness of solution for the higher-order partial differential equations is studied. For a bounded domains with non-smooth boundary earlier considered partial differential equations is generalized.*

Keywords. Smoothness of solution, higher-order partial differential equation

Mathematics Subject Classification (2010): 35Q35, 35A15

1 Introduction and preliminaries

Let us consider the problems on smoothness of solutions of partial differential higher-order type equations

$$\sum_{\substack{\left(\alpha_k,\frac{1}{l_k^i}\right)\le 1,\\ \left(\beta_k,\frac{1}{l_k^i}\right)\le 1,\\ k\in e_s, i\in Q}} D^{\alpha}\left(a^i_{\alpha\beta}(x)\,D^{\beta}u(x)\right)=\sum_{\substack{\left(\alpha_k,\frac{1}{l_k^i}\right)\le 1,\\ k\in e_s, i\in Q}} D^{\alpha}f_{\alpha}(x), \tag{1.1}$$

where $e_s=\{1,2,...,s\}$, $1\le s\le n$, s and n be positive integers, Q the set of vectors $i=(i_1,i_2,...,i_s)$, $i_k=0,1,...,n_k$; $x=(x_1,...,x_s)\in R^n$, $x_k=(x_{k,1},...,x_{k,n_k})$, $(k\in e_s)$; $l^i=\left(l_1^{i_1},...,l_s^{i_s}\right)$-correspondence vectors i associated with the fixed positive integers vector $l=(l_1,...,l_s)$ in the following way: $l_k^0=(0,...,0)$, $l_k^1=(l_{k,1},0,...,0),...,l_k^{n_k}=(0,...,0,l_{k,n_k})$ for all $k\in e_s$;

$$\left(\alpha_k^i,\frac{1}{l_k^i}\right)=\begin{cases}\sum\limits_{j=1}^{n_k}\frac{\alpha_j}{l_j^i}, & |i_k|=1,\\ \frac{\alpha_j}{l_j^i}, & |i_k|>1,\end{cases}$$

⋆ Corresponding author

Leyla Sh. Kadimova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Az-1141, V.Bahabzade 9, Baku, Azerbaijan
E-mail: kleylusha1@rambler.ru

Rena E. Kerbalayeva
Institute of Mathematics and Mechanics of NAS of Azerbaijan, AZ 1141, V.Bahabzade 9, Baku, Azerbaijan
E-mail: rena-kerbalayeva@mail.ru

$|i_k|$ -of set indexes component vectors i_k, $\alpha = (\alpha_1, ..., \alpha_s)$; $\beta = (\beta_1, ..., \beta_s)$, $\alpha_j \geq 0$, $\beta_j \geq 0$ - be an integers. Suppose, that the coefficients $a_{\alpha\beta}(x)$ are bounded, measurable functions are in the domain G, $a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$ and

$$\sum_{\substack{\left(\alpha_k, \frac{1}{l_k^i}\right)=1,\\ \left(\beta_k, \frac{1}{l_k^i}\right)=1,\\ k\in e_s, i\in Q}} (-1)^{|\alpha|} a_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq C_0 \sum_{\substack{\left(\alpha_k, \frac{1}{l_k^i}\right)=1,\\ k\in e_s, i\in Q}} |\xi_\alpha|^2, \quad C_0 = const > 0, \tag{1.2}$$

where $|\alpha| = \sum\limits_{j\in e_s} |\alpha_j|$. We assume that $f_\alpha \in L_2(G)$ for $\left(\alpha_k, \frac{1}{l_k^i}\right) < 1$; $f_\alpha(x) \in L_2$, a, χ for $\left(\alpha_k, \frac{1}{l_k^i}\right) = 1$.

A generalized solution to (1.1) in G is a function $u(x) \in W_2^l(G, s)$ such that,

$$\sum_{\substack{\left(\alpha_k, \frac{1}{l_k^i}\right)\leq 1,\\ \left(\beta_k, \frac{1}{l_k^i}\right)\leq 1,\\ k\in e_s, i\in Q}} (-1)^{|\alpha|} \int\limits_G a_{\alpha\beta}(x) D^\beta u(x) D^\alpha v(x)\,dx$$

$$= \sum_{\substack{\left(\alpha_k, \frac{1}{l_k^i}\right)\leq 1,\\ k\in e_s, i\in Q}} (-1)^{|\alpha|} \int\limits_G f_\alpha(x) D^\beta v(x)\,dx. \tag{1.3}$$

Existence of solution to (1.1) is proved with the help of the variational method in [6] for every function $v(x) \in \overset{\circ}{W}{}_2^l(G, s)$.

Definition [7]. *The Sobolev - Morrey spaces of many groups* $W_{p,a,\chi,\tau}^l(G, s)$ *is the Banach space of locally summable on G functions f with finite norm* $(1 \leq p < \infty, l \in N^n)$*:*

$$\|f\|_{W_{p,a,\chi,\tau}^l(G,s)} = \sum_{i\in Q} \left\| D^{l_1^i} f \right\|_{L_{p,a,\chi,\tau}(G)}, \tag{1.4}$$

where

$$\|f\|_{L_{p,a,\chi,\tau}(G)} = \sup_{x\in G} \left\{ \int\limits_0^\infty \cdots \int\limits_0^\infty \left[\mathop{\Pi}_{k\in e_s} [t_k]_1^{-\frac{|\chi_k|a}{p}} \|f\|_{p,G_{t\chi}^{(x)}} \right]^\tau \mathop{\Pi}_{k\in l_s} \frac{dt_k}{t_k} \right\}^{\frac{1}{\tau}},$$

$$\|f\|_{L_{p,a,\chi}(G)} = \|f\|_{L_{p,a,\chi,\infty}(G)} = \sup_{\substack{x\in G\\ t_k>0,\\ k\in e_s}} \left(\mathop{\Pi}_{k\in e_s} [t_k]_1^{-\frac{|\chi_k|a}{p}} \|f\|_{p,G_{t\chi}^{(x)}} \right),$$

$$D^{l^i} f = D_1^{l_1^{l_1}} ... D_s^{l_s^{l_s}} f, \quad D_k^{l_k^{i_k}} f = D_{k,1}^{l_k^{i_k}} ... D_{k,n_k}^{l_k^{i_k}} f,$$

$$G_{t^\chi}(x) = G \cap I_{t^\chi}(x), \quad I_{t^\chi}(x) = I_{t_1^{\chi_1}}(x_1) \times I_{t_2^{\chi_2}}(x_2) \times \cdots \times I_{t_s^{\chi_s}}(x_s),$$

$$I_{t_k^{\chi_k}}(x_k) = \left\{ y_k : |y_k - x_k| < \frac{1}{2} t_k^{\chi_k}, k \in e_s \right\}, \quad \frac{dt_k}{t_k} = \underset{j \in e_k^i}{\Pi} \frac{dt_{k,j}}{t_{k,j}}, \quad l_k^i = \sup p l_k^i;$$

$$a \in [0,1], \quad \chi = (\chi_1, ..., \chi_s), \quad \chi_k = (\chi_{k,1}, ..., \chi_{k,n_k}),$$

$$\chi_{k,j} > 0 \, (k \in e_s, j = 1, ..., n_k), \quad 1 \le \tau \le \infty, \quad [t_k]_1 = \min\{1, t_k\}_l, \quad k \in e_s.$$

The spaces $W^l_{p,a,\chi,\tau}(s,G)$ in case $s = 1$ coincides with the space type Sobolev-Morrey $W^l_{p,a,\chi,\tau}(G)$ studied in [9], $s = n$ with s spaces type Sobolev-Morrey with dominant inixed derivatives $S^l_{p,a,\chi,\tau}W(G)$ studied in [10], in case $a = 0$, $\tau = \infty$ coincides with the spaces of Sobolev of many groups of variables $W^l_p(s,G)$ studied in [2]. Note that for $s = 1$ the equation (1.1) converts into the following (quasi-elliptic type)

$$\sum_{\substack{(\alpha, \frac{1}{l}) \le 1 \\ (\beta, \frac{1}{l}) \le 1}} D^\alpha \left(a_{\alpha\beta}(x) D^\beta u(x) \right) = \sum_{(\alpha, \frac{1}{l}) \le 1} D^\alpha f_\alpha, \tag{1.5}$$

where $\left|\alpha, \frac{1}{l}\right| = \sum\limits_{j=1}^{n} \frac{\alpha_j}{l_j}$, for $s = n$ the equation (1.1) converts into the following (hypoelliptic type)

$$\sum_{\substack{\alpha_j \le l_j \\ \beta_j \le l_j \\ j \in e \le e_n}} D^{\alpha^e} \left(a_{\alpha^l \beta^l}(x) D^{\beta^l} u(x) \right) = \sum_{\substack{\alpha_j \le l_j \\ j \in l \le e_n}} D^{\alpha^e} f_{\alpha^e}(x) \tag{1.6}$$

studied in [7], [9], [11]. Here $\alpha^l = \left(\alpha_1^l, ..., \alpha_n^l\right)$, $\alpha_j^l = \alpha_j$ for $j \in e$; $\alpha_j^l = 0$ for $j \in e_n \backslash e$. In particular $n = 2$, $l = (1,1)$ $a_{\alpha\beta}(x) = 1$ the equation (1) has the form

$$u_x^{(1)} + u_y^{(1)} + u = f$$

the equation (1.6) has the form

$$u_{xy}^{(2)} + u_x^{(1)} + u_y^{(1)} + u = f.$$

The problem of the local smoothness of solutions of equations of type (1) was considered by several authors. In [4] the Holder continuity of solutions of quasi-elliptic equations with continuous or Holder continuous coefficients of the leading derivatives is studied. In [1] L_p -estimates for solutions were studied under the condition that the coefficients of the leading derivatives are infinitely differentiable. In [5] a theorem was proven claiming that the solution belongs to the Holder class inside the domain, and in [3] local "interior" Holder estimates were obtained for solutions to a quasi-elliptic type equation in the case when the right-hand side satisfies the anisotropic Holder condition. In this article and [5], as proved [9], [11], [13] theorems stating that the solution belongs to the Holder class inside the domain, and has a zero boundary Dirichlet condition up to bounds without any smoothness condition on coefficients. However, observe that unlike [5] here

1) $\nu \ne 0$;

2) f_α for $\left(\alpha_k, \frac{1}{l^i}\right) = 1$ belongs to a broader class, i.e. $f_\alpha \in L_{r,a,\chi}(G)$.

Definition [2]. *The domain $G \subset R^n$ satisfies the condition "σ-semi-horn", i.e. $G \subset A(T^\sigma)$, if we have finite subdomains $G_1, G_2, ..., G_N \subset G$, satisfying the condition "σ-semi-horn" and*

$$G = \bigcup_{j=1}^{N} G_j. \tag{1.7}$$

We suppose that $G \in A_\varepsilon(T^\sigma)$ $(\varepsilon > 0)$ if we substitute the condition $G = \bigcup_{j=1}^{N} G_{j,\varepsilon}$ into condition (1.7). Note that $G_{j,\varepsilon} = \{x : \rho_\sigma(x, \partial G_j \backslash \partial G) > \varepsilon\}$.

For the prove of the main results. Let us formulate Theorem 2.1 and 2.2 which were proved in [8], [13] $\left(l \in N^n,\ B^l_{2,2,a,\chi,\tau}(s,G) \equiv W^l_{2,a,\chi,\tau}(s,G)\right)$.

2 Main results.

Theorem 2.1. *Let $G \in A(T^\sigma)$, i.e. the domain $G \subset R^n$ satisfies the condition "σ-semi-norm", $1 \le p \le q \le \infty$, $\nu = (\nu_1, ..., \nu_s)$, $\nu_j \ge 0$ be an integer $(j = 1,2,...,n_k, k \in e_s)$ and*

a) $\nu_{k,j} \ge l^0_{k,j}$ $(j = 1,2,...,n_k, k \in e_s)$,

b) $\nu_{k,j} \ge l^{i_k}_{k,j} + 1$; $\nu_{k,i_k} < l^{i_k}_{k,i_k} + 1$; *let* $|\chi_k| \le |\sigma_k|$ $(k \in e_s)$; $1 \le \tau_1 \le \tau_2 \le \infty$, $\mu^i_{k,i_k} = l^i_{k,k_k}\sigma_{k,i_k} - (\nu_k, \sigma_k) - (|\sigma_k| - |\chi_k|\,a)\left(\frac{1}{p} - \frac{1}{q}\right) > 0$ $(i_k = 1,2,...,n_k, k \in e_s)$, $i \in Q$, $\sigma_k > 0$, $k \in e_s$ and let $f \in B^l_{p,\theta,a,\chi,\tau_1}(G,s)$. *Then there exist the generalized derivatives $D^\nu f$ on the domain Gstaifying the inequalities*

$$\|D^\nu f\|_{q,G} \le C^1 \sum_{i\in Q} \mathop{\Pi}_{k\in e_s} T_k^{\mu^i_{k,i_k}} \|f\|_{L^{\langle l^i\rangle}_{p,\theta,a,\chi,\tau_1}(s,G)}, \tag{2.1}$$

$$\|D^\nu f\|_{q,b,\chi,\tau_2,G} \le C^2 \|f\|_{B^l_{p,\theta,a,\chi,\tau_1}(s,G)}\ (p \le q < \infty). \tag{2.2}$$

In particular, if

$$\mu^i_{k,i_k,o} = l^i_{k,i_k}\sigma_{k,i_k} - (\nu_k, \sigma_k) - (|\sigma_k| - |\chi_k|\,a)\frac{1}{p} > 0\ (i_k = 1,2,...,n_k, k \in e_s),$$

then $D^\nu f$ is continuous on G and in addition,

$$\sup_{x\in G} |D^\nu f(x)| \le C^1 \sum_{i\in Q} \mathop{\Pi}_{k\in e_s} T_k^{\mu^i_{k,i_k,o}} \|f\|_{L^{l^i}_{p,\theta,a,\chi,\tau}(s,G)}, \tag{2.3}$$

where $T_k \in (O, \min(1, t_{0,k})]$, $k \in e_s$; $t_0 = (t_{01}, ..., t_{0s})$ is a fixed positive vector, B is an arbitrary number satisfying:

$0 \le b < 1$ if $\mu^i_{k,i_k} > 0$,

$0 \le b \le 1$ if $\mu^i_{k,i_k} = 0$,

$0 \le b < a + \frac{\mu^i_{k,i_k} q(1-a)}{|\sigma_k| - |\chi_k| a}$ if $\mu^i_{k,i_k} < 0$,

C^1 and C^2 are constants independent of f, and C^1 is independent of T.

Theorem 2.2. *Let all conditions of Theorem 2.1 be satisfied and besides $G \in A_\varepsilon(T^\sigma)$. Then for $\mu^i_{k,i_k} > 0$ $(i_k = 1,2,...,n_k, k \in e_s)$, $i \in Q$ the derivative $D^\nu f$satisfies the Holder condition on the domain G, for the metric of L_q, with index ε. More precisely,*

$$\|\Delta(\gamma, G)\, D^\nu f\|_{q,G} \le C\,\|f\|_{B^l_{p,\theta,a\chi,\tau}(s,G)} \mathop{\Pi}_{k\in e_s} |\gamma_k|^{\varepsilon_k}, \tag{2.4}$$

where ε is an arbitrary number satisfying the conditions $0 \le \varepsilon_k \le 1$ if $\frac{\mu^0_{k,i_k}}{\sigma_k} > 1$, $0 \le \varepsilon_k < 1$ if $\frac{\mu^0_{k,i_k}}{\sigma_k} = 1$, $0 \le \varepsilon_k \le \frac{\mu^0_{k,i_k}}{\sigma_k}$, $\frac{\mu^0_{k,i_k}}{\sigma_k} < 1$ $(k \in l_s)$ $\mu^0_{k,i_k} = \min \mu^i_{k,i_k}$, $i \in Q$.

In particular, if $\mu^i_{k,i_k,0} > 0$, $(i_k = 1, 2, ..., n_k, k \in e_s)$, $i \in Q$, then

$$\sup_{x\in G} |\Delta (\gamma, G) D^\nu f (x)| \le C \|f\|_{B^l_{p,\theta,a,\chi,\tau}(s,G)} \underset{k\in e_s}{\Pi} |\gamma_k|^{\varepsilon^0_k}, \tag{2.5}$$

where ε^0_k satisfies the same conditions, but we must substitute $\mu^i_{k,i_k,0}$ instead of μ^i_{k,i_k}.

Theorem 2.3. *If* $(\nu_k, \sigma_k) + \frac{|\sigma_k|}{2} \le 1$, $\sigma^{-1}_k = l^i_k \in N^k$, $k \in e_s$ *then any generalized solution of equation* (1.1) *from* $W^l_2 (s, G)$ *belongs to the space* $C_{\nu,\varepsilon} (G_d)$, $\overline{G}_d \subset G$.

Proof of theorem 2.3. First let all $a_{\alpha,\beta} (x)$ is satisfies condition (1.2) and except for the ones for which $\left(\alpha_k, \frac{1}{l^i_k}\right) = 1$, $i = (i_1, i_2, ..., i_s) \in Q$ and $f_x = 0$. Let $d = (d_1, ..., d_s)$-fixed vector, $d_k = (d_{k,1}, ..., d_{k,n_k})$ and $b = (b_1, ..., b_s)$, $b_k = (b_{k,1}, ..., b_{k,n_k})$ and $b_k \le d_k$, $k \in e_s$. Let $x_0 \in G$ and $\Pi_b (x_0)$ be the parallelepiped in R^n

$$\Pi_b (x_0) = \left\{x : |x_j - x_{j0}| < b^{\sigma_j}_j,\ j \in e_n\right\}$$

and G_d be a subdomain of the domain G such that $(0 < d_j < 1, j \in e_n)$

$$G_d = \left\{y : |y_j - x_j| < d^{\sigma_j}_j,\ j \in e_n\right\}.$$

From the variational principle it follows that

$$\int\limits_{\Pi_b(x_0)} \sum_{\substack{\left(\alpha_k,\frac{1}{l^i_k}\right)=1,\\ \left(\beta_k,\frac{1}{l^i_k}\right)=1,\\ k\in e_s,\ i\in Q}} (-1)^{|\alpha|} a_{\alpha\beta} (x) D^\beta (\theta (x) (u (x) - P (x))) D^\alpha (\theta (x) (u (x) - P (x))) dx$$

$$\ge \int\limits_{\Pi_b(x_0)} \sum_{\substack{\left(\alpha_k,\frac{1}{l^i_k}\right)=1,\\ \left(\beta_k,\frac{1}{l^i_k}\right)=1,\\ k\in e_s,\ i\in Q}} (-1)^{|\alpha|} a_{\alpha\beta} (x) D^\beta (u (x) - P (x)) D^\alpha (u (x) - P (x)) dx$$

$$= A (u (x) - P (x) , \Pi_b (x_0)) \tag{2.6}$$

for any $\theta (x) \in C^\infty (\Pi_b (x_0))$, such that $\theta (x) \equiv 1$ in the neighborhood of $\partial\Pi_b (x_0)$, in any polynomial $P (x)$ of the form $P (x) = \sum\limits_{\substack{\left(\alpha_k,\frac{1}{l^i_k}\right)=1\\ k\in e_s,\ i\in Q}} C_\alpha x^\alpha$ and for an arbitrary solution $u (x)$ of equation (1.1).

Assume in (2.6)

$$\theta (x) = 1 - \underset{j\in e_n}{\Pi} \delta_j \left(\frac{x_j - x_{j0}}{b^{\sigma_j}_j}\right),$$

where $\delta_j (t_j) \in C^\infty (R)$, $\delta_j (t_j) = 1$ for $|t_j| < \frac{1}{2}$, $\delta_j (t_j) = 0$ for $|t_j| \ge 1$. It is clear that $\theta (x) \equiv 1$ in $\Pi_{\frac{b}{2}} (x_0)$. We have taken the coefficients $P (x)$ as

$$\int\limits_{\Pi_b(x_0)\backslash\Pi_{\frac{b}{2}}(x_0)} (u (x) - P (x)) x^\alpha dx = 0.$$

From inequality (2.6) with help of (1.7) we obtain

$$A\left(u\left(x\right)-P\left(x\right),\Pi_b\left(x_0\right)\right)\le A\left(u\left(x\right)-P\left(x\right),\Pi_b\left(x_0\right)\backslash\Pi_b\left(x_0\right)\backslash\Pi_{\frac{b}{2}}\left(x_0\right)\right)$$

$$+C_1\int\limits_{\Pi_b(x_0)\backslash\Pi_{\frac{b}{2}}(x_0)}\sum\limits_{\substack{\left(\alpha_k+\rho_k,\frac{1}{l_k^i}\right)=1\\ k\in e_s,\ i\in Q}}\underset{k\in e_s}{\Pi}b_k^{-2\rho_k}\left(D^{\alpha}\left(u\left(x\right)-P\left(x\right)\right)^2\right)dx$$

$$\le rA\left(u\left(x\right)-P\left(x\right),\Pi_b\left(x_0\right)\backslash\Pi_{\frac{b}{2}}\left(x_0\right)\right).$$

Since $A\left(u\left(x\right)-P\left(x\right),G\right)=A\left(u\left(x\right),G\right)$, then

$$A\left(u\left(x\right),\Pi_{\frac{b}{2}}\left(x_0\right)\right)_2\le\left(1-\frac{1}{r}\right)A\left(u\left(x\right),\Pi_b\left(x_0\right)\right),$$

hence by induction we obtain that

$$A\left(u\left(x\right),\Pi_{\frac{b}{2^m}}\left(x_0\right)\right)\le\left(1-\frac{1}{r}\right)^m A\left(u\left(x\right),\Pi_b\left(x_0\right)\right)(m>1).$$

Let $0<\eta_k<\frac{b_k}{2^m}$, it follows $\Pi_\eta\left(x_0\right)\subset\Pi_{\frac{b}{2^m}}\left(x_0\right)$, $\eta=(\eta_1,...,\eta_k)$, $\eta_k=(\eta_{k,1},...,\eta_{k,n_k})$, $k\in e_s$. Further $m\ln 2<\ln\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}$, we take $m=\left[\frac{\ln\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}}{\ln 2}\right]$, $\omega=1-\frac{1}{r}$ then

$$A\left(u\left(x\right),\Pi_\eta\left(x_0\right)\right)\le\omega^k A\left(u\left(x\right),G\right)<\omega^{\frac{\ln\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}}{\ln 2}-1}A\left(u\left(x\right),G\right)$$

$$=e^{\frac{\ln\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}}{\ln 2}\ln\omega-\ln\omega}A\left(u\left(x\right),G\right)=e^{\ln\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}\left(\frac{\ln\omega}{\ln 2}-\frac{\ln\omega}{\ln\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}}\right)}A\left(u\left(x\right),G\right)$$

$$=\left(\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}\right)^{\left(\frac{\ln\omega}{\ln 2}-\frac{\ln\omega}{\ln\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}}\right)}A\left(u\left(x\right),G\right)=\left(\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}\right)^{\left|\frac{\ln\omega}{\ln 2}-\frac{\ln\omega}{\ln\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}}\right|}\times A\left(u\left(x\right),G\right)$$

$$\le\left(\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}\right)^{\left|\frac{\ln\omega}{\ln 2}\right|-\left|\frac{\ln\omega}{\ln\underset{k\in l_s}{\Pi}\frac{b_k}{\eta_k}}\right|}A\left(u\left(x\right),G\right),$$

for any $x_0\in G_d$, $b_k\le d_k$, $k\in l_s$. Denote $\left|\frac{\ln\omega}{\ln 2}\right|=|\chi|\,a<1$, $\left|\frac{\ln\omega}{\ln\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}}\right|=\rho_k$, then

$$A\left(u\left(x\right)\Pi_\eta\left(x_0\right)\right)\le\left(\underset{k\in e_s}{\Pi}\frac{b_k}{\eta_k}\right)^{|\chi_k|a-\rho_k}A\left(u\left(x\right),G\right),$$

and

$$\left[\underset{k\in e_s}{\Pi}\eta_k^{-|\chi_k|a}\int\limits_{\Pi_\eta(x_0)}u^2\left(x\right)dx\right]^{\frac{1}{2}}\le\left(\underset{k\in e_s}{\Pi}\frac{1}{b_k}\right)^{|\chi_k|a-\rho_k}A\left(u\left(x\right),G\right)\le C_1.$$

This means that $u \in L_{2,a,\chi,\infty}(G_d) = L_{2,a,\chi}(G_d)$ and also $D^{l^i}u \in L_{2,a,\chi}(G_d)$ for all $i \in Q$, then it follows that $u(x) \in W^l_{2,a,\chi}(s,G)$. If we check the conditions Theorems 2.1 and 2.2, it turns out that $\mu^i_{k,i_k} > 0$, $\mu^i_{k,i_k,0} > 0$ $(k \in e_s, i_k = 1,2,...,\eta_k)$ and Theorems 2.1 and 2.2 are satisfied. Thus by Theorem 2.1 $D^\nu u(x)$ is continuous on G_d and by Theorem 2.2 $D^\nu u(x)$ satisfies the Holder condition, i.e. $u(x) \in C_{\nu,\varepsilon}(G_d)$.

Now we consider the nonhomogeneous quasi-elliptic equation

$$\sum_{\substack{\left(\alpha_k,\frac{1}{l^i_k}\right)=1 \\ \left(\beta_k,\frac{1}{l^i_k}\right)=1 \\ k\in e_s,\ i\in Q}} D^\alpha\left(a_{\alpha\beta}(x)\,D^\beta u(x)\right) = \sum_{\substack{\left(\alpha_k,\frac{1}{l^i_k}\right)=1 \\ k\in e_s,\ i\in Q}} D^\alpha f_\alpha(x), \tag{2.7}$$

where $a_{\alpha\beta}(x)$ satisfies earlier imposed restrictions, inequality (1.2) is satisfied, $f_\alpha(x) \in L_2(G)$ for $\left(\alpha_k, \frac{1}{l^i_k}\right) < 1$, $f_\alpha(x) \in L_{2,a,\chi}(G)$ for $\left(\alpha_k, \frac{1}{l^i_k}\right) = 1$. We again consider some substitution G_d. Let $x_0 \in G_d$ and u_{b,x_0} be a generalized solution of equation (2.7) from the space $\overset{\circ}{W}{}^l_2(s, \Pi_b(x_0))$ i.e.

$$\sum_{\substack{\left(\alpha_k,\frac{1}{l^i_k}\right)=1 \\ \left(\beta_k,\frac{1}{l^i_k}\right)=1 \\ k\in e_s,\ i\in Q}} (-1)^{|\alpha|}\int_G a_{\alpha\beta}(x)\,D^\beta u(x)\,D^\alpha v(x)\,dx$$

$$= \sum_{\substack{\left(\alpha_k,\frac{1}{l^i_k}\right)=1 \\ k\in e_s,\ i\in Q}} (-1)^{|\alpha|}\int_G a_\alpha(x)\,D^\alpha v(x)\,dx. \tag{2.8}$$

Assuming $v(x) \equiv u_{b,x_0}$ in (2.8) by virtue of (1.2) we obtain

$$\int_{\Pi_b(x_0)} \sum_{\substack{\left(\alpha_k,\frac{1}{l^i_k}\right)=1 \\ k\in e_s,\ i\in Q}} \left(D^\alpha u_{b,x_0}\right)^2 dx \le C_2 \sum_{\substack{\left(\alpha_k+\rho_k,\frac{1}{l^i_k}\right)=1 \\ k\in e_s,\ i\in Q}} \underset{k\in l_s}{\Pi}\, b_k^{\rho_k} \int_{\Pi_b(x_0)} f_2^2 dx$$

$$+ \sum_{\substack{\left(\alpha_k,\frac{1}{l^i_k}\right)=1 \\ k\in e_s,\ i\in Q}} \int_{\Pi_b(x_0)} f_2^2 dx \le C_3 \underset{k\in e_s}{\Pi}\, b_k^{\Delta_k}, \tag{2.9}$$

where $\Delta_k = \max\left\{\rho_k = 2 - 2\left(\alpha_k, \frac{1}{l^i_k}\right), |\chi_k|\,a\right\}$, C_3 and Δ_k not depend on $u(x)$ and x_0. Since $\overline{u(x)} = u(x) - u_{b,x_0}$ is a solution of equation (1.1) when the right hand side is zero, therefore for it

$$A\left(\overline{u(x)}, \Pi_\eta(x_0)\right) \le C_4 \left(\underset{k\in l_s}{\Pi} \frac{\eta_k}{b_k}\right)^{|\chi_k|a-\rho_k} A(u(x), G), \tag{2.10}$$

is valid for any $\eta_k < b_k$, $k \in e_s$ if $x_0 \in G_d$. Then from (2.9) and (2.10) we obtain

$$A\left(u\left(x\right),\mathit{\Pi}_{\eta}\left(x_0\right)\right) \le C_4 A\left(\overline{u},\mathit{\Pi}\eta\left(x_0\right)\right) + C_5 A\left(u_{b,x_0}\mathit{\Pi}_{\eta}\left(x_0\right)\right)$$

$$\le C_6\left(\underset{k\in e_s}{\mathit{\Pi}}\frac{\eta_k}{b_k}\right)^{|\chi_k|a-\rho_k} A\left(u\left(x\right),G\right).$$

Further, we again apply Theorems 2.1 and 2.2 and in this case we obtain the required results.

Finally, we consider equation (1.1) all of whose coefficients are different from zero and exist for small derivatives of the solution. Then we transfer such members to the right hand side of the equation and obtain the required result. The following theorem on smoothness of solution under the conditions of Theorem 2.3 holds when the generalized solution satisfies the Dirichlet boundary condition.

Theorem 2.4. *Let the domain $G \subset R^n$ such that there exists $\vartheta = const > 0$ for any point $x_0 \in \partial G$ and the number $\epsilon < 1$ there exists a parallelepiped $\mathit{\Pi}_{v\epsilon}\left(x^1\right)$ such that $\mathit{\Pi}_{v\epsilon}\left(x^1\right) \subset \mathit{\Pi}_{\epsilon}\left(x_0\right) \cap \left(R^n \backslash G\right)$ and $u\left(x\right)$ is a solution of equation (1.1) from the $\overset{\circ}{W}{}_2^{\,l}\left(s,G\right)$. If $\left(\nu_k,\sigma_k\right)+\frac{|\sigma_k|}{2} \le 1$, $\sigma_k^{-1} = l_k^i \in N^k$, $k \in e_s$, then $u\left(x\right)$ belongs to the space $C_{\nu,\varepsilon}\left(\overline{G}\right)$.*

Proof of Theorem 2.4. It is sufficiently in this case, to let all $a_{\alpha\beta}\left(x\right) = 0$ except for ones for which $\left(\alpha_k, \frac{1}{l_k^i}\right) = 1$, $k \in l_s^i$. Let $x_0 \in \partial G$ and $f_\alpha\left(x\right) \equiv 0$ in $\mathit{\Pi}_b\left(x_0\right)$, $u\left(x\right) \equiv 0$ outside of G.

From the variational principle it follows that

$$A\left(u\left(x\right),\mathit{\Pi}_b\left(x_0\right)\right) \le A\left(\theta\left(x\right)u\left(x\right),\mathit{\Pi}_b\left(x_0\right)\right).$$

As $\theta\left(x\right) \equiv 0$ in $\mathit{\Pi}_{\frac{b}{2}}\left(x_0\right)$, then

$$A\left(u\left(x\right),\mathit{\Pi}_b\left(x_0\right)\right) \le A\left(u\left(x\right),\mathit{\Pi}_b\left(x_0\right)\backslash\mathit{\Pi}_{\frac{b}{2}}\left(x_0\right)\right)$$

$$+C_1 \sum_{\substack{\left(\alpha_k,\frac{1}{l_k^i}\right)<1 \\ k\in e_s,\ i\in Q}} \underset{k\in l_s}{\mathit{\Pi}} b_k^{2+2\left(\alpha_k,\frac{1}{l_k^i}\right)} \int\limits_{\mathit{\Pi}_b\left(x_0\right)\backslash\mathit{\Pi}_{\frac{b}{2}}\left(x_0\right)} \left(D^\alpha u\left(x\right)\right)^2 dx.$$

As $u\left(x\right) \equiv 0$ in $\mathit{\Pi}_{\upsilon b}\left(x^1\right)$, then we have $\left(\mathit{\Pi}_{\upsilon b}\left(x^1\right) \subset \mathit{\Pi}_b\left(x_0\right)\backslash\mathit{\Pi}_{\frac{b}{2}}\left(x_0\right)\right)$

$$A\left(u\left(x\right),\mathit{\Pi}_{\eta}\left(x_0\right)\right) \le C\left(\underset{k\in e_s}{\mathit{\Pi}}\frac{\eta_k}{b_k}\right)^{|\chi_k|a-\rho_k} A\left(u\left(x\right),G\right) \tag{2.11}$$

where $\eta_k < b_k$, $k \in e_s$ for all $x_0 \in \partial G$, $f_\alpha \equiv 0$ in $\mathit{\Pi}_b\left(x_0\right)$.

Let's estimate now $A\left(u\left(x\right),\mathit{\Pi}_{\eta}\left(x_0\right)\right)$ at given $0 < \eta_k < 1$, $k \in e_s$, $x_0 \in G$ and $f_\alpha\left(x\right) \ne 0$. Let us consider two cases:

a) $x_0 \in G_{\sqrt{\eta}}$;

b) $x_0 \notin G_{\sqrt{\eta}}$.

a) In this case for all $0 \le \eta_k \le b_k \le 1$, $k \in l_s$ assuming that $b_k = \sqrt{\eta_k}$ we have

$$A\left(u\left(x\right),\mathit{\Pi}_{\eta}\left(x_0\right)\right) \le C_1\left(\underset{k\in e_s}{\mathit{\Pi}}\frac{\eta_k}{b_k}\right)^{|\chi_k|a-\rho_k} A\left(u\left(x\right),G\right) + C_2 \underset{k\in l_s}{\mathit{\Pi}} b_k^{\Delta_k}$$

$$\leq C_3 \underset{k\in e_s}{\Pi} \left(\frac{\eta_k}{b_k}\right)^{|\chi_k|a-\rho_k} \left(A\left(u\left(x\right),G\right)+1\right), \tag{2.12}$$

b) In this case there exist a point $x^1 \in \partial G$, such that $\Pi_{2\sqrt{\eta}}\left(x^1\right) \supset \Pi_{\sqrt{\eta}}\left(x_0\right)$. Let $b_k > 2\sqrt{\eta_k}$, $k \in l_s$. For all b_k consider u_{b,x^1}-solution of equation (1.1) in $\Pi_b\left(x^1\right) \cap G$ from the space $\overset{\circ}{W}{}_2^l\left(s, \Pi_b\left(x^1\right) \cap G\right)$, for which inequality

$$A\left(u_{b,x^1}, \Pi_b\left(x_0\right)\right) \leq C_4 \underset{k\in e_s}{\Pi} b_k^{\rho_k} \tag{2.13}$$

is valid, of assuming that $u_{b,x^1} \equiv 0$ outside of $\Pi_b\left(x^1\right) \cap G$.

The function $u\left(x\right) - u_{b,x^1}$ solution of equation (1.1) $\Pi_b\left(x^1\right)$, where for all $a_{\alpha\beta}\left(x\right) = 0$, except for the ones for which $\left(\alpha_k, \frac{1}{l'_k}\right) = 1$, $k \in e_s$, $i \in Q$ and $f_\alpha \equiv 0$. From inequalities (2.11) and (2.13) we have

$$A\left(u\left(x\right), \Pi_{2\sqrt{\eta}}\left(x^1\right)\right) \leq C_5 A\left(u - u_{b,x^1}, \Pi_{2\sqrt{\eta}}\left(x^1\right)\right)$$

$$+C_6 A\left(u_{b,x^1}, \Pi_{2\sqrt{\eta}}\left(x^1\right)\right) \leq C_7 \left(\underset{k\in e_s}{\Pi} \frac{\eta_k}{b_k}\right)^{|\chi_k|a-\rho_k} A\left(u\left(x\right),G\right),$$

consequently

$$A\left(u\left(x\right), \Pi_\eta\left(x_0\right)\right) \leq C_8 \left(\underset{k\in e_s}{\Pi} \frac{\eta_k}{b_k}\right)^{|\chi_k|a-\rho_k} A\left(u\left(x\right),G\right).$$

This implies that $u\left(x\right) \in L_{2,a\chi,\infty}\left(\overline{G}\right)$, and also $u\left(x\right) \in W_{2,a,\chi}^l\left(s, \overline{G}\right)$. Then in case the conditions in Theorems 2.1 and 2.2 are satisfied. Thus by Theorems 2.1 and 2.2 it follows that $u\left(x\right) \in C_{\nu,\varepsilon}\left(\overline{G}\right)$.

References

1. Akkeryd, L.: *On L^p estimates for quasi elliptic boundary problems*, Math. Scand., **24** (1), 141–144 (1969).
2. Djabrailov, A.D., Maksudov, F.Q.: *The method of integral representation in the theory of spaces*, Baku, 199 p. (2000) (in Russian).
3. Filatov, P.S.: *Local anisotropic Holder estimates for solutions to a quasielliptic equation*, Sib. Math. J., **38** (6), 1397–3409 (1997) (in Russian)
4. Giusti, E.: *Equazioni quasi elliptic spazi $S^{P,\theta}\left(\Omega, \delta\right)$*, Ann. Math. Pure Appl. Ser. **4** (75), 313–353 (1967).
5. Huseynov, R.V.: *On the smoothness of solutions of one class of quasielliptic equations*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **6**, 10–14 (1992) (in Russian).
6. Kerbalayeva, R.E.: *Embedding theorems for functions spaces of many groups of variables with parameters*, PhD thesis, Baku, 126 p. (2017).
7. Najafov, A.M.: *Interpolation theorems of Besov-Morrey type spaces and some its applications*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **24** (4), 125–134 (2004).
8. Najafov, A.M.: *Embedding theorems for spaces with parameters functions of many groups of variables*, Vestnik Baku State Univ. Ser. Fiz. Mat. Sci. (3),56-63 (2005) (in Russian).

9. Nadzhafov, A.M.: *On some properties of functions in the Sobolev-Morrey type spaces* $W^{l}_{p,a,\chi,\tau}(G)$, Sib. Math. J. **46** (3), 634–648 (2005).
10. Najafov, A.M.: *Embedding theorems in spaces type Sobolev-Morrey* $S^{l}_{p,a,\chi,\tau}W(G)$ *with dominant mixed derivatives*, Sib. Math. J. **47** (3), 613–625 (2006).
11. Najafov, A.M.: *Problem on smoothness of solution of one class hypoelliptic equations*, Proc. A. Razmadze Math. Inst. **140**, 131–139 (2006).
12. Najafov, A.M., Orujova, A.T.: *On solutions of one class of partial differential equations*, Electron. J. Qual. Theory Differ. Equ. 44, 1-9 (2017).
13. Najafov, A.M., Kerbalayeva, R.E.: *Embedding theorems for Besov-Morrey spaces of many groups of variables*, Georgian Math. J. 26 (1), 125-131 (2019).