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A unified approach to the certain integrals of k-Mittag-Leffler type function of two variables

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Abstract. We establish certain integral formulas of extended type k-Mittag-Leffler function (k-MLf) of two and one variables by using MacRobert, Lavoie and Trottier, and Oberhettinger integrals. These integral formulas are expressed in terms of some families in the Literature of special functions such as Fox H-function and Fox-Wright. We also consider some useful examples as particular cases of k-MLf to give the applications of our main results.

Keywords. k-Mittag-Leffler function of two variables · Fox-Wright function · Fox H-function of two variables and Integrals

Mathematics Subject Classification (2010): 33C20 · 33B15

1 Introduction

Integral transforms appear as an important tool with special functions, which deal with the most intensively developing areas of applied science. Its fields of applications range from mathematics, physics, biology and electro-chemistry to economics, probability theory and statistics. Excellent works on the topics of integral transforms have given useful account of the theory and applications with special functions in many different areas of mathematical analysis. Numerous advantageous applications of the Mittag-Leffler function $E_\alpha(z)$ and its generalizations were found in many diverse fields such as mathematical analysis, mathematical physics, biology, chemistry, engineering and other applied sciences. Lately, the attention of mathematicians and applied scientists like as Arshad et al. [2], Choi [4], Kamarujjama et al. [9–12] go through the function of Bessel, Mittag-leffler, Struve type has extended, chiefly due to their association to the integrals such are Lavoie and Trottier [14],

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MacRobert [15] and Oberhettinger [18] and its applications that is attracting advanced interest in distinct area of dynamical system, control system, engineering and applied sciences.

The Mittag-Leffler function was introduced by the Swedish Mathematician Gösta Mittag-Leffler [16] and its generalization $E_{\mu,\tau}$(z) given by Wiman [24] as: The Mittag-Leffler function was introduced by the Swedish Mathematician Gösta Mittag-Leffler [16] and its generalization $E_{\mu,\tau}$(z) given by Wiman [24] as:

$$E_{\mu}(z)=\sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\mu r+1)},\quad E_{\mu,\tau}(z)=\sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\mu r+\tau)},\tag{1.1}$$

where $\mu,\tau,z\in\mathbb{C}$; $R(\mu)>0$, $R(\tau)>0$.

New generalization of $E_{\mu,\tau}$(z) was introduced by Prabhakar [19] and further its generalization $E^{\eta,q}_{\mu,\tau}(z)$ is given by Shukla and Prajapati [21] in the following series form:

$$E^{\eta}_{\mu,\tau}(z)=\sum_{r=0}^{\infty}\frac{(\eta)_r\ z^r}{\Gamma(\mu r+\tau)r!},\quad E^{\eta,q}_{\mu,\tau}(z)=\sum_{r=0}^{\infty}\frac{(\eta)_{qr}\ z^r}{\Gamma(\mu r+\tau)r!},\tag{1.2}$$

where $\mu,\tau,\eta\in\mathbb{C}$, $R(\mu)>0$, $R(\tau)>0$, $q\in(0,1)$ and $(\eta)_r$ is the Pochhammer symbol [22].

In 2012, Dorrego and Cerutti [6] introduced k-Mittag-Leffler function and further its generalization given by Chand et al. [3] as follows:

$$E^{\eta}_{k,\mu,\tau}(z)=\sum_{r=0}^{\infty}\frac{(\eta)_{r,k}\ z^r}{\Gamma_k(\mu r+\tau)r!},\qquad E^{\eta,q}_{k,\mu,\tau}(z)=\sum_{r=0}^{\infty}\frac{(\eta)_{qr,k}\ z^r}{\Gamma_k(\mu r+\tau)r!},\tag{1.3}$$

where k $\in R^+$; $\mu,\tau,\eta\in\mathbb{C}$; $R(\mu)>0$, $R(\tau)>0$, $q\in(0,1)$ and $(\eta)_{r,k}$ is the k-Pochhammer symbol is defined as:

$$(x)_{r,k}=\begin{cases}\frac{\Gamma_k(x+rk)}{\Gamma_k(x)}, & (k\in R^+;x\in\mathbb{C})\\ x(x+k),....,(x+(r-1)k), & (r\in\mathbb{N};x\in\mathbb{C})\end{cases}\tag{1.4}$$

and the relation with classical gamma function as

$$\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}),\qquad (x\in\mathbb{C},R(x)>0;k\in R^+).\tag{1.5}$$

Further, let $x\in\mathbb{C}$; $k,p\in R^+$ and $n,q\in\mathbb{N}$, then the following identity holds true:

$$(x)_{rq,k}=(k)^{rq}(\frac{x}{k})_{rq}.\tag{1.6}$$

A Mittag-Leffler type function of two variables is defined by [8] as:

$$E_1(x,y)=E_1\left(\begin{matrix}\eta_1,\mu_1\ ;\ \eta_2,\nu_1\\ \tau_1,\mu_2,\nu_2\ ;\ \tau_2,\mu_3\ ;\ \tau_3,\nu_3\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\eta_1)_{\mu_1 r}\ (\eta_2)_{\nu_1 s}}{\Gamma(\tau_1+\mu_2 r+\nu_2 s)}\frac{x^r}{\Gamma(\tau_2+\mu_3 r)}\frac{y^s}{\Gamma(\tau_3+\nu_3 s)},\tag{1.7}$$

where η_1, η_2, τ_1, τ_2, τ_3, x, $y\in\mathbb{C}$ and $\min\{\mu_1,\mu_2,\mu_3,\nu_1,\nu_2,\nu_3\}>0$.

Recently, Kamarujjama et al. [12] introduced extended type k-Mittag-Leffler function of two variables

$$E_k(x,y) = E_k\left(\begin{matrix} \eta_1,\mu_1\ ;\ \eta_2,\nu_1 \\ \tau_1,\mu_2,\nu_2\ ;\ \tau_2,\mu_3\ ;\ \tau_3,\nu_3 \end{matrix}\middle| \begin{matrix} x \\ y \end{matrix}\right)$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\eta_1)_{\mu_1 r,k}\,(\eta_2)_{\nu_1 s,k}}{\Gamma_k(\tau_1+\mu_2 r+\nu_2 s)} \frac{x^r}{\Gamma_k(\tau_2+\mu_3 r)} \frac{y^s}{\Gamma_k(\tau_3+\nu_3 s)}, \quad (1.8)$$

where $k \in R^+$; $\eta_1,\eta_2,\tau_1,\tau_2,\tau_3,x,y \in \mathbb{C}$ with $\min\{\mu_1,\mu_2,\mu_3,\nu_1,\nu_2,\nu_3\} > 0$ and $(x)_{nq,k}$ is the generalized k-Pochhammer symbol [5].

Remark 1.1 Assuming $k = 1$ in (1.8) gives (1.7).

Remark 1.2 Letting $y \to 0$ in (1.8), we get new generalization of k-Mittag-Leffler function of one variable as follows:

$$E_k(x,y)\mid_{y\to 0} = \Gamma_k(\tau_3) E^{\eta_1,\mu_1}_{k,\mu_2,\tau_1;\mu_3,\tau_2}(x), \quad (1.9)$$

where

$$E^{\eta_1,\mu_1}_{k,\mu_2,\tau_1;\mu_3,\tau_2}(x) = \sum_{r=0}^{\infty} \frac{(\eta_1)_{\mu_1 r,k}}{\Gamma_k(\tau_1+\mu_2 r)} \frac{x^r}{\Gamma_k(\tau_2+\mu_3 r)},\ (\eta_1,\tau_1,\tau_2,x \in \mathbb{C}, \min\{\mu_1,\mu_2,\mu_3\} > 0). \quad (1.10)$$

By using (1.8) we consider following special cases and connections with other special functions as follows:

1. For $k = \tau_2 = \mu_3 = 1$ and $\nu_1 = \nu_2 = 0$ in (1.8) gives the product of generalized Mittag-Leffler functions of one variable as:

$$E_1\left(\begin{matrix} \eta_1,\mu_1\ ;\ \eta_2,0 \\ \tau_1,\mu_2,0\ ;\ 1,1\ ;\ \tau_3,\nu_3 \end{matrix}\middle| \begin{matrix} x \\ y \end{matrix}\right) = E^{\eta_1,\mu_1}_{\mu_2,\tau_1}(x)\ E_{\nu_3,\tau_3}(y). \quad (1.11)$$

2. Taking $k = 1$, $\tau_2 = \tau_2 + 1$, $\tau_1 = \mu_2 = 1$ and $\mu_1 = \nu_1 = \nu_2 = 0$ in (1.8) becomes the products of Bessel-Maitland and Mittag-Leffler functions as:

$$E_1\left(\begin{matrix} \eta_1,0\ ;\ \eta_2,0 \\ 1,1,0\ ;\ \tau_2+1,\mu_3\ ;\ \tau_3,\nu_3 \end{matrix}\middle| \begin{matrix} x \\ y \end{matrix}\right) = J^{\mu_3}_{\tau_2}(x)\ E_{\nu_3,\tau_3}(y). \quad (1.12)$$

3. Putting $k = 1$, $\tau_2 = \mu_1 = \mu_2 = \mu_3 = \nu_3 = 1$ and $\nu_1 = \nu_2 = 0$ in (1.8) yields the product of confluent hypergeometric functions (Kummer function) of one variables as:

$$E_1\left(\begin{matrix} \eta_1,1\ ;\ \eta_2,0 \\ \tau_1,1,0\ ;\ 1,1\ ;\ \tau_3,1 \end{matrix}\middle| \begin{matrix} x \\ y \end{matrix}\right) = \frac{1}{\Gamma(\tau)}\ {}_1F_1\left[\begin{matrix}\eta_1 \\ \tau_1\end{matrix};x\right]\ {}_1F_1\left[\begin{matrix}1 \\ \tau_3\end{matrix};y\right]. \quad (1.13)$$

4. Connection with H-functions of two variables is as follows:

$$E_k\left(\begin{matrix} \eta_1,\mu_1\ ;\ \eta_2,\nu_1 \\ \tau_1,\mu_2,\nu_2\ ;\ \tau_2,\mu_3\ ;\ \tau_3,\nu_3 \end{matrix}\middle| \begin{matrix} x \\ y \end{matrix}\right) = \frac{k^{(\frac{\eta_1+\eta_2-\tau_1-\tau 2-\tau_3}{k}+1)}}{\Gamma_k(\eta_1)\Gamma_k(\eta_2)}$$

$$\times H^{0,0:1,2:1,2}_{0,1:2,2:2,2}\left[\begin{array}{c}-xk^{(\mu_1-\frac{\mu_2}{k}-\frac{\mu_3}{k})}\\ -yk^{(\nu_1-\frac{\nu_2}{k}-\frac{\nu_3}{k})}\end{array}\middle|\begin{array}{c}\overline{\qquad\qquad}:\ (0,1),(1-\frac{\eta_1}{k},\mu_1);(0,1),(1-\frac{\eta_2}{k},\nu_1)\\ (1-\frac{\tau_1}{k};\frac{\mu_2}{k},\frac{\nu_2}{k}):(0,1),(1-\frac{\tau_2}{k},\frac{\mu_3}{k});(0,1),(1-\frac{\tau_3}{k},\frac{\nu_3}{k})\end{array}\right]. \tag{1.14}$$

Remark 1.3 If we take $k = \mu_1 = \mu_2 = \mu_3 = \nu_1 = \nu_2 = \nu_3 = 1$ in (1.14), we can deduce the connection between k-MLf and Meijer G-function of two variables. Further, upon setting y$\to 0$ in (1.14), we can derive the relation between k-MLf and H-function of one variable.

The H-function of two variables (see [17], [23]) is defined as:

$$H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{array}{c}-x\\ -y\end{array}\middle|\begin{array}{c}(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2};(e_j,E_j)_{1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2};(f_j,F_j)_{1,q_3}\end{array}\right]$$

$$= \frac{-1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\Theta_2(\xi)\Theta_3(\eta)x^{\xi}y^{\xi}d\xi d\eta, \tag{1.15}$$

where

$$\phi_1(\xi,\eta) = \frac{\prod\limits_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod\limits_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod\limits_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)},$$

$$\Theta_2(\xi) = \frac{\prod\limits_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)\prod\limits_{j=1}^{n_2}\Gamma(1-c_j-\gamma_j\xi)}{\prod\limits_{j=m_2+1}^{q_2}\Gamma(1-d_j-\delta_j\xi)\prod\limits_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)},$$

$$\Theta_3(\eta) = \frac{\prod\limits_{j=1}^{m_3}\Gamma(f_j-F_j\eta)\prod\limits_{j=1}^{n_3}\Gamma(1-e_j-E_j\eta)}{\prod\limits_{j=m_3+1}^{q_3}\Gamma(1-f_j-F_j\eta)\prod\limits_{j=n_3+1}^{q_3}\Gamma(e_j-E_j\eta)},$$

x and y are not equal to zero and empty product is interpreted as unity, and p_i, q_i, n_i (i=1,2,3) and m_2, m_3 are not negative integers such that $0 \leq n_i \leq$, $q_i \geq 0$, $0 \leq m_j \leq q_j$ (i=1,2,3; j=2,3) and all letters $\alpha, \beta, \gamma, \delta$ A, B, E and F are assumed to be positive. The contour L_1 is in the ξ-plane and runs from $-i\infty$ to $+i\infty$, with the loops, if necessary, that the poles of $\Gamma(d_j - \delta_j\xi)$ (j=1,..,m_2) lie to the right and those of $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ (j=1,..,n_1), $\Gamma(1 - c_j + \gamma_j\xi)$ (j=1,..,n_3) to the left of the contour. The contour L_2 is in the η-plane and runs from $-i\infty$ to $+i\infty$, with the loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$ (j=1,..,m_3) lie to the right and those of $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ (j=1,..,n_1), $\Gamma(1 - e_j + E_j\xi)$ (j=1,..,n_3) to the left of the contour.

For $A_j = B_j = E_j = F_j = \gamma_j = \delta_j = 1$, (1.15) reduces to Meijer G-function of two variables [1]. Further taking $y \to 0$, (1.15) reduces to Fox H-function of one variable [7].

Also we recall here the following interesting and useful results as follows:

• Oberhettinger [18] established following integral formula (1.16) for $0 < R(\rho) < R(\sigma)$

$$\int_0^{\infty} u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma}du = 2\sigma\, a^{-\sigma}\left(\frac{a}{2}\right)^{\rho}\frac{\Gamma(2\rho)\Gamma(\sigma-\rho)}{\Gamma(\sigma+\rho+1)}. \tag{1.16}$$

• MacRobert [15] obtained the result (1.17) for $\mathbb{R}(\rho) > 0$, $\mathbb{R}(\sigma) > 0$; a and b are non zero constants

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[au+b(1-u)]^{-\rho-\sigma}du = \frac{1}{a^\rho b^\sigma}\frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}, \ (0 \le u \le 1). \quad (1.17)$$

• Lavoie and Trottier [14] introduced following integral formula:

$$\int_0^\infty u^{\rho-1}(1-u)^{2\sigma-1}\left(1-\frac{u}{3}\right)^{2\rho-1}\left(1-\frac{u}{4}\right)^{\sigma-1}du$$
$$= \left(\frac{2}{3}\right)^\rho \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma+1)}, \ (R(\rho) > 0, R(\sigma) > 0). \quad (1.18)$$

2 Main results

We evaluate following certain unified integral formulas of extended type k-Mittag-Leffler function (1.8) of one and two variables as given in Theorem 2.1-2.3.

Theorem 2.1 *Let $k \in R^+$; x, y, η_1, η_2, τ_1, τ_2, $\tau_3 \in \mathbb{C}$ with $R(\rho) > 0$, $R(\sigma) > 0$ and u> 0. Then integral formula holds true:*

$$\int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma}E_k\left(\begin{matrix}\eta_1,\mu_1\ ;\ \eta_2,\nu_1\\ \tau_1,\mu_2,\nu_2\ ;\ \tau_2,\mu_3\ ;\ \tau_3,\nu_3\end{matrix}\left|\begin{matrix}\frac{x}{(u+a+\sqrt{u^2+2au})^{\frac{\mu_2}{k}}}\\ \frac{y}{(u+a+\sqrt{u^2+2au})^{\frac{\nu_2}{k}}}\end{matrix}\right.\right)du$$

$$= \frac{a^{\rho-\sigma}\Gamma(2\rho)k^{(\frac{\eta_1+\eta_2-\tau_1-\tau 2-\tau_3}{k}+1)}}{2^{\rho-1}\Gamma_k(\eta_1)\Gamma_k(\eta_2)}H^{0,2:1,2:1,2}_{2,3:2,2:2,2}\left[\begin{matrix}-\frac{x}{a^{\frac{\mu_2}{k}}}k^{(\mu_1-\frac{\mu_2}{k}-\frac{\mu_3}{k})}\\ -\frac{y}{a^{\frac{\nu_2}{k}}}k^{(\nu_1-\frac{\nu_2}{k}-\frac{\nu_3}{k})}\end{matrix}\left|\begin{matrix}(1+\rho-\sigma,\frac{\mu_2}{k},\frac{\nu_2}{k}),\\ (1-\sigma,\frac{\mu_2}{k},\frac{\nu_2}{k}),\end{matrix}\right.\right.$$
$$\left.\begin{matrix}(-\sigma,\frac{\mu_2}{k},\frac{\nu_2}{k}): \quad (0,1),(1-\frac{\eta_1}{k},\mu_1);(0,1),(1-\frac{\eta_2}{k},\nu_1)\\ (-\rho-\sigma,\frac{\mu_2}{k},\frac{\nu_2}{k}),(1-\frac{\tau_1}{k};\frac{\mu_2}{k},\frac{\nu_2}{k}):(0,1),(1-\frac{\tau_2}{k},\frac{\mu_3}{k});(0,1),(1-\frac{\tau_3}{k},\frac{\nu_3}{k})\end{matrix}\right]. \quad (2.1)$$

Proof. In order to derive (2.1), we indicate the left-hand side of (2.1) by I_1 and then expanding $E_k(x,y)$ in series form, we get

$$I_1 = \sum_{r=0}^\infty\sum_{s=0}^\infty \frac{(\eta_1)_{\mu_1 r,k}\,(\eta_2)_{\nu_1 s,k}}{\Gamma_k(\tau_1+\mu_2 r+\nu_2 s)}\frac{x^r}{\Gamma_k(\tau_2+\mu_3 r)}\frac{y^s}{\Gamma_k(\tau_3+\nu_3 s)}$$
$$\times\int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-(\sigma+\frac{\mu_2}{k}+\frac{\nu_2}{k})}du. \quad (2.2)$$

In view of the conditions given in Theorem 2.1, we can apply the integral formula (1.16) to the integral of (2.2), we have

$$I_1 = \sum_{r=0}^\infty\sum_{s=0}^\infty \frac{(\eta_1)_{\mu_1 r,k}\,(\eta_2)_{\nu_1 s,k}}{\Gamma_k(\tau_1+\mu_2 r+\nu_2 s)}\frac{\left(\frac{x}{a^{\frac{\mu_2}{k}}}\right)^r}{\Gamma_k(\tau_2+\mu_3 r)}\frac{\left(\frac{y}{a^{\frac{\nu_2}{k}}}\right)^r}{\Gamma_k(\tau_3+\nu_3 s)}\frac{\Gamma(\sigma+1+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s)}{\Gamma(\sigma+\rho+1+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s)}$$
$$\times\frac{\Gamma(\sigma-\rho+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s)}{\Gamma(\sigma+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s)}$$

now interpreting the above expression with the help of (1.15), we achieve the require result (2.1) of Theorem 2.1. □

Corollary 2.1 *Under the conditions stated already with Theorem 2.1, then following integral formula holds true:*

$$\int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma}E_1\left(\begin{array}{c}\eta_1,\mu_1\ ;\ \eta_2,\nu_1\\ \tau_1,\mu_2,\nu_2\ ;\ \tau_2,\mu_3\ ;\ \tau_3,\nu_3\end{array}\middle|\begin{array}{c}\frac{x}{(u+a+\sqrt{u^2+2au})^{\mu_2}}\\ \frac{y}{(u+a+\sqrt{u^2+2au})^{\nu_2}}\end{array}\right)du$$

$$=\frac{a^{\rho-\sigma}\Gamma(2\rho)}{2^{\rho-1}\Gamma(\eta_1)\Gamma(\eta_2)}H^{0,2:1,2:1,2}_{2,3:2,2:2,2}\left[\begin{array}{c}-\frac{x}{a^{\mu_2}}\\ -\frac{y}{a^{\nu_2}}\end{array}\middle|\begin{array}{c}(1+\rho-\sigma,\mu_2,\nu_2),\\ (1-\sigma,\mu_2,\nu_2),\end{array}\right.$$

$$\left.\begin{array}{c}(-\sigma,\mu_2,\nu_2):\quad(0,1),(1-\eta_1,\mu_1);\ (0,1),(1-\eta_2,\nu_1)\\ (-\rho-\sigma,\mu_2,\nu_2),(1-\tau_1;\mu_2,\nu_2):(0,1),(1-\tau_2,\mu_3);\ (0,1),(1-\tau_3,\nu_3)\end{array}\right].\quad(2.3)$$

Remark 2.1 Upon setting y to 0 in result (2.1), we deduce the following consequence of Theorem 2.1.

Corollary 2.2 *Under the conditions stated already with Theorem 2.1, then following integral formula holds true:*

$$\int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma}E^{\eta_1,\mu_1}_{k,\mu_2,\tau_1;\mu_3,\tau_2}\left(\frac{x}{(u+a+\sqrt{u^2+2au})^{\frac{\mu_2}{k}}}\right)du$$

$$=\frac{a^{\rho-\sigma}\Gamma(2\rho)k^{(\frac{\eta_1+\eta_2-\tau_1-\tau_2}{k}+1)}}{2^{\rho-1}\Gamma_k(\eta_1)\Gamma_k(\tau_3)}$$

$$H^{1,3}_{3,5}\left[\frac{x}{a^{\mu_2}k^{(\mu_1-\frac{\mu_2}{k}-\frac{\mu_3}{k})}}\middle|\begin{array}{c}(1-\sigma+\rho,\frac{\mu_2}{k}),\ (-\sigma,\frac{\mu_2}{k}),\ (1-\frac{\eta_1}{k},\mu_1)\\ (0,1),(1-\sigma-\rho,\frac{\mu_2}{k}),\ (1-\sigma,\frac{\mu_2}{k}),(1-\tau_1,\mu_2),(1-\tau_2,\mu_3)\end{array}\right].\quad(2.4)$$

Theorem 2.2 *Let $k\in R^+$; x, y, η_1, η_2, τ_1, τ_2, $\tau_3\in\mathbb{C}$ with $R(\rho)>0$, $R(\sigma)>0$ and $u>0$. Then integral formula holds true:*

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[au+b(1-u)]^{-\rho-\sigma}E_k\left(\begin{array}{c}\eta_1,\mu_1\ ;\ \eta_2,\nu_1\\ \tau_1,\mu_2,\nu_2\ ;\ \tau_2,\mu_3\ ;\ \tau_3,\nu_3\end{array}\middle|\begin{array}{c}\frac{x(1-u)^{\frac{\mu_2}{k}}}{(au+b(1-u))^{\frac{\mu_2}{k}}}\\ \frac{y(1-u)^{\frac{\nu_2}{k}}}{(au+b(1-u))^{\frac{\nu_2}{k}}}\end{array}\right)du$$

$$=\frac{\Gamma(\rho)k^{(\frac{\eta_1+\eta_2-\tau_1-\tau_2-\tau_3}{k}+1)}}{a^\rho b^\sigma\Gamma_k(\eta_1)\Gamma_k(\eta_2)}H^{0,1:1,1:1,1}_{1,2:1,1:1,1}\left[\begin{array}{c}-\frac{x}{b^{\frac{\mu_2}{k}}}k^{(\mu_1-\frac{\mu_2}{k}-\frac{\mu_3}{k})}\\ -\frac{y}{b^{\frac{\nu_2}{k}}}k^{(\nu_1-\frac{\nu_2}{k}-\frac{\nu_3}{k})}\end{array}\middle|\begin{array}{c}(1-\sigma,\frac{\mu_2}{k},\frac{\nu_2}{k})\\ (1-\sigma-\rho,\frac{\mu_2}{k},\frac{\nu_2}{k}),\end{array}\right.$$

$$\left.\begin{array}{c}:\quad(0,1),(1-\frac{\eta_1}{k},\mu_1);\ (0,1),(1-\frac{\eta_2}{k},\nu_1)\\ (1-\frac{\tau_1}{k};\frac{\mu_2}{k},\frac{\nu_2}{k}):(0,1),(1-\frac{\tau_2}{k},\frac{\mu_3}{k});\ (0,1),(1-\frac{\tau_3}{k},\frac{\nu_3}{k})\end{array}\right].\quad(2.5)$$

Proof. In order to derive (2.5), we indicate the left-hand side of (2.5) by I_1 and then expanding $E_k(x,y)$ in series form, we get

$$I_1=\sum_{r=0}^\infty\sum_{s=0}^\infty\frac{(\eta_1)_{\mu_1 r,k}\,(\eta_2)_{\nu_1 s,k}}{\Gamma_k(\tau_1+\mu_2 r+\nu_2 s)}\frac{x^r}{\Gamma_k(\tau_2+\mu_3 r)}\frac{y^s}{\Gamma_k(\tau_3+\nu_3 s)}$$

$$\times\int_0^1 u^{\rho-1}(1-u)^{\sigma+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s-1}[au+b(1-u)]^{-\rho-(\sigma+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s)}du.\quad(2.6)$$

In view of the conditions given in Theorem 2.2, we can apply the integral formula (1.17) to the integral of (2.6), then

$$I_1=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\eta_1)_{\mu_1 r,k}\,(\eta_2)_{\nu_1 s,k}}{\Gamma_k(\tau_1+\mu_2 r+\nu_2 s)}\frac{\left(\frac{x}{a^{\frac{\mu_2}{k}}}\right)^r}{\Gamma_k(\tau_2+\mu_3 r)}\frac{\left(\frac{y}{a^{\frac{\nu_2}{k}}}\right)^r}{\Gamma_k(\tau_3+\nu_3 s)}$$
$$\times\frac{1}{a^{\rho}b^{\sigma+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s}}\frac{\Gamma(\rho)\Gamma(\sigma+1+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s)}{\Gamma(\sigma+\rho+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s)}.$$

Finally solving the above expression with the help of (1.15), we achieve the required result (2.5) of Theorem 2.2. □

Corollary 2.3 *Under the conditions stated already with Theorem 2.2, then following integral formula holds true:*

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[au+b(1-u)]^{-\rho-\sigma}E_1\left(\begin{matrix}\eta_1,\mu_1\,;\,\eta_2,\nu_1\\ \tau_1,\mu_2,\nu_2\,;\,\tau_2,\mu_3\,;\,\tau_3,\nu_3\end{matrix}\middle|\begin{matrix}\frac{x(1-u)^{\mu_2}}{(au+b(1-u))^{\mu_2}}\\ \frac{y(1-u)^{\nu_2}}{(au+b(1-u))^{\nu_2}}\end{matrix}\right)du$$
$$=\frac{\Gamma(\rho)}{a^{\rho}b^{\sigma}\Gamma(\eta_1)\Gamma(\eta_2)}H^{0,1:1,1:1,1}_{1,2:1,1:1,1}\left[\begin{matrix}-\frac{x}{b^{\mu_2}}\\ -\frac{y}{b^{\nu_2}}\end{matrix}\middle|\begin{matrix}(1-\sigma,\mu_2,\nu_2)\\ (1-\sigma-\rho,\mu_2,\nu_2),\end{matrix}\right.$$
$$\left.\begin{matrix}:\quad(0,1),(1-\eta_1,\mu_1);\,(0,1),(1-\eta_2,\nu_1)\\ (1-\tau_1;\mu_2,\nu_2):(0,1),(1-\tau_2,\mu_3);\,(0,1),(1-\tau_3,\nu_3)\end{matrix}\right].\tag{2.7}$$

Corollary 2.4 *Under the conditions stated already with Theorem 2.2, then following integral formula holds true:*

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[au+b(1-u)]^{-\rho-\sigma}E^{\eta_1,\mu_1}_{k,\mu_2,\tau_1;\mu_3,\tau_2}\left(\frac{x}{(u+a+\sqrt{u^2+2au})^{\frac{\mu_2}{k}}}\right)du$$
$$=\frac{\Gamma(\rho)k^{(\frac{\eta_1+\eta_2-\tau_1-\tau_2}{k}+1)}}{a^{\rho}b^{\sigma}\Gamma_k(\eta_1)\Gamma_k(\tau_3)}$$
$$H^{1,3}_{3,5}\left[\frac{x}{a^{\mu_2}k^{(\mu_1-\frac{\mu_2}{k}-\frac{\mu_3}{k})}}\middle|\begin{matrix}(1-\sigma+\rho,\frac{\mu_2}{k}),\,(-\sigma,\frac{\mu_2}{k}),\,(1-\frac{\eta_1}{k},\mu_1)\\ (0,1),(1-\sigma-\rho,\frac{\mu_2}{k}),\,(1-\sigma,\frac{\mu_2}{k}),(1-\tau_1,\mu_2),(1-\tau_2,\mu_3)\end{matrix}\right].\tag{2.8}$$

Theorem 2.3 *Let $k\in R^+$; x, y, η_1, η_2, τ_1, τ_2, $\tau_3\in\mathbb{C}$ with $R(\rho)>0$, $R(\sigma)>0$ and $u>0$. Then integral formula holds true:*

$$\int_0^{\infty}u^{\rho-1}(1-u)^{2\sigma-1}\left(1-\frac{u}{3}\right)^{2\rho-1}\left(1-\frac{u}{4}\right)^{\sigma-1}$$
$$E_k\left(\begin{matrix}\eta_1,\mu_1\,;\,\eta_2,\nu_1\\ \tau_1,\mu_2,\nu_2\,;\,\tau_2,\mu_3\,;\,\tau_3,\nu_3\end{matrix}\middle|\begin{matrix}x(1-u)^{\frac{2\mu_2}{k}}(1-\frac{u}{4})^{\frac{\mu_2}{k}}\\ y(1-u)^{\frac{2\nu_2}{k}}(1-\frac{u}{4})^{\frac{\nu_2}{k}}\end{matrix}\right)du$$
$$=\frac{(2/3)^{2\rho}k^{(\frac{\eta_1+\eta_2-\tau_1-\tau_2-\tau_3}{k}+1)}}{\Gamma_k(\eta_1)\Gamma_k(\eta_2)}H^{0,1:1,1:1,1}_{1,2:1,1:1,1}\left[\begin{matrix}-xk^{(\mu_1-\frac{\mu_2}{k}-\frac{\mu_3}{k})}\\ -yk^{(\nu_1-\frac{\nu_2}{k}-\frac{\nu_3}{k})}\end{matrix}\middle|\begin{matrix}(1-\sigma,\frac{\mu_2}{k},\frac{\nu_2}{k})\\ (-\sigma-\rho,\frac{\mu_2}{k},\frac{\nu_2}{k}),\end{matrix}\right.$$
$$\left.\begin{matrix}:\quad(0,1),(1-\frac{\eta_1}{k},\mu_1);\,(0,1),(1-\frac{\eta_2}{k},\nu_1)\\ (1-\frac{\tau_1}{k};\frac{\mu_2}{k},\frac{\nu_2}{k}):(0,1),(1-\frac{\tau_2}{k},\frac{\mu_3}{k});\,(0,1),(1-\frac{\tau_3}{k},\frac{\nu_3}{k})\end{matrix}\right].\tag{2.9}$$

Proof. In order to derive (2.9), we indicate the left-hand side of (2.9) by I_1 and then expanding $E_k(x,y)$ in series form, we get

$$I_1 = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\eta_1)_{\mu_1 r,k}\,(\eta_2)_{\nu_1 s,k}}{\Gamma_k(\tau_1+\mu_2 r+\nu_2 s)} \frac{x^r}{\Gamma_k(\tau_2+\mu_3 r)} \frac{y^s}{\Gamma_k(\tau_3+\nu_3 s)}$$
$$\times \int_0^{\infty} u^{\rho-1}(1-u)^{2(\sigma+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s)-1}\left(1-\frac{u}{3}\right)^{2\rho-1}\left(1-\frac{u}{4}\right)^{\sigma+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s-1} du. \tag{2.10}$$

In view of the conditions given in Theorem 2.3, we can apply the integral formula (1.18) to the integral of (2.10), then

$$I_1 = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\eta_1)_{\mu_1 r,k}\,(\eta_2)_{\nu_1 s,k}}{\Gamma_k(\tau_1+\mu_2 r+\nu_2 s)} \frac{\left(\frac{x}{a^{\frac{\mu_2}{k}}}\right)^r}{\Gamma_k(\tau_2+\mu_3 r)} \frac{\left(\frac{y}{a^{\frac{\nu_2}{k}}}\right)^r}{\Gamma_k(\tau_3+\nu_3 s)}$$
$$\times \left(\frac{2}{3}\right)^{\rho} \frac{\Gamma(\rho)\Gamma(\sigma+1+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s)}{\Gamma(\sigma+\rho+1+\frac{\mu_2}{k}r+\frac{\nu_2}{k}s)}.$$

Now interpreting the above expression with the help of (1.15), we achieve the require result (2.9) of Theorem 2.3. □

Corollary 2.5 *Under the conditions stated already with Theorem 2.3, then following integral formula holds true:*

$$\int_0^{\infty} u^{\rho-1}(1-u)^{2\sigma-1}\left(1-\frac{u}{3}\right)^{2\rho-1}\left(1-\frac{u}{4}\right)^{\sigma-1}$$
$$E_k\left(\begin{matrix}\eta_1,\mu_1\ ;\ \eta_2,\nu_1\\ \tau_1,\mu_2,\nu_2\ ;\ \tau_2,\mu_3\ ;\ \tau_3,\nu_3\end{matrix}\middle|\begin{matrix} x(1-u)^{2\mu_2}(1-\frac{u}{4})^{\mu_2}\\ y(1-u)^{2\nu_2}(1-\frac{u}{4})^{\nu_2}\end{matrix}\right) du$$
$$= \frac{(2/3)^{2\rho}}{\Gamma(\eta_1)\Gamma(\eta_2)} H^{0,1:1,1:1,1}_{1,2:1,1:1,1}\left[\begin{matrix}-x\\-y\end{matrix}\middle|\begin{matrix}(1-\sigma,\mu_2,\nu_2)\\(-\sigma-\rho,\mu_2,\nu_2),\end{matrix}\right.$$
$$\left.\begin{matrix} : & (0,1),(1-\eta_1,\mu_1);\ (0,1),(1-\eta_2,\nu_1)\\ (1-\tau_1;\mu_2,\nu_2): & (0,1),(1-\tau_2,\mu_3);\ (0,1),(1-\tau_3,\nu_3)\end{matrix}\right]. \tag{2.11}$$

Corollary 2.6 *Under the conditions stated already with Theorem 2.3, then following integral formula holds true:*

$$\int_0^{\infty} u^{\rho-1}(1-u)^{2\sigma-1}\left(1-\frac{u}{3}\right)^{2\rho-1}\left(1-\frac{u}{4}\right)^{\sigma-1} E^{\eta_1,\mu_1}_{k,\mu_2,\tau_1;\mu_3,\tau_2}\left(\frac{x}{(u+a+\sqrt{u^2+2au})^{\frac{\mu_2}{k}}}\right) du$$
$$= \frac{(2/3)^{2\rho} k^{(\frac{\eta_1+\eta_2-\tau_1-\tau_2}{k}+1)}}{\Gamma_k(\eta_1)\Gamma_k(\tau_3)}$$
$$H^{1,3}_{3,5}\left[-xk^{(\mu_1-\frac{\mu_2}{k}-\frac{\mu_3}{k})}\middle|\begin{matrix}(1-\sigma+\rho,\frac{\mu_2}{k}),\ (-\sigma,\frac{\mu_2}{k}),\ (1-\frac{\eta_1}{k},\mu_1)\\ (0,1),(1-\sigma-\rho,\frac{\mu_2}{k}),(1-\sigma,\frac{\mu_2}{k}),(1-\tau_1,\mu_2),(1-\tau_2,\mu_3)\end{matrix}\right]. \tag{2.12}$$

3 Particular examples

Example 3.1 If we set $k = \tau_2 = \mu_3 = 1$ and $\nu_1 = \nu_2 = 0$ in (1.8), then k-MLf of two variables reduces to the product of k-MLf of one variable. Therefore, in view of (1.11) using these values in Theorems 2.1, we conclude following integral formula:

$$\int_0^\infty u^{\rho-1}(u+a+\sqrt{u^2+2au})^{-\sigma} E^{\eta_1,\mu_1}_{\mu_2,\tau_1}\left(\frac{x}{(u+a+\sqrt{u^2+2au})^{\mu_2}}\right) E_{\nu_3,\tau_3}(y)du$$

$$= \frac{a^{\rho-\sigma}\Gamma(2\rho)}{2^{\rho-1}\Gamma(\eta_1)} H^{1,3}_{3,4}\left[-\frac{x}{a^{\mu_2}}\middle| \begin{matrix}(1+\rho-\sigma,\mu_2),\ (-\sigma,\mu_2),(1-\eta_1,\mu_1)\\ (0,1),(1-\sigma,\mu_2),\ (-\rho-\sigma,\mu_2),(1-\tau_1,\mu_2)\end{matrix}\right]$$

$$\times H^{1,0}_{0,2}\left[-y\middle|\ \begin{matrix}\overline{\quad}\\ (0,1),(1-\tau_3,\nu_3)\end{matrix}\right]. \tag{3.1}$$

• Similarly, if we taking these values in Theorem 2.2 and 2.3, we can compute more integral formulae for the product of generalized Mittag-Leffler function of one variables, which are expressed in terms of the product of Fox H-function of one variable.

Example 3.2 For k=1, $\tau_2 = \tau_2 + 1$, $\tau_1 = \mu_2 = 1$ and $\mu_1 = \nu_1 = \nu_2 = 0$, (1.8) gives the products of generalized Bessel and Mittag-Leffler functions of one variable. Therefore, in view of (1.12) using these values in Theorems 2.2, we compute following integral formula

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}[au+b(1-u)]^{-\rho-\sigma} J^{\mu_3}_{\tau_2}\left(\frac{x(1-u)}{[au+b(1-u)]}\right) E_{\nu_3,\tau_3}(y)du$$

$$= \frac{\Gamma(\rho)}{a^\rho b^\sigma \Gamma(\eta_1)\Gamma(\eta_2)} {}_1\Psi_2\left[\begin{matrix}(\sigma,1)\\ (\sigma+\rho,1),(\tau_2+1,\mu_3)\end{matrix}\middle|\frac{x}{b}\right] {}_1\Psi_1\left[\begin{matrix}(1,1)\\ (\tau_3,\nu_3)\end{matrix}\middle| y\right]. \tag{3.2}$$

• Further setting k=1, $\tau_2 = \tau_2 + 1$, $\tau_1 = \mu_2 = 1$ and $\mu_1 = \nu_1 = \nu_2 = 0$ in the result of Theorem 2.1 and 2.3, we can derive more integral formulae for the product of generalized Bessel and Mittag-Leffler function of one variables, which are expressed in terms of the product of Fox-Wright function of one variable.

Example 3.3 For k=1, $\tau_2 = \mu_1 = \mu_2 = \mu_3 = \nu_3 = 1$ and $\nu_1 = \nu_2 = 0$, (1.8) gives the product of confluent hypergeometric functions (Kummer function) of one variables as Therefore, in view of (1.13) using these values in Theorems 2.3, we compute following integral formula

$$\int_0^\infty u^{\rho-1}(1-u)^{2\sigma-1}\left(1-\frac{u}{3}\right)^{2\rho-1}\left(1-\frac{u}{4}\right)^{\sigma-1} {}_1F_1\left[\begin{matrix}\eta_1\\ \tau_1\end{matrix}; x(1-u)(1-\frac{u}{4})\right] {}_1F_1\left[\begin{matrix}1\\ \tau_3\end{matrix}; y\right] du$$

$$= \frac{(2/3)^{2\rho}}{\Gamma(\tau_1)\Gamma(\eta_1)\Gamma(\eta_2)} {}_2F_2\left[\begin{matrix}\rho,\eta_1\\ \sigma+\rho+1,\tau_1\end{matrix}; x\right] {}_1F_1\left[\begin{matrix}1\\ \tau_3\end{matrix}; y\right]. \tag{3.3}$$

• Again setting these values in the result of Theorem 2.1 and 2.2, in view of above example, we can evaluate more results for the product of confluent hypergeometric functions of one variable.

4 Conclusion

In this manuscript, we have evaluated certain integral formulas involving extended type k-Mittag-Leffler function of two variable. These integral formulas are expressed in terms of Fox H-function of two and one variables . Also, the k-MLf of two variables can express in terms of Meijer G-function of two variables [1] and Kamp$\acute{e}$ de F$\acute{e}$riet double hypergeometric function [20] by considering specific values of the parameters. Therefore, the results presented in this paper are easily converted in terms of a similar type of new interesting integrals with different arguments.

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