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Oscillation and basis properties of a fourth order differential operator with spectral parameter in the boundary conditions

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Abstract. *In this paper we consider a spectral problem with describes bending vibrations of a homogeneous rod, on the right end of which a tracing force acts and on the right end an inertial mass is concentrated. We study oscillation properties of eigenfunctions and basis properties of subsystems of eigenfunctions in the space* $L_p(0,1),\ 1 < p < \infty$ *of this problem.*

Keywords. homogeneous rod, fourth order differential operator, spectral parameter, eigenvalue, eigenfunction

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1 Introduction

We consider the following eigenvalue problem

$$y^{(4)}(x) = \lambda y(x),\ 0 < x < 1, \tag{1.1}$$

$$y''(0) = 0, \tag{1.2}$$

$$y'''(0) - a\lambda y(0) = 0, \tag{1.3}$$

$$y''(1) - b\lambda y'(1) = 0, \tag{1.4}$$

$$y'''(1) - c\lambda y(1) = 0, \tag{1.5}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $a,\ b$ and c are real constants such that $a > 0,\ b > 0$ and $c < 0$.

The eigenvalue problem (1.1)-(1.5) describes the bending vibrations of a homogeneous rod, on the right end of which a tracing force acts and on the right end an inertial mass is concentrated (see [10, 25]).

Eigenvalue problems for ordinary differential operators with a spectral parameter in the boundary conditions of different statements have been studied in many papers (see, for example, [1-4, 6-11, 14-14, 26]). The oscillation properties of eigenfunctions of ordinary differential operators of second and fourth orders with a spectral parameter in boundary

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conditions play a fundamental role in the study of basis properties of root functions in the space L_p, $1 < p < \infty$, of these operators. The study of the oscillation properties of ordinary differential operators has a long history. These properties of differential operators of second and fourth orders well studied in papers [2, 4-7, 9-12, 15, 16, 19-22]. Basis properties of root functions in the space L_p, $1 < p < \infty$, of ordinary differential operators of second and fourth orders operators studied in detail in [1-3, 6-10, 16-21, 23]. In recent papers [9, 10] were investigated spectral problems of the fourth order two boundary conditions of which contained a spectral parameter. It should be noted that for the first time in this work is study the spectral properties of ordinary differential operators fourth order with a spectral parameter contained in three of the boundary conditions.

The purpose of this paper is to study the location of eigenvalues on the real axis, oscillation and basis properties of eigenfunctions of the spectral problem (1.1)-(1.5).

2 Operator interpretation of the spectral problem (1.1)-(1.4)

The considered problem (1.1)-(1.4) can be reduced to the eigenvalue problem for the linear operator L in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^3$ with the inner product

$$(\hat{u},\hat{v}) = (\{y,m,n,\tau\},\{v,s,t,\varkappa\}) = (y,v)_{L_2} + |a|^{-1}m\bar{s} + |b|^{-1}n\bar{t} + |c|^{-1}\tau\bar{\varkappa}, \quad (2.1)$$

where

$$L\hat{y} = L\{y,m,n,\tau\} = \{y^{(4)}(x),\, ay(0),\, by'(1),\, cy(1)\}$$

is an operator with the domain

$$D(L) = \{\{y\ (x),\ m,\ n\} : y \in W_2^4(0,1), y''(0) = 0, m = ay(0), n = by'(1), \tau = cy(1)\}$$

dense everywhere in H. Problem (1.1)-(1.5) takes the form

$$L\hat{y} = \lambda\hat{y}, y \in D(L)$$

i.e., the eigenvalues $\lambda_k,$, $k \in \mathbb{N}$, of the operator L and problem (1.1)-(1.5) coincide taking into account their multiplicities and between the root functions, there is a one-to-one correspondence

$$y_k(x) \leftrightarrow\ \{y_k(x),\, m_k,\, n_k, \tau_k\},\ m_k = ay_k(0),\, n_k = by_k'(1),\, \tau_k = cy_k(1).$$

By direct computation, we make sure that L is a self-adjoint discrete lower-semibounded operator in H. Hence this operator has a system of eigenvectors $\{y_k(x), m_k, n_k, \tau_k\}_{k=1}^{\infty}$ that forms an orthogonal basis in H.

3 The existence and properties of the solution of problem (1.1)-(1.3), (1.5)

We introduce the boundary conditions (see [4, 12, 19])

$$y'(1)\cos\gamma - y''(1)\sin\gamma = 0, \quad (3.1)$$

where $\gamma \in [0, \frac{\pi}{2}]$.

Alongside the problem (1.1)-(1.5) we also consider the problem (1.1)-(1.3), (3.1), (1.5). The spectral properties of problem (1.1)-(1.3), (3.1), (1.5) in the case of $\gamma = \frac{\pi}{2}$ have been considered in [10].

Using the method of [10] it can be shown that for problem (1.1)-(1.3), (3.1), (1.5) the following result holds.

Theorem 3.1 *The eigenvalues of the boundary value problem* (1.1)-(1.3), (3.1), (1.5) *form an infinitely nondecreasing sequence* $0 = \lambda_1(\gamma) = \lambda_2(\gamma) < \lambda_3(\gamma) < \ldots < \lambda_k(\gamma) < \ldots$. *The eigenfunction* $y_{k,\gamma}(x)$, *corresponding to the eigenvalue* $\lambda_k(\gamma)$, *for* $k \geq 3$ *has* $k-1$ *simple zeros in* $(0,1)$. *Moreover, for any* $\gamma \in [0, \frac{\pi}{2}]$ *we have the following relation*

$$\begin{gathered}\lambda_1(\pi/2) = \lambda_1(\gamma) = \lambda_1(0) = 0 = \lambda_2(\pi/2) \\ < \lambda_2(\gamma) < \lambda_2(0) < \lambda_3(\pi/2) < \lambda_3(\gamma) < \lambda_2(0) < \ldots .\end{gathered} \tag{3.2}$$

Theorem 3.2 *For each fixed* $\lambda \in \mathbb{C}\backslash\{0\}$ *there exists a nontrivial solution* $y(x,\lambda)$ *of the problem* (1.1)-(1.3), (1.5) *which is unique up to a constant coefficient.*

Proof. We denote by $\varphi_k(x,\lambda), k = \overline{1,4}$, be solutions of equation (1.1), normalized for $x = 0$ by the Cauchy conditions

$$\varphi_k^{(s-1)}(0,\lambda) = \delta_{ks}, \ s = \overline{1,4}, \tag{3.3}$$

where δ_{ks} is the Kronecker delta.

The function $y(x,\lambda)$ will be sought in the following form

$$y(x,\lambda) = \sum_{k=1}^{4} C_k \varphi_k(x,\lambda), \tag{3.4}$$

where $C_k, k = \overline{1,4}$, are constants.

By (3.3), (3.4) and boundary conditions (1.2), (1.3) it follows that $C_3 = 0$, $C_4 = a\lambda C_1$. Hence the function $y(x,\lambda)$ can be rewritten as follows:

$$y(x,\lambda) = C_1(\varphi_1(x,\lambda) + a\lambda\varphi_4(x,\lambda)) + C_2(\varphi_2(x,\lambda). \tag{3.5}$$

Then in view of boundary conditions (1.5) from (3.5) we obtain

$$\begin{gathered}C_1\left\{(\varphi_1'''(1,\lambda) + a\lambda\varphi_4'''(1,\lambda)) - c\lambda\left(\varphi_1(1,\lambda) + a\lambda\varphi_4(1,\lambda)\right)\right\} \\ +C_2\left\{\varphi_2'''(1,\lambda) - c\lambda\varphi_2(1,\lambda)\right\} = 0.\end{gathered} \tag{3.6}$$

Let $\lambda > 0$. Then by virtue of (1.5) it follows from [12, Lemma 2.1] that $\varphi_2(1,\lambda) > 0$, $\varphi_2'''(1,\lambda) > 0$. Since $c < 0$ it follows that $\varphi_2'''(1,\lambda) - c\lambda\varphi_2(1,\lambda) > 0$. Consequently, by (3.6) the function $y(x,\lambda)$ takes the form

$$\begin{gathered}y(x,\lambda) = C_1\left\{\varphi_1(x,\lambda) + a\lambda\varphi_4(x,\lambda)\right. \\ \left. -\tfrac{\varphi_1'''(1,\lambda)+a\lambda\varphi_4'''(1,\lambda)-c\lambda(\varphi_1(1,\lambda)+a\lambda\varphi_4(1,\lambda))}{\varphi_2'''(1,\lambda)-c\lambda\varphi_2(1,\lambda)}\varphi_2(x,\lambda)\right\}.\end{gathered} \tag{3.7}$$

Let now $\lambda \in \mathbb{C}\backslash\,[0,+\infty)$. If in this case

$$\varphi_1'''(1,\lambda) + a\lambda\varphi_4'''(1,\lambda) - c\lambda\left(\varphi_1(1,\lambda) + a\lambda\varphi_4(1,\lambda)\right) = 0$$

and

$$\varphi_2'''(1,\lambda) - c\lambda\varphi_2(1,\lambda) = 0,$$

then the functions $\varphi_1(x,\lambda) + a\lambda\varphi_4(x,\lambda)$ and $\varphi_2(x,\lambda)$ are solutions of problem (1.1)-(1.3). We consider the function

$$\begin{gathered}v(x,\lambda) = \varphi_2'(1,\lambda)\left(\varphi_1(x,\lambda) + a\lambda\varphi_4(1,\lambda)\right) \\ -\left(\varphi_1'(1,\lambda) + a\lambda\varphi_4'(1,\lambda)\right)\varphi_2(x,\lambda).\end{gathered}$$

Then $v(x,\lambda)$ is an eigenfunction of problem (1.1)-(1.3), (3.1), (1.5) with $\gamma=0$ corresponding to the eigenvalue $\lambda\in\mathbb{C}\backslash\,[0,+\infty)$ which contradicts to (3.2). Thus

$$\{\varphi_1'''(1,\lambda)+a\lambda\varphi_4'''(1,\lambda)-c\lambda\left(\varphi_1(1,\lambda)+a\lambda\varphi_4(1,\lambda)\right)\}^2$$
$$+\{\varphi_2'''(1,\lambda)-c\lambda\varphi_2(1,\lambda)\}^2>0.$$

Cosequently, the function $y(x,\lambda)$ can be represented as follows:

$$\begin{aligned}y(x,\lambda)=C\,\{&(\varphi_2'''(1,\lambda)-c\lambda\varphi_2(1,\lambda))\,\{\varphi_1(x,\lambda)+a\lambda\varphi_4(x,\lambda)\}\\&-\left(\varphi_1'''(1,\lambda)+a\lambda\varphi_4'''(1,\lambda)-c\lambda\left(\varphi_1(1,\lambda)+a\lambda\varphi_4(1,\lambda)\right)\right)\varphi_2(x,\lambda)\},\end{aligned}\tag{3.8}$$

where

$$C=\frac{C_1}{\varphi_2'''(1,\lambda)-c\lambda\varphi_2(1,\lambda)}\ \text{ if }\ \varphi_2'''(1,\lambda)-c\lambda\varphi_2(1,\lambda)\neq 0,$$

$$C=\tfrac{-C_2}{\varphi_1'''(1,\lambda)+a\lambda\varphi_4'''(1,\lambda)-c\lambda(\varphi_1(1,\lambda)+a\lambda\varphi_4(1,\lambda))}\ \text{ if}$$

$$\varphi_1'''(1,\lambda)+a\lambda\varphi_4'''(1,\lambda)-c\lambda\left(\varphi_1(1,\lambda)+a\lambda\varphi_4(1,\lambda)\right)\neq 0.$$

The proof of this lemma is complete.

Remark 3.1 Since the functions $\varphi_i(x,\lambda)$, $\imath=\overline{1,4}$, are entire function of λ it follows from (3.7) and (3.8) that without loss of generality we can regard solution $y(x,\lambda)$ of the problem (1.1)-(1.3), (1.5) for each fixed $x\in[0,1]$ as an entire function of λ for $\lambda\in\mathbb{C}\backslash\{0\}$ of the following form

$$\begin{aligned}y(x,\lambda)=&\left(\varphi_2'''(1,\lambda)-c\lambda\varphi_2(1,\lambda)\right)\{\varphi_1(x,\lambda)+a\lambda\varphi_4(x,\lambda)\}\\&-\left(\varphi_1'''(1,\lambda)+a\lambda\varphi_4'''(1,\lambda)-c\lambda\left(\varphi_1(1,\lambda)+a\lambda\varphi_4(1,\lambda)\right)\right)\varphi_2(x,\lambda).\end{aligned}\tag{3.9}$$

Remark 3.2 It is obvious that if $\lambda=0$ then problem (1.1)-(1.3) has two linearly independent solutions $y_1(x,0)=1$ and $y_2(x,0)=x$.

Remark 3.3 Let $\lambda=\rho^4$. Then solution $y(x,\lambda)$ problem (1.1)-(1.3), (1.5) represented as follows:

$$y(x,\lambda)=\cos\rho(x)+\cosh\rho x+C(\rho)\sin\rho x+(C(\rho)+2a\rho)\sinh\rho x,\tag{3.10}$$

where

$$C(\rho)=\frac{\sin\rho+\sinh\rho+2a\rho\cosh\rho-c\rho(\cos\rho+\cosh\rho+2a\rho\sinh)}{\cos\rho-\cosh\rho+c\rho(\sin\rho+\sinh\rho)}.\tag{3.11}$$

By virtue of (3.10) and (3.11) we have

$$\lim_{\lambda\to 0}y(x,\lambda)=2\left(1+\frac{2(1+a-c)}{2c-1}\,x\right).\tag{3.12}$$

Remark 3.4 Now we can define a function $y(x,\lambda)$ everywhere on $[0,1]\times\mathbb{C}$ by putting

$$y(x,0)=2\left(1+\frac{2(1+a-c)}{2c-1}\,x\right).\tag{3.13}$$

Remark 3.5 It follows from (3.13) that the function $y(x,0)$ has one zero $x = \frac{1-2c}{2(1+a-c)} = \frac{1-2c}{1+2a+1-2c}$ in the interval $(0,1)$. This fact plays a fundamental role in the study of the oscillation properties of the eigenfunctions of the problem (1.1)-(1.5).

Remark 3.6 By virtue of (3.13) we have

$$y'(x,0) = \frac{2(1+a-c)}{2c-1} < 0, \;\; y''(x,0) = 0, \;\; x \in [0,1]. \tag{3.14}$$

Remark 3.7 If $\lambda > 0$, then by (1.2), (1.3), (1.5) and the first part of [12, Lemma 2.1] we have $y(0,\lambda)\, y'(0,\lambda) \neq 0$.

Lemma 3.1 *The zeros in* $(0,1]$ *of functions* $y(x,\lambda)$ *and* $y'(x,\lambda)$ *are simple and* C^1 *function of* $\lambda > 0$.

Proof. Let $\lambda > 0$ and $x_0 \in (0,1)$ such that $y(x_0,\lambda) = y'(x_0,\lambda) = 0$ ($y'(x_0,\lambda) = y''(x_0,\lambda) = 0$). If $y''(x_0,\lambda)y'''(x_0,\lambda) \geq 0$ ($y(x_0,\lambda)y'''(x_0,\lambda) \geq 0$), then by virtue of the first part of [12, Lemma 2.1] we have $y(1,\lambda) = y'''(1,\lambda) > 0$ in contradiction with (1.5). If $y''(x_0,\lambda)y'''(x_0,\lambda) < 0$ ($y(x_0,\lambda)y'''(x_0,\lambda) < 0$), then by virtue of the second part of [12, Lemma 2.1] we have $y''(0,\lambda) \neq 0$ in contradiction with (1.2).

Let $\lambda > 0$ such that $y(1,\lambda) = y'(1,\lambda) = 0$ ($y'(1,\lambda) = y''(1,\lambda) = 0$). Then by (1.5) it follows from the second part of [1, Lemma 2.1] that $y''(0,\lambda) \neq)$ in contradiction with (1.2).

Let $\lambda \leq 0$ and $x_0 \in (0,1]$ such that $y(x_0,\lambda) = y'(x_0,\lambda) = 0$. Then multiplying both sides of equation (1.1) by $\overline{y(x,\lambda)}$, and integrating obtaining equality in the range from 0 to x_0, using the formula of integration by parts and taking boundary conditions (1.2), (1.3) into account, we have

$$\int\limits_0^{x_0} y''^2(x,\lambda)dx = \lambda \left\{ \int\limits_0^{x_0} y^2(x,\lambda)dx + ay^2(0,\lambda) \right\}$$

which implies that $\lambda > 0$ in contradiction with $\lambda \leq 0$.

Let $\lambda \leq 0$ and $x_0 \in (0,1]$ such that $y'(x_0,\lambda) = y''(x_0,\lambda) = 0$. Since $y''(0,\lambda) = 0$ there exists a point ξ closest to x_0 such that $y'''(\xi,\lambda) = 0$. We can assume without loss of generality that $y''(x,\lambda) < 0$ for $x \in [\xi, x_0)$. Then $y'(\xi,\lambda) > 0$, $y'''(\xi-0,\lambda) < 0$ and $y'''(\xi+0,\lambda) < 0$. Hence by Eq. (1.1) $y(\xi,\lambda) < 0$. Consequently, by the above argument we have

$$\int\limits_0^{\xi} y''^2(x,\lambda)dx - y'(\xi,\lambda)\, y''(\xi,\lambda) = \lambda \left\{ \int\limits_0^{\xi} y^2(x,\lambda)dx + ay^2(0,\lambda) \right\}$$

which implies that $\lambda > 0$ in contradiction with $\lambda \leq 0$.

The continuous differentiability of the function $x(\lambda)$ follows from the well-known implicit function theorem. The proof of this lemma is complete.

Corollary 3.1 *As* $\lambda > 0$ ($\lambda \leq 0$) *varies the function* $y(x,\lambda)$ *and* $y'(x,\lambda)$ *can lose or gain zeros only by these zeros leaving or entering the interval* $[0,1]$ *through its endpoint* $x = 1$ ($x = 0$).

Lemma 3.2 *Let* $y(x,\lambda)$ *be a nontrivial solution of problem* (1.1), (1.2), (1.5) *for* $\lambda > 0$. *If* $y(x_0,\lambda) = 0$ *or* $y''(x_0,\lambda) = 0$, *then* $y'(x,\lambda)Ty(x,\lambda) < 0$ *in a some neighborhood of* $x_0 \in (0,1)$; *if* $y'(x_0,\lambda) = 0$ *or* $Ty(x_0,\lambda) = 0$, *then* $y(x,\lambda)y''(x,\lambda) < 0$ *in a some neighborhood of* $x_0 \in (0,1)$.

The proof of this lemma is similar to that of [2, Lemma 2.2].

Lemma 3.3 *Let $\lambda > 0$. Then between consecutive zeros of function $y'(x,\lambda)$ in half-open interval $(0,1]$, there is exactly one zero of function $y(x,\lambda)$.*

The proof of Lemmas 3.3 is similar to that of [9, Lemma 2.5] using Lemma 3.2.
We consider the function

$$H(x,\lambda) = \frac{y'(x,\lambda)}{y''(x,\lambda)}.$$

By virtue of Theorem 3.2, and Remarks 3.1-3.4 that the function $H(x,\lambda)$ is a finite order meromorphic function of λ for all finite λ and fixed $x \in (0,1]$.

Remark 3.8 It is obvious that the eigenvalues $\lambda_n(0)$ and $\lambda_n(\pi/2)$, $n \in \mathbb{N}$, $n \neq 1$, of the spectral problem (1.1)-(1.3), (3.1), (1.5) for $\gamma = 0$ and $\gamma = \pi/2$ are zeros of the entire functions $y'(1,\lambda)$ and $y''(1,\lambda)$, respectively.

To study the spectral properties of problem (1.1)-(1.5), we need to investigate the behavior of the function

$$F(\lambda) = \frac{1}{H(1,\lambda)} = \frac{y''(1,\lambda)}{y'(1,\lambda)}$$

on the real axis. In view of Theorem 3.1, (3.14) and Remark 3.8 this function is well defined for

$$\lambda \in \mathcal{D} \equiv (\mathbb{C}\backslash\mathbb{R}) \cup (-\infty,\, \lambda_2(0)) \cup \left(\bigcup_{k=3}^{\infty} (\lambda_{k-1}(0), \lambda_k(0)) \right)$$

and is a meromorphic function of finite order. The eigenvalues $\lambda_k\,(\pi/2)$ and $\lambda_k(0)$ of problem (1.1)-(1.3), (3.1), (1.5) for $k \geq 2$ are zeros and poles of this function, respectively.

Remark 3.9 It follows from (3.14) that $F(0) = 0$.

Lemma 3.4 *For each $\lambda \in \mathcal{D}\backslash\{0\}$ the relation*

$$\frac{dF(\lambda)}{d\lambda} = -\,\frac{1}{y'^2(1,\lambda)} \left\{ \int_0^1 y^2(x,\lambda)\,dx + ay^2(0,\lambda) -\, cy^2(1,\lambda) \right\} \tag{3.15}$$

holds.

Proof. By virtue of Eq. (1.1) we have

$$\left(y'''(x,\mu)\right)' y(x,\lambda) - \left(y'''(x,\lambda)\right)' y(x,\mu) = (\mu-\lambda) y(x,\mu) y(x,\lambda). \tag{3.16}$$

Integrating equality (3.16) from 0 to 1, using the formula for the integration by parts and taking boundary conditions (1.1), (1.3) and (1.5) into account we obtain

$$\begin{gathered} -\,y''(1,\mu)\,y'(1,\lambda) + y''(1,\lambda)\,y'(1,\mu) \\ = (\mu-\lambda)\left\{ \int_0^1 y(x,\mu)\,y(x,\lambda)dx + ay(0,\mu)\,y(0,\lambda) - cy(1,\mu)\,y(1,\lambda) \right\}. \end{gathered} \tag{3.17}$$

By (3.8) for $\mu, \lambda \in \mathcal{D},\ \mu \neq \lambda$, we have

$$\frac{y''(1,\mu)}{y''(1,\mu)} - \frac{y''(1,\lambda)}{y'(1,\lambda)} = -\,(\mu-\lambda)\,\frac{\int_0^l y(x,\mu)\,y(x,\lambda)\,dx + a\,y'(1,\mu)\,y'(1,\lambda)}{y(1,\mu)\,y(1,\lambda)}\,. \tag{3.18}$$

Dividing both sides of relation (3.18) by $\mu-\lambda\,(\mu \neq \lambda)$ and by passing to the limit as $\mu \to \lambda$ we obtain (3.15). The proof of this lemma is complete.

Remark 3.10 By (3.10) and (3.11) we have

$$F(\lambda) = \rho\, \frac{-\cos\rho+\cosh\rho-C(\rho)\sin\rho+(C(\rho)+2a\rho)\sinh\rho}{-\sin\rho+\sinh\rho+C(\rho)\cos\rho+(C(\rho)+2a\rho)\cosh\rho}, \tag{3.19}$$

which implies that

$$F(\lambda) = \frac{4c-1+4a(3c-1)}{12(1+a-c)}\,\lambda + o(\lambda) \text{ as } \lambda \to 0. \tag{3.20}$$

Then it follows from (3.20) that

$$\frac{dF(0)}{d\lambda} = \frac{4c-1+4a(3c-1)}{12(1+a-c)} < 0. \tag{3.21}$$

Corollary 3.2 *The function $F(\lambda)$ strictly decreases on each of intervals $(-\infty, \lambda_2(0))$ and $(\lambda_{k-1}(0), \lambda_k(0)),\ k = 3, 4, \ldots$.*

Lemma 3.5 *The following relation holds:*

$$\lim_{\lambda \to -\infty} F(\lambda) = +\infty. \tag{3.22}$$

Proof. We denote: $\tau = \frac{\sqrt[4]{|\lambda|}}{\sqrt{2}}$. Note that if $\lambda < 0$, then $\rho = \sqrt[4]{\lambda} = (1+i)\ \tau$. Hence by direct calculation from (3.19) we get

$$F(\lambda) = 2\,\tau\left(1 + O\left(\frac{1}{\tau}\right)\right) = \sqrt{2}\,\sqrt[4]{|\lambda|}\left(1 + O\left(\frac{1}{\sqrt[4]{|\lambda|}}\right)\right) \text{ as } \lambda \to -\infty.$$

The proof of this lemma is complete.

Lemma 3.6 *If there exist $x \in (0,1]$ and $\lambda > 0$ such that $y'(x,\lambda) = 0$, then*

$$\frac{\partial H(x,\lambda)}{\partial x} > 0.$$

The proof of this lemma is similar to that of [9, Lemma 3.5] with the use of Lemmas 3.1.

The following comparison type theorem is valid.

Lemma 3.7 *Let $0 < \nu < \eta$. If $y'(x,\mu)$ has m zeros in the interval $(0,1)$, then $y'(x,\eta)$ has at least m zeros in this interval.*

The proof of this lemma is similar to that of [9, Lemma 3.6] with the use of Theorem 3.2, Remarks 3.1-3.4, Corollary 3.1, Lemmas 3.3-3.6.

Denote by $\tau(\lambda)$ and $s(\lambda)$ we denote the number of zeros in the interval $(0,1)$ of functions $y(x,\lambda)$ and $y'(x,\lambda)$, respectively.

Theorem 3.3 *If $\lambda \in (0, \lambda_2(0)]$, then $\tau(\lambda) = 1,\ s(\lambda) = 0$, if $\lambda \in (\lambda_{k-1}(0),\ \lambda_k(\pi/2))$ and $k \geq 3$, then $\tau(\lambda) = k-2$ or $\tau(\lambda) = k-1$, if $\lambda \in [\lambda_k(\pi/2), \lambda_k(0)]$ and $k \geq 3$, then $\tau(\lambda) = k-1$, if $\lambda \in (\lambda_{k-1}(0), (\lambda_k(0)]$ and $k \geq 3$, then $s(\lambda) = k-2$.*

The proof of this lemma is similar to that of [9, Theorem 3.2] with the use of Remarks 3.5 , 3.6, Lemmas 3.1, 3.3, 3.4, 3.7, Corollary 3.1.

Remark 3.11 If $\lambda \in (\lambda_{k-1}(0), \lambda_k(\pi/2)),\ k \geq 3$, is sufficiently close to $\lambda_{k-1}(0)$, then $\tau(\lambda) = k-2$, and if λ is sufficiently close to $\lambda_k(\pi/2)$, then $\tau(\lambda) = k-1$.

4 Main properties of eigenvalues and eigenfunctions of problem (1.1)-(1.4), (1.5)

Lemma 4.1 *The eigenvalues of the boundary value problem* (1.1)-(1.5) *are real and form an at most countable set without finite limit point.*

Proof. It's obvious that the eigenvalues of problem (1.1)-(1.5) are the roots of the equation

$$y''(1,\lambda) - c\lambda\, y'(1,\lambda) = 0. \tag{4.1}$$

Let λ be the nonreal eigenvalue of problem (1.1)-(1.4). Since the coefficients $q(x),\ a,\ b,\ c$ are real it follows that $\bar{\lambda}$ is also an eigenvalue of problem (1.1)-(1.4). In this case $y(x,\bar{\lambda}) = \overline{y(x,\lambda)}$, so that if equality (4.1) holds for λ, then it also holds for $\bar{\lambda}$.

Putting $\mu = \bar{\lambda}$ in (3.17), we get

$$\begin{gathered} -\overline{y''(1,\lambda)}\, y'(1,\lambda) + y''(1,\lambda)\,\overline{y'(1,\mu)} \\ = (\bar{\lambda} - \lambda)\left\{ \int\limits_0^1 |y(x,\lambda)|^2 dx + a|y(0,\lambda)|^2 - c|y(1,\lambda)|^2 \right\}. \end{gathered} \tag{4.2}$$

In view of (1.4) from (4.2) we obtain

$$\begin{gathered} -b\,(\bar{\lambda} - \lambda)\,|y'(1,\lambda)|^2 \\ = (\bar{\lambda} - \lambda)\left\{ \int\limits_0^1 |\,y(x,\lambda)|^2 dx + a|y(0,\lambda)|^2 - c|y(1,\lambda)|^2 \right\}. \end{gathered} \tag{4.3}$$

Since $\bar{\lambda} \neq \lambda$ from (4.3) we obtain

$$\int\limits_0^1 |\,y(x,\lambda)|^2 dx + a\,|\,y(0,\lambda)|^2 + b\,|y'(1,\lambda)|^2 - c\,|\,y(1,\lambda)|^2 = 0 \tag{4.4}$$

which implies that $y(x,\lambda) \equiv 0$.

The entire function on the left-hand side in equation (4.1) does not vanish for non-real λ. Consequently, it does not vanish identically. Therefore, its zeros form an at most countable set without finite limit points.

Lemma 4.2 *The nonzero eigenvalues of the boundary value problem* (1.1)-(1.5) *are simple.*

Let λ be a nonzero eigenvalue of problem (1.1)-(1.4). Then by Lemma 4.1 from (4.4) we get

$$\begin{gathered} \int\limits_0^1 y''^2(x,\lambda)dx + \int\limits_0^1 q(x)\, y'^2(x,\lambda)dx \\ = \lambda\left\{ \int\limits_0^1 y^2(x,\lambda)dx + a\, y^2(0,\lambda) + b\, y'^2(1,\lambda) - c\, y^2(1,\lambda) \right\}. \end{gathered} \tag{4.5}$$

Hence it follows from (4.5) that $\lambda > 0$. Then by virtue of boundary condition (1.1) and Lemma 2.1 we have $y'(1,\lambda) \neq 0$. Therefore each root (with regard of multiplicities) of equation (4.1) is a root of the equation

$$F(\lambda) = b\lambda. \tag{4.6}$$

To prove the lemma, it suffices to show that the equation (4.6) has only simple roots. Indeed, if $\lambda = \lambda^*$ is a multiple root of (4.6), then we have

$$F(\lambda^*) = b\lambda^*,\ F'(\lambda^*) = b. \tag{4.7}$$

In view of (3.15) from the second relation in (4.7) we obtain

$$\int_0^1 y^2(x,\lambda^*)dx + a\,y^2(0,\lambda^*) + b\,y'^2(1,\lambda^*) - c\,y^2(1,\lambda^*) = 0. \tag{4.8}$$

Consequently, $y(x,\lambda^*) \equiv 0$. The proof of this lemma is complete.

We have the following oscillation theorem for the problem (1.1)-(1.5).

Theorem 4.1 *There exists an infinitely nondecreasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ of eigenvalues of the spectral problem* (1.1)-(1.5) *such that $\lambda_1 = \lambda_2 = 0$ and $\lambda_k > 0,\ k \geq 3$. Moreover, the corresponding eigenfunctions and their derivatives have the following oscillation properties:*

(i) the eigenfunction $y_k(x)$, corresponding to the eigenvalue λ_k for $k \geq 3$ has either $k-2$ or $k-1$ simple zeros in $(0,1)$;

(ii) the function $y'_k(x)$ for $k \geq 3$ has exactly $k-2$ simple zeros in the interval $(0,1)$.

Proof. By virtue of Remark 3.10 and Corollary 3.2 the function $F(\lambda) = y''(1,\lambda)/y'(1,\lambda)$ is a continuous strictly decreasing function in the intervals $(-\infty, \lambda_2(0))$ and $(\lambda_{k-1}(0), \lambda_k(0))$, $k \geq 3$. In view of relations (3.22) and $y'(1,\lambda_k(0)) = 0,\ k \geq 2$, we have

$$\lim_{\lambda\to-\infty} F(\lambda) = +\infty, \quad \lim_{\lambda\to\lambda_k(0)-0} F(\lambda) = -\infty, \quad \lim_{\lambda\to\lambda_k(0)+0} F(\lambda) = +\infty. \tag{4.9}$$

Consequently, the function $F(\lambda)$ takes each value in $(-\infty,+\infty)$ at a unique point in each of intervals $(-\infty, \lambda_2(0))$ and $(\lambda_{k-1}(0), \lambda_k(0))$, $k \geq 3$. The function $Q(\lambda) = b\lambda$ is strictly increasing on $(-\infty,+\infty)$ in view of condition $b > 0$. Then it follows from the preceding considerations that in in each of intervals $(-\infty, \lambda_2(0))$ and $(\lambda_{k-1}(0), \lambda_k(0))$, $k \geq 3$ the equation

$$F(\lambda) = Q(\lambda)$$

has unique solution $\lambda = \lambda_k^*$, i.e. (1.4) holds. By virtue of Remark 3.9 $\lambda = 0$ is an eigenvalue of the problem (1.1)-(1.5). It is easy to verify that this eigenvalue has geometric multiplicity 2. Therefore, we can assume that $\lambda_1 = \lambda_2 = \lambda_1^* = 0$, and corresponding eigespace consists of functions $px+q,\ p,q \in \mathbb{R},\ x \in [0,1]$. Then λ_k^* for $k \geq 2$ is the $(k+1)$th eigenvalue of problem (1.1)-(1.5), i.e. $\lambda_k = \lambda_{k-1}^*$ for $k \geq 3$, and consequenly, $y_k(x) = y(x,\lambda_k)$ is the corresponding eigenfunction.

By (4.9) it follows from Lemmas 3.4 that $F(\lambda) > 0$ for $\lambda \in (\lambda_{k-1}(0), \lambda_k(\pi/2))$, $k = 3,\ 4,\ \ldots$. Then by the above arguments we have

$$\lambda_1 = \lambda_2 = 0 < \lambda_2(0) < \lambda_3 < \lambda_3(\pi/2) < \lambda_3(0) < \lambda_4 < \lambda_2(\pi/2) < \ldots . \tag{4.10}$$

Hence it follows from Theorem 3.3 that

$$k-2 \leq s(\lambda_k) \leq k-1 \text{ and } \tau(\lambda_k) = k-2,\ k = 3,\ 4,\ \ldots . \tag{4.11}$$

The proof of this theorem is complete.

Theorem 4.2 *The following asymptotic formulas hold:*

$$\sqrt[4]{\lambda_k} = (k - 11/4)\,\pi + O\,(1/k)\,, \tag{4.12}$$

$$y_k(x) = \sin(k-11/4)\pi x + (-1)^k e^{-(k-11/4)\pi(1-x)} + O(1/k), \tag{4.13}$$

where relation (4.13) *holds uniformly for* $x \in [0,1]$.

Proof. By virtue of (4.9) from relation (4.13) it follows that for sufficiently large $k \in \mathbb{N}$ the eigenvalue λ_k of problem (1.1)-(1.5) is sufficiently close to $\lambda_{k-1}(0)$. Following the corresponding reasoning carried out in [1, § 3] we are convinced that

$$\lambda_k(0) = \left(k - \tfrac{7}{4}\right) + O\left(\tfrac{1}{k}\right),$$

$$y_{k,0}(x) = \sin(k - 7/4)\pi x + (-1)^{k+1} e^{-(k-7/4)\pi(1-x)} + O(1/k),$$

where $y_{k,0}(x)$ is an eigenfunction corresponding to the eigenvalue $\lambda_k(0)$. Hence using oscillation properties of eigenfunctions of problem (1.1)-(1.5) and by following the arguments on pp. 909-911 of [19] we obtain (4.12) and (4.13). The proof is complete.

5 Basis properties of eigenfunctions of problem (1.1)-(1.5)

Let $\hat{y}_k = \{y_k(x), m_k, n_k, \tau_k\}_{k=1}^{\infty}$, $m_k = a y_k(0)$, $n_k = b y'_k(1)$, $\tau_k = c y_k(1)$, be the system of eigenvectors of the operator L. By the arguments in § 2 this system forms an orthogonal basis in H.

We denote by

$$\delta_k = (\hat{y}_k, \hat{y}_k).$$

By virtue of (2.1) we have

$$\delta_k = ||y_k||_{L_2}^2 + a^{-1} m_k^2 + b^{-1} n_k^2 - c^{-1} \tau_k^2 > 0. \tag{5.1}$$

Then the system $\hat{v}_k = \delta_k^{-\frac{1}{2}} \hat{y}_k$ forms an Riesz basis in the space in H.

Let j, r and l be different arbitrary fixed natural numbers and

$$\begin{aligned} \Delta_{j,r,l} &= \begin{vmatrix} \delta_j^{-1/2} m_j & \delta_j^{-1/2} n_j & \delta_j^{-1/2} \tau_j \\ \delta_r^{-1/2} m_r & \delta_r^{-1/2} n_r & \delta_r^{-1/2} \tau_r \\ \delta_l^{-1/2} m_l & \delta_l^{-1/2} n_l & \delta_l^{-1/2} \tau_l \end{vmatrix} \\ &= \delta_j^{-1/2} \delta_r^{-1/2} \delta_l^{-1/2} abc \begin{vmatrix} y_j(0) & y'_j(1) & y_j(1) \\ y_r(0) & y'_r(1) & y_r(1) \\ y_l(0) & y'_l(1) & y_l(1) \end{vmatrix}. \end{aligned} \tag{5.2}$$

Denote by

$$\tilde{\Delta}_{j,r,l} = \begin{vmatrix} y_j(0) & y'_j(1) & y_j(1) \\ y_r(0) & y'_r(1) & y_r(1) \\ y_l(0) & y'_l(1) & y_l(1) \end{vmatrix}. \tag{5.3}$$

By (5.1)-(5.3) we have

$$\Delta_{r,l} \neq 0 \Leftrightarrow \tilde{\Delta}_{r,l} \neq 0. \tag{5.4}$$

Theorem 5.1 *If $\Delta_{j,r,l} \neq 0$, then the system $\{y_k(x)\}_{k=1,\, k\neq j,r,l}^{\infty}$ of eigenfunctions of problem* (1.1)-(1.5) *forms a basis in the space $L_p(0,1), 1 < p < \infty$, which is an unconditional basis for $p = 2$. If $\Delta_{r,l} = 0$, then this system is incomplete and nonminimal in the space $L_p(0,1)$, $1 < p < \infty$.*

The proof of Theorem 5.1 for $p = 2$ follows from [1, Theorem 3.1 and Corollary 3.1], for $p \in (1, +\infty) \backslash \{2\}$ is similar to that of [19, Theorem 5.1] by using asymptotic formulas (4.12)-(4.13).

It should be noted that using Theorems 4.1 and 4.2 from Theorem 5.1 one can obtain various sufficient conditions for the system $\{y_k(x)\}_{k=1,\, n\neq r,l}^{\infty}$ to form a basis in $L_p(0,1)$, $1 < p < \infty$.

Remark 5.1 Recall that the eigenvalue $\lambda = 0$ of problem (1.1)-(1.5) has a geometric multiplicity of 2 (in this case $\lambda_1 = \lambda_2 = 0$) and the corresponding eigenspace consists of functions of the form $y(x) = px + q,\ p,\ q \in \mathbb{R},\ x \in [0, 1]$. Hence the functions $y_1(x)$ and $y_2(x)$ can be chosen arbitrarily of the form $y_1(x) = p_1x + q_1$ and $y_2(x) = p_2x + q_2$, $p_1,\ p_2,\ q_1,\ q_2 \in \mathbb{R}$.

Now let $j = 1,\ r = 2$ and $l \geq 3$ be the arbitrary fixed natural number. Moreover, let $y_1(x) = 1$ and $y_2(x) = 1 - x$, $x \in [0, 1]$. Then we have

$$\tilde{\Delta}_{j,\,r,\,l} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ y_l(0) & y_l'(1) & y_l(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ y_l(0) + y_l'(1) & y_l'(1) & y_l(1) \end{vmatrix}$$

$$= y_l(0) + y_l'(1) - y_l(1).$$

If l is a sufficiently large number of odd multiplicity, then by virtue of Theorem 4.1 and Remark 3.7 we have $y_l(0)y_l(1) < 0$. Moreover, by virtue of Remark 3.7 and [12, Lemma 2.1] it follows that $y_l(0)y_l'(0) < 0$ and consequently, $y_l(1)y_l'(1) > 0$.

We can assume without of generality that $y_l(0) > 0$. Then it follows from the preceding considerations that $y_l'(1) > 0$ and $y_l(1) < 0$. Hence $\tilde{\Delta}_{j,\,r,\,l} > 0$. Thus by (5.2)-(5.4) it follows from Theorem 5.1 the following result.

Theorem 5.2 *If $j = 1,\ r = 2$ and l is a sufficiently large number of odd multiplicity, then the system of eigenfunctions $\{y_n(x)\}_{n=1,\,n\neq j,\,r,\,l}^{\infty}$ of problem* (1.1)-(1.5) *forms a basis in the space $L_p(0, 1),\ 1 < p < \infty$ (which is an unconditional basis for $p = 2$).*

References

1. Aliyev, Z.S.: *On the defect basicity of the system of root functions of differential operators with spectral parameter in the boundary conditions*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **28**, 3–14 (2008).
2. Aliyev, Z.S.: *Basis properties of a fourth order differential operator with spectral parameter in the boundary condition*, Cent. Eur. J. Math. **8** (2), 378–388 (2010).
3. Aliev, Z.S.: *Basis properties in L_p of systems of root functions of a spectral problem with spectral parameter in a boundary condition*, Differ. Equ. **47**(6), 766–777 (2011).
4. Aliev, Z.S.: *Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order*, Sb. Math. **207**(12), 1625–1649 (2016).
5. Aliev, Z.S., Agaev, E.A.: *Oscillation properties of the eigenfunctions of fourth-order completely regular sturmian systems*, Dokl. Math. **90**(3), 657–659 (2014).
6. Aliev, Z.S., Dunyamalieva, A.A.: *Basis properties of root functions of the Sturm-Liouville problem with a spectral parameter in the boundary conditions*, Dokl. Math. **88**(1), 441–445 (2013).
7. Aliev, Z.S., Dunyamalieva, A.A.: *Defect basis property of a system of root functions of a Sturm-Liouville problem with spectral parameter in the boundary conditions*, Differ. Equ. **51**(10), 1249–1266 (2015).
8. Aliyev, Z.S., Dunyamaliyeva, A.A., Mehraliyev, Y.T.: *Basis properties in L_p of root functions of Sturm-Liouville problem with spectral parameter-dependent boundary conditions*, Mediterr. J. Math. **14**(3), 1–23 (2017).
9. Aliyev, Z.S., Guliyeva, S.B.: *Properties of natural frequencies and harmonic bending vibrations of a rod at one end of which is concentrated inertial load*, J. Differential Equations, **263**(9), 5830–5845 (2017).

10. Aliyev, Z.S., Namazov, F.M.: *Spectral properties of a fourth-order eigenvalue problem with spectral parameter in the boundary conditions*, Electron. J. Qual. Theory Differ. Equ. **2017** (307), 1-11 (2017).
11. Amara, J. Ben, Vladimirov, A.A: *On oscillation of eigenfunctions of a fourth-order problem with spectral parameters in the boundary conditions*, J. Math. Sci., **150**(5), 2317–2325 (2008).
12. Banks, D.O., Kurowski, G.J.: *A Prüfer transformation for the equation of the vibrating beam*, Trans. Amer. Math. Soc. **199**, 203-222 (1974).
13. Bolotin, B.B.: Vibrations in technique: Handbook in 6 volumes, The vibrations of linear systems, I, *Engineering Industry*, Moscow (1978).
14. Fulton, C.T.: *Two-point boundary value problems with eigenvalue parameter in the boundary conditions*, Proc. Roy. Soc. Edinburgh, Sect.**A**, **77**(3-4), 293–308 (1977).
15. Gao, G.,, Li, X., Ma, R.: *Eigenvalues of a Linear Fourth-Order Differential Operator with Squared Spectral Parameter in a Boundary Condition*, Mediterr. J. Math, **15** (7), 1–14 (2018).
16. Kapustin, N.Yu.: *Oscillation properties of solutions to a nonselfadjoint spectral problem with spectral parameter in the boundary condition*, Differ. Equ. **35**(8), 1031–1034 (1999).
17. Kapustin, N.Yu.: *On a spectral problem arising in a mathematical model of torsional vibrations of a rod with pulleys at the ends*, Differ. Equ., **41** (10), 1490–1492 (2005).
18. Kapustin, N.Yu., Moiseev, E.I.: *On the basis property in the space* L_p *of systems of eigenfunctions corresponding to two problems with spectral parameter in the boundary condition*, Differ. Equ., **36**(10), 1357–1360 (2000).
19. Kerimov, N.B., Aliev, Z.S.: *On the basis property of the system of eigenfunctions of a spectral problem with spectral parameter in a boundary condition*, Differ. Equ., **43**(7), 905–915 (2007).
20. Kerimov, N.B., Aliev, Z.S.: *Spectral properties of the differential operators of the fourth-order with eigenvalue parameter dependent boundary condition*, Int. J. Math. Math. Sci., **2012** (2012), Article ID 456517, 28 pages.
21. Kerimov, N.B., Aliev, Z.S.: *The oscillation properties of the boundary value problem with spectral parameter in the boundary condition*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **25**(7), 61–68 (2005).
22. Kerimov, N.B., Aliev, Z.S., Agayev, E.A.: *On the oscillation of eigenfunctions of a fourth-order spectral problem*, Dokl. Math. **85**(3), 355–357 (2012).
23. Moiseev, E.I, Kapustin, N.Yu.: *On singularities of the root space of a spectral problem with spectral parameter in a boundary condition*, Dokl. Math., **385**(1), 20–24 (2002).
24. Möller, M., Zinsou, B.: *Self-adjoint fourth order differential operators with eigenvalue parameter dependent boundary conditions*, Quaest. Math. **34**(3), 393–406 (2011).
25. Roseau, M.: Vibrations in mechanical systems. Analytical methods and applications, *Springer-Verlag, Berlin* (1987).
26. Walter, J.: *Regular eigenvalue problems with eigenvalue parameter in the boundary condition*, Math. Z. **133**(4), 301–312 (1973).