Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **39** (1), 121-131 (2019).

Boundedness of the commutator of Dunkl-type fractional integral operators in the Dunkl-type modified Morrey spaces

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Received: 02.11.2018 / Revised: 28.04.2019 / Accepted: 06.05.2019

Abstract. *In this paper, we find necessary and sufficient conditions for the boundedness of the commutator of the Dunkl-type fractional integral operator* $[b, I_{\beta,\alpha}]$ *from the Dunkl-type modified Morrey space* $\widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R})$ *to* $\widetilde{\mathcal{M}}_{q,\lambda,\alpha}(\mathbb{R})$*, where* $0 < \beta < 2\alpha + 2$, $0 \le \lambda < 2\alpha + 2 - \beta$, $\frac{\beta}{2\alpha+2} \le \frac{1}{p} - \frac{1}{q} \le \frac{\beta}{2\alpha+2-\lambda}$, $1 < p < \frac{2\alpha+2-\lambda}{\beta}$ *and* $b \in BMO_\alpha(\mathbb{R})$.

Keywords. Commutator, Dunkl-type fractional integral operator, Dunkl-type modified Morrey space, BMO_α space.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35

1 Introduction

In the theory of partial differential equations, together with weighted $L_{p,w}(\mathbb{R}^n)$ spaces, Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ play an important role. Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [27]). Later, Morrey spaces found important applications to Navier-Stokes ([33]) and Schrödinger equations ([4], [21], [22], [31]), elliptic problems with discontinuous coefficients ([11], [17], [18], [19], [20]) and potential theory (see [3]).

On the real line, the Dunkl operators D_ν are differential-difference operators introduced in 1989 by Dunkl [10]. For a real parameter $\nu \ge -1/2$, we consider the *Dunkl operator*, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1)\frac{f(x) - f(-x)}{2x}, \ \ \forall\, x \in \mathbb{R}.$$

Note that $D_{-1/2} = d/dx$.

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It is well known that the the fractional integral, associated with D_ν differential-difference Dunkl operators play an important role in Dunkl harmonic analysis, differentiation theory and PDE's. The harmonic analysis of the one-dimensional Dunkl operator and Dunkl transform was developed in [1], [15], [16], [10], [29], [34]. The Dunkl operator and Dunkl transform considered here are the rank-one case of the general Dunkl theory, which is associated with a finite reflection group acting on a Euclidean space.

In this paper, in the framework of the Dunkl analysis in the setting $\mathbb{R}$, we find necessary and sufficient conditions for the boundedness of the commutator of the Dunkl-type fractional integral operator $[b, I_{\beta,\alpha}]$ from the Dunkl-type modified Morrey space $\widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R}$ to $\widetilde{\mathcal{M}}_{q,\lambda,\alpha}(\mathbb{R})$ where $\beta/(2\alpha+2) = 1/p - 1/q$, $1 < p < (2\alpha+2)/\beta$ and $b \in BMO_\alpha(\mathbb{R})$.

2 Definitions, notation and preliminaries

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1}\, dx.$$

For every $1 \le p \le \infty$, we denote by $L_{p,\alpha}(\mathbb{R}) = L_p(\mathbb{R}, d\mu_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} \equiv \|f\|_{L_{p,\alpha}} = \left(\int_{\mathbb{R}} |f(x)|^p\, d\mu_\alpha(x)\right)^{1/p} < \infty \quad \text{if} \quad p \in [1,\infty),$$

and

$$\|f\|_{\infty,\alpha} \equiv \|f\|_{L_\infty} = ess \sup_{x\in\mathbb{R}} |f(x)| \quad \text{if} \quad p = \infty.$$

For $1 \le p < \infty$ we denote by $WL_{p,\alpha}(\mathbb{R})$, the weak $L_{p,\alpha}(\mathbb{R})$ spaces defined as the set of locally integrable functions f with the finite norm

$$\|f\|_{WL_{p,\alpha}} = \sup_{r>0} r\, \left(\mu_\alpha\{x \in \mathbb{R} \,:\, |f(x)| > r\}\right)^{1/p}.$$

Note that $L_{p,\alpha} \subset WL_{p,\alpha}$ and $\|f\|_{WL_{p,\alpha}} \le \|f\|_{L_{p,\alpha}}$ for all $f \in L_{p,\alpha}(\mathbb{R})$.

Let $B(x,t) = \{y \in \mathbb{R} : |y| \in]\max\{0, |x|-t\}, |x|+t[\,\}$ and $B_t \equiv B(0,t) =]-t, t[$, $t > 0$. Then $\mu_\alpha B_t = b_\alpha\, t^{2\alpha+2}$, where $b_\alpha = \left[2^{\alpha+1}\,(\alpha+1)\,\Gamma(\alpha+1)\right]^{-1}$.

Definition 2.1 *Let $1 \le p < \infty$, $0 \le \lambda \le 2\alpha+2$. We denote by $\mathcal{M}_{p,\lambda,\alpha}(\mathbb{R})$ Dunkl-type Morrey space ($\equiv$ D-Morrey space) as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$, with the finite norm*

$$\|f\|_{\mathcal{M}_{p,\lambda,\alpha}} = \sup_{t>0,\, x\in\mathbb{R}} \left(t^{-\lambda} \int_{B_t} [\tau_x|f(y)|\,]^p d\mu_\alpha(y)\right)^{1/p}.$$

Definition 2.2 *Let $1 \le p < \infty$, $0 \le \lambda \le 2\alpha+2$ and $[t]_1 = \min\{1, t\}$. We denote by $\widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R})$ the Dunkl-type modified Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$, with finite norm*

$$\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} = \sup_{t>0,\, x\in\mathbb{R}} \left([t]_1^{-\lambda} \int_{B_t} [\tau_x|f(y)|\,]^p d\mu_\alpha(y)\right)^{1/p}.$$

Definition 2.3 *Let $1 \le p < \infty$ and $0 \le \lambda \le 2\alpha + 2$. A measurable function f on $\mathbb{R}$ is said to belong to the Dunkl-type modified weak Morrey space $W\widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R})$ if the quasi-norm*

$$\|f\|_{W\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}} \left([t]_1^{-\lambda} \int_{\{y\in B_t:\, \tau_x|f(y)|\}} d\mu_\alpha(y) \right)^{1/p}$$

is finite.

Let $M_\alpha^\sharp$ be the Dunkl-type sharp maximal function defined by

$$M_\alpha^\sharp f(x) := \sup_{r>0} \frac{1}{\mu_\alpha B_r} \int_{B_r} |\tau_x f(y) - f_{B_r}(x)|\, d\mu_\alpha(y),$$

where $f_{B_r}(x) = \frac{1}{\mu_\alpha B_r} \int_{B_r} \tau_x f(y)\, d\mu_\alpha(y)$.

We denote by $BMO_\alpha(\mathbb{R})$ (Dunkl-type BMO space) the set of locally integrable functions f with finite norm

$$\|f\|_{BMO_\alpha} := \sup_{r>0,\ x\in\mathbb{R}} \frac{1}{\mu_\alpha B_r} \int_{B_r} |\tau_x f(y) - f_{B_r}(x)|\, d\mu_\alpha(y) < \infty.$$

Theorem 2.1 *[25] 1) Let $f \in L_{1,\alpha}^{loc}(\mathbb{R})$. If*

$$\|f\|_{BMO_{p,\alpha}} := \sup_{t>0,\ x\in\mathbb{R}} \left(\mu_\alpha(B_t)^{-1} \int_{B_t} |\tau_y f(x) - f_{B_t}|^p d\mu_\alpha(y) \right)^{1/p} < \infty$$

then for any $1 < p < \infty$,

$$\|f\|_{BMO_\alpha} \le \|f\|_{BMO_{p,\alpha}} \le A_p \|f\|_{BMO_\alpha},$$

where the constant A_p depends only on p.

2) Let $f \in BMO_\alpha(\mathbb{R})$. Then, there is a constant $C > 0$ such that

$$|f_{B_r} - f_{B_t}| \le C\|f\|_{BMO_\alpha} \ln \frac{t}{r}, \ \ 0 < 2r < t,$$

where C is independent of f, x, r and t.

For all $x,\ y,\ z \in \mathbb{R}$, we put

$$W_\alpha(x,y,z) = (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}) \varDelta_\alpha(x,y,z)$$

where

$$\sigma_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x,\ y \in \mathbb{R}\setminus\{0\}, \\ 0 & \text{otherwise} \end{cases}$$

and $\varDelta_\alpha$ is the Bessel kernel given by

$$\varDelta_\alpha(x,y,z) = \begin{cases} d_\alpha \frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\alpha-1/2}}{|xyz|^{2\alpha}}, & \text{if } |z| \in A_{x,y}, \\ 0, & \text{otherwise,} \end{cases}$$

where $d_\alpha = (\varGamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\, \varGamma(\alpha+\frac{1}{2}))$ and $A_{x,y} = [||x|-|y||, |x|+|y|]$.

Property 2.1 *(see Rösler [28]) The signed kernel W_α is even with respect to all variables and satisfies the following properties*

$$W_\alpha(x,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y),$$
$$W_\alpha(x,y,z) = W_\alpha(-z,y,-x) = W_\alpha(-x,-y,-z)$$

and

$$\int_{\mathbb{R}} |W_\alpha(x,y,z)|\, d\mu_\alpha(z) \le 4.$$

In the sequel we consider the signed measure $\nu_{x,y}$, on $\mathbb{R}$, given by

$$\nu_{x,y} = \begin{cases} W_\alpha(x,y,z)\, d\mu_\alpha(z) & \text{if } x,\ y \in \mathbb{R}\setminus\{0\}, \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0. \end{cases}$$

Definition 2.4 *For $x,\ y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, we put*

$$\tau_x f(y) = \int_{\mathbb{R}} f(z)\, d\nu_{x,y}(z).$$

The operators τ_x, $x \in \mathbb{R}$, are called Dunkl translation operators on $\mathbb{R}$ and it can be expressed in the following form (see [28])

$$\tau_x f(y) = c_\alpha \int_0^\pi f_e\left((x,y)_\theta\right)\, h_1(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta$$
$$+ c_\alpha \int_0^\pi f_o\left((x,y)_\theta\right)\, h_2(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta,$$

where $(x,y)_\theta = \sqrt{x^2+y^2-2|xy|\cos\theta}$, $f = f_e + f_o$, f_o and f_e being respectively the odd and the even parts of f, with

$$c_\alpha \equiv \left(\int_0^\pi (\sin\theta)^{2\alpha}\, d\theta\right)^{-1} = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)},$$
$$h_1(x,y,\theta) = 1 - sgn(xy)\cos\theta$$

and

$$h_2(x,y,\theta) = \begin{cases} \frac{(x+y)\,[1-sgn(xy)\cos\theta]}{(x,y)_\theta}, & \text{if } xy \neq 0, \\ 0, & \text{if } xy = 0. \end{cases}$$

Using the change of variable $z = (x,y)_\theta$, we have also (see [2])

$$\tau_x f(y) = c_\alpha \int_0^\pi \Big\{ f\left((x,y)_\theta\right) + f\left(-(x,y)_\theta\right) + \frac{x+y}{(x,y)_\theta}\left[f\left((x,y)_\theta\right) - f\left(-(x,y)_\theta\right)\right] \Big\}(1-\cos\theta)(\sin\theta)^{2\alpha}\, d\theta.$$

Now we define the Dunkl-type fractional maximal function $M_{\beta,\alpha}f$ and Dunkl-type fractional integral $I_{\beta,\alpha}f$ by

$$M_{\beta,\alpha}f(x) = \sup_{r>0} \big(\mu_\alpha B_r\big)^{\frac{\beta}{2\alpha+2}-1} \int_{B_r} \tau_x|f|(y)\, d\mu_\alpha(y), \quad 0 \le \beta < 2\alpha+2$$

and

$$I_{\beta,\alpha}f(x) = \int_{\mathbb{R}} |y|^{\beta-2\alpha-2}\, \tau_y f(x) d\mu_\alpha(y), \quad 0 < \beta < 2\alpha + 2$$

respectively.

If $\beta = 0$, then $M_\alpha \equiv M_{0,\alpha}$ is the Hardy-Littlewood maximal operator associated with the Dunkl operator (see [1], [15], [30]).

The following theorem were proved in [26], which obtained the necessary and sufficient conditions for the Dunkl-type fractional integral operator $I_{\beta,\alpha}$ to be bounded from the spaces $\widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R})$ to $\widetilde{\mathcal{M}}_{q,\lambda,\alpha}(\mathbb{R})$, $1 < p < q < \infty$ and from the spaces $\widetilde{\mathcal{M}}_{1,\lambda,\alpha}(\mathbb{R})$ to the weak spaces $W\widetilde{\mathcal{M}}_{q,\lambda,\alpha}(\mathbb{R})$, $1 < q < \infty$.

Theorem 2.2 *[26] Let $0 < \beta < 2\alpha + 2$, $0 \le \lambda < 2\alpha + 2 - \beta$ and $1 \le p < \frac{2\alpha+2-\lambda}{\beta}$.*

1) If $1 < p < \frac{2\alpha+2-\lambda}{\beta}$, then condition $\frac{\beta}{2\alpha+2} \le \frac{1}{p} - \frac{1}{q} \le \frac{\beta}{2\alpha+2-\lambda}$ is necessary and sufficient for the boundedness of the operators $M_{\beta,\alpha}$ and $I_{\beta,\alpha}$ from $\widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R})$ to $\widetilde{\mathcal{M}}_{q,\lambda,\alpha}(\mathbb{R})$.

2) If $p = 1 < \frac{2\alpha+2-\lambda}{\beta}$, then condition $\frac{\beta}{2\alpha+2} \le 1 - \frac{1}{q} \le \frac{\beta}{2\alpha+2-\lambda}$ is necessary and sufficient for the boundedness of the operators $M_{\beta,\alpha}$ and $I_{\beta,\alpha}$ from $\widetilde{\mathcal{M}}_{1,\lambda,\alpha}(\mathbb{R})$ to $W\widetilde{\mathcal{M}}_{q,\lambda,\alpha}(\mathbb{R})$.

For $1 \le p, \theta \le \infty$, $0 \le \lambda \le 2\alpha + 2$ and $0 < s < 1$, the Besov-Morrey space $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ and modified Besov-Morrey space $\widetilde{B}^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ for the Dunkl operators on $\mathbb{R}$ consists of all functions f in $\widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R})$ so that

$$\|f\|_{B^s_{p\theta,\lambda,\alpha}} = \|f\|_{\mathcal{M}_{p,\lambda,\alpha}} + \left(\int_{\mathbb{R}} \frac{\|\tau_x f(\cdot) - f(\cdot)\|^\theta_{\mathcal{M}_{p,\lambda,\alpha}}}{|x|^{2\alpha+2+s\theta}} d\mu_\alpha(x) \right)^{1/\theta} < \infty$$

and

$$\|f\|_{\widetilde{B}^s_{p\theta,\lambda,\alpha}} = \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} + \left(\int_{\mathbb{R}} \frac{\|\tau_x f(\cdot) - f(\cdot)\|^\theta_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}}}{|x|^{2\alpha+2+s\theta}} d\mu_\alpha(x) \right)^{1/\theta} < \infty.$$

Besov spaces in the setting of the Dunkl operators studied by C. Abdelkefi and M. Sifi [1,2], R. Bouguila, M.N. Lazhari and M. Assal [6], L. Kamoun [24], V.S. Guliyev, Y.Y. Mammadov [15]. In the following theorem we prove the boundedness of the Dunkl-type fractional maximal operator $M_{\beta,\alpha}$ in the Dunkl-type Besov spaces.

Theorem 2.3 *([15]) For $1 < p < q < \infty$, $0 \le \lambda < 2\alpha + 2 - \beta$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2-\lambda}$, $1 \le \theta \le \infty$ and $0 < s < 1$ the Dunkl-type fractional maximal operator $M_{\beta,\alpha}$ is bounded from $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ to $B^s_{q\theta,\lambda,\alpha}(\mathbb{R})$. More precisely, there is a constant $C > 0$ such that*

$$\|M_{\beta,\alpha} f\|_{B^s_{q\theta,\lambda,\alpha}} \le C \|f\|_{B^s_{p\theta,\lambda,\alpha}}$$

hold for all $f \in B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$.

3 Commutators of the Dunkl-type fractional integral operator in Dunkl-type modified Morrey spaces

In this section we consider commutators of the Dunkl-type fractional integral operator defined as the following equality

$$[b, I_{\beta,\alpha}]f(x) = \int_{\mathbb{R}} |y|^{\beta-2\alpha-2}\, \tau_y \left((b(x) - b(y))f(x)\right) d\mu_\alpha(y), \quad 0 < \alpha < 2\alpha + 2.$$

Given a measurable function b the operator $|b, I_{\beta,\alpha}|$ is defined by

$$|b, I_{\beta,\alpha}|f(x) = \int_{\mathbb{R}} |y|^{\beta-2\alpha-2}\, \tau_y \left|(b(x) - b(y))f(x)\right| d\mu_\alpha(y), \quad 0 < \alpha < 2\alpha + 2.$$

Theorem 3.1 *Let* $0 < \beta < 2\alpha + 2$, $0 \le \lambda < 2\alpha + 2 - \beta$, $\frac{\beta}{2\alpha+2} \le \frac{1}{p} - \frac{1}{q} \le \frac{\beta}{2\alpha+2-\lambda}$ *and* $1 < p < \frac{2\alpha+2-\lambda}{\beta}$. *Then the commutator* $|b, I_{\beta,\alpha}|$ *is bounded from* $\widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R})$ *to* $\widetilde{\mathcal{M}}_{q,\lambda,\alpha}(\mathbb{R})$ *if and only if* $b \in BMO_\alpha$.

Proof. *Sufficiency:* Let $f \in \widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R})$. Then

$$\begin{aligned} |b, I_{\beta,\alpha}|f(x) &= \left(\int_{B_t} + \int_{\mathbb{R}\setminus B_t}\right) |y|^{\beta-2\alpha-2}\tau_y \left|(b(x) - b(y))f(x)\right| d\mu_\alpha(y) \\ &\equiv F_1(x,t) + F_2(x,t). \end{aligned} \tag{3.1}$$

Firstly, we estimate $F_1(x,t)$. By using Hölder's inequality we have

$$\begin{aligned} F_1(x,t) &= \int_{B_t} |y|^{\beta-2\alpha-2}\tau_y \left|(b(x) - b(y))f(x)\right| d\mu_\alpha(y) \\ &\le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\beta-2\alpha-2} \int_{B_{2^{j+1}t}\setminus B_{2^j t}} \tau_y \left|(b(x) - b(y))f(x)\right| d\mu_\alpha(y) \\ &\le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\beta-2\alpha-2} \left(\int_{B_{2^{j+1}t}\setminus B_{2^j t}} |\tau_y b(x) - b|^{r'} d\mu_\alpha(y)\right)^{1/r'} \\ &\times \left(\int_{B_{2^{j+1}t}\setminus B_{2^j t}} \tau_y \left[|f|\right]^r (x) d\mu_\alpha(y)\right)^{1/r} \\ &\le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{2\alpha+2} \left(2^j t\right)^{\beta-2\alpha-2} \|b\|_{BMO_\alpha} \left(M_\alpha(|f|^r)(x)\right)^{1/r}. \end{aligned}$$

Hence

$$F_1(x,t) \le Ct^\beta \|b\|_{BMO_\alpha} \left(M_\alpha(|f|^r)(x)\right)^{1/r}. \tag{3.2}$$

Now we estimate $F_2(x,t)$. By using Hölder's inequality we get

$$\begin{aligned} F_2(x,t) &\le \int_{\mathbb{R}\setminus B_t} \tau_y(|b-b(x)||f(x)|)|y|^{\beta-2\alpha-2} d\mu_\alpha(y) \\ &\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\beta-2\alpha-2} \int_{B_{2^{j+1}t}\setminus B_{2^j t}} \tau_y(|b-b(x)||f(x)|) d\mu_\alpha(y) \\ &\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\beta-2\alpha-2} \left(\int_{B_{2^{j+1}t}\setminus B_{2^j t}} |\tau_y b(x) - b|^{p'} d\mu_\alpha(y)\right)^{1/p'} \\ &\times \left(\int_{B_{2^{j+1}t}\setminus B_{2^j t}} \tau_y \left[|f|\right]^p (x) d\mu_\alpha(y)\right)^{1/p} \\ &\le C t^{\beta-\frac{2\alpha+2}{p}} \|b\|_{BMO_\alpha} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} \sum_{j=0}^{\infty} 2^{j\left(\beta-\frac{2\alpha+2}{p}\right)} [2^j t]_1^{\frac{\lambda}{p}} \end{aligned}$$

$$\begin{aligned} &\le C t^{\beta-\frac{2\alpha+2}{p}} \|b\|_{BMO_\alpha} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} \\ &\times \begin{cases} \left(2^\lambda t^\lambda \sum\limits_{j=0}^{\log_2[\frac{1}{2t}]} 2^{j\left(\beta-\frac{2\alpha+2-\lambda}{p}\right)} + \sum\limits_{j=\log_2[\frac{1}{2t}]+1}^{\infty} 2^{j\left(\beta-\frac{2\alpha+2}{p}\right)}\right)^{1/p}, & 0<t<1, \\ \left(\sum\limits_{j=0}^{\infty} 2^{j\left(\beta-\frac{2\alpha+2}{p}\right)}\right)^{1/p}, & t \ge 1 \end{cases} \\ &\le C t^{\beta-\frac{2\alpha+2}{p}} \|b\|_{BMO_\alpha} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} \begin{cases} (C_1 t^\lambda + C_2 t^{\beta-\frac{2\alpha+2}{p}})^{\frac{1}{p}}, & 0<t<1, \\ C_3, & t\ge 1 \end{cases}. \end{aligned}$$

Thus

$$F_2(x,t) \le C t^{\beta-\frac{2\alpha+2}{p}} [t]_1^{\frac{\lambda}{p}} \|b\|_{BMO_\alpha} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}}. \tag{3.3}$$

So, from (3.2) and (3.3) we have

$$\begin{aligned} ||b, I_{\beta,\alpha}|f(x)| &\le C t^\beta \|b\|_{BMO_\alpha} \left(M_\alpha(|f|^r)(x)\right)^{1/r} + C t^{\beta-\frac{2\alpha+2}{p}} [t]_1^{\frac{\lambda}{p}} \|b\|_{BMO_\alpha} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} \\ &\le C \min\Big\{ t^\beta \|b\|_{BMO_\alpha} \left(M_\alpha(|f|^r)(x)\right)^{1/r} + C t^{\beta-\frac{2\alpha+2-\lambda}{p}} \|b\|_{BMO_\alpha} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}}, \\ &\quad t^\beta \|b\|_{BMO_\alpha} \left(M_\alpha(|f|^r)(x)\right)^{1/r} + C t^{\beta-\frac{2\alpha+2}{p}} \|b\|_{BMO_\alpha} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} \Big\}. \end{aligned}$$

Minimizing with respect to $t = \left[\left(M_\alpha(|f|^r)(x)\right)^{-1/r} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}}\right]^{p/(2\alpha+2-\lambda)}$ or $t = \left[\left(M_\alpha(|f|^r)(x)\right)^{-1/r} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}}\right]^{p/(2\alpha+2)}$ we obtain

$$||b, I_{\beta,\alpha}|f(x)| \le C \|b\|_{BMO_\alpha} \times$$

$$\min\left\{ \left(\frac{\left(M_\alpha(|f|^r)(x)\right)^{1/r}}{\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}}}\right)^{1-\frac{p\alpha}{2\alpha+2-\lambda}}, \left(\frac{\left(M_\alpha(|f|^r)(x)\right)^{1/r}}{\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}}}\right)^{1-\frac{p\alpha}{2\alpha+2}} \right\}.$$

Hence, by Theorem 2.2, we get

$$\begin{aligned}
&\int_{B_t} \tau_y \left[|[b, I_{\beta,\alpha}|f|]^q \right](x) d\mu_\alpha(y) \\
&\le C\|b\|^q_{BMO_\alpha} \|f\|^{q-p}_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} \int_{B_t} \tau_y \left(M_\alpha(|f|^r)(x)\right)^{p/r} d\mu_\alpha(y) \\
&\le C[t]_1^\lambda \|b\|^q_{BMO_\alpha} \|f\|^{q-p}_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} \||f|^r\|^p_{\widetilde{L}_{p/r,\lambda,\gamma}} = C[t]_1^\lambda \|b\|^q_{BMO_\alpha} \|f\|^{q-p}_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} \|f\|^p_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} \\
&= C[t]_1^\lambda \|b\|^q_{BMO_\alpha} \|f\|^q_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}}.
\end{aligned}$$

Finally $|b, I_{\beta,\alpha}|f \in \widetilde{\mathcal{M}}_{q,\lambda,\alpha}(\mathbb{R})$ and

$$\||b, I_{\beta,\alpha}|f\|_{\widetilde{\mathcal{M}}_{q,\lambda,\alpha}} \le C\|b\|_{BMO_\alpha} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}}.$$

Necessity: Let $|b, I_{\beta,\alpha}|$ be bounded from $\widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R})$ to $\widetilde{\mathcal{M}}_{q,\lambda,\alpha}(\mathbb{R})$, $1 < p < \frac{2\alpha+2-\lambda}{\beta}$. We have

$$\begin{aligned}
&\frac{1}{\mu_\alpha B_t} \int_{B_t} |\tau_z b(x) - f_{B_t}| \, d\mu_\alpha(z) \\
&= \frac{1}{\mu_\alpha B_t} \int_{B_t} \left| \tau_z b(x) - \frac{1}{\mu_\alpha B_t} \int_{B_t} \tau_z b(y) d\mu_\alpha(y) \right| d\mu_\alpha(z) \\
&\le \frac{1}{(\mu_\alpha B_t)^{1+\frac{\beta}{2\alpha+2}}} \int_{B_t} \frac{1}{|B_t|_\gamma^{1-\frac{\beta}{2\alpha+2}}} \int_{B_t} |\tau_z b(x) - \tau_z b(y)| \, d\mu_\alpha(y) d\mu_\alpha(z) \\
&\le \frac{1}{(\mu_\alpha B_t)^{1+\frac{\beta}{2\alpha+2}}} \int_{B_t} \int_{B_t} \frac{|\tau_z(b(x) - b(y))|}{|y|^{Q-\alpha}} d\mu_\alpha(y) d\mu_\alpha(z) \\
&\le \frac{1}{(\mu_\alpha B_t)^{1+\frac{\beta}{2\alpha+2}}} \int_{B_t} |b, I_{\beta,\alpha}|\chi_{B_t}(z) d\mu_\alpha(z) \\
&\le C t^{-2\alpha-2-\beta} [t]_1^\lambda \||b, I_{\beta,\alpha}|\chi_{B_t}\|_{\widetilde{\mathcal{M}}_{q,\lambda,\alpha}} \|\chi_{B_t}\|_{\widetilde{\mathcal{M}}_{q',\lambda,\alpha}} \\
&\le C t^{\frac{2\alpha+2}{q'} + \frac{2\alpha+2}{p} - 2\alpha - 2 - \alpha} [t]_1^{\frac{-\lambda}{q'} + \frac{-\lambda}{p} + \lambda} \le C,
\end{aligned}$$

which shows that $b \in BMO_\alpha(\mathbb{R})$.

Theorem 3.2 *Let* $0 < \beta < 2\alpha + 2$, $0 \le \lambda < 2\alpha + 2 - \beta$, $\frac{\beta}{2\alpha+2} \le 1 - \frac{1}{q} \le \frac{\beta}{2\alpha+2-\lambda}$ *and* $b \in BMO_\alpha(\mathbb{R})$. *Then the commutator* $|b, I_{\beta,\alpha}|$ *is bounded from* $\widetilde{\mathcal{M}}_{1,\lambda,\alpha}(\mathbb{R})$ *to* $W\widetilde{\mathcal{M}}_{q,\lambda,\alpha}(\mathbb{R})$.

Proof. Let $f \in \widetilde{\mathcal{M}}_{1,\lambda,\alpha}(\mathbb{R})$. We have

$$\begin{aligned}
\mu_\alpha\left(\{x \in B_t \; : \; ||b, I_{\beta,\alpha}|f(x)| > 2\delta\}\right) &\le \mu_\alpha(\{x \in B_t \; : \; F_1(x,t) > \delta\}) \\
&\quad + \mu_\alpha(\{x \in B_t \; : \; F_2(x,t) > \delta\}),
\end{aligned}$$

where $F_1(x,t)$ and $F_2(x,t)$ are defined in (3.2). Now taking into account (3.2) and Theorem 2.2 we have

$$\begin{aligned}
&\mu_\alpha(\{x \in B_t \; : \; |F_1(x,t)| > \delta\}) \\
&\le \mu_\alpha\left(\left\{x \in B_t \; : \; (M_\alpha(|f|^r)(x))^{1/r} > \frac{\delta}{Ct^\beta \|b\|_{BMO_\alpha}}\right\}\right) \\
&\le \frac{Ct^\beta}{\delta} \cdot \|b\|_{BMO_\alpha} \|f\|_{\widetilde{\mathcal{M}}_{1,\lambda,\alpha}}.
\end{aligned}$$

Also taking into account (3.3), for $\delta = Ct^{\beta-\frac{2\alpha+2}{p}}[t]_1^{\frac{\lambda}{p}}\|b\|_{BMO_\alpha}\|f\|_{\widetilde{\mathcal{M}}_{1,\lambda,\alpha}}$ we have $F_2(x,t) \leq \delta$, that is, $\mu_\alpha(\{x \in B_t \ : \ F_2(x,t) > \delta\}) = 0$.

Finally

$$\mu_\alpha(\{x \in B_t \ : \ ||b, I_{\beta,\alpha}|f(x)| > 2\delta\})$$
$$\leq \frac{C}{\delta} t^\beta \|b\|_{BMO_\alpha}\|f\|_{\widetilde{\mathcal{M}}_{1,\lambda,\alpha}} = C\left(\frac{\|b\|_{BMO_\alpha}\|f\|_{\widetilde{\mathcal{M}}_{1,\lambda,\alpha}}}{\delta}\right)^q.$$

Theorem 3.3 *Let* $0 \leq \lambda < 2\alpha+2$, $1 < p < \frac{2\alpha+2-\lambda}{\alpha}$, $0 < s < 1$, $1 \leq \theta \leq \infty$, $\frac{\beta}{2\alpha+2} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\beta}{2\alpha+2-\lambda}$ *and* $b \in BMO_\alpha(\mathbb{R})$. *Then the commutator* $[b, I_{\beta,\alpha}]$ *is bounded from* $\widetilde{B}^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ *to* $\widetilde{B}^s_{q\theta,\lambda,\alpha}(\mathbb{R})$.

Proof. From the definition of the Dunkl-type modified Besov-Morrey type spaces it suffices to show that

$$\|\tau_x[b, I_{\beta,\alpha}]f - [b, I_{\beta,\alpha}]f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}} \leq C\|b\|_{BMO_\alpha}\|\|\tau_x f - f\|_{\widetilde{\mathcal{M}}_{p,\lambda,\alpha}}.$$

It is easy to see that τ_x commutes with $[b, I_{\beta,\alpha}]$, i.e. $\tau_x[b, I_{\beta,\alpha}]f = [b, I_{\beta,\alpha}](\tau_x f)$. Hence we have

$$|\tau_x[b, I_{\beta,\alpha}]f - [b, I_{\beta,\alpha}]f| = |[b, I_{\beta,\alpha}](\tau_x f) - [b, I_{\beta,\alpha}]f| \leq [b, I_{\beta,\alpha}](|\tau_x f - f|).$$

Taking $\widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R})$ norm of both sides of the above inequality, from the boundedness of $[b, I_{\beta,\alpha}]$ from $\widetilde{\mathcal{M}}_{p,\lambda,\alpha}(\mathbb{R})$ to $\widetilde{\mathcal{M}}_{q,\lambda,\alpha}(\mathbb{R})$, we obtain the desired result. Thus Theorem 3.3 is proved.

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