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A movable optimal control problem with additional constraint for one heat process

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Abstract. *A movable optimal control problem with additional constraint for a system described by a parabolic type equation is considered. At first the solution of a mixed problem for every fixed possible controller is determined. Then the found general solution of the mixed problem is used in studying optimal control problem. Then optimal control problem is reduced to the L- moment problem in the subspace of the space $L_2(0,T)$ (with respect to newly determined scalar product), necessary and sufficient condition for the solvability of the obtained L- moment problem is proved.*

Keywords. movable optimal control problem · quadratic functional · positive-definite kernel · integral equation.

Mathematics Subject Classification (2010): 49K15, 49K21, 49M35

1 Introduction

A lot of real processes are described by distributed parameter equations: partial differential equations, integro-differential equations, functional equations, etc. The processes described by hyperbolic and parabolic type equations occupy special place among them. The oscillation processes, diffusion processes, technological processes and so on are related to these processes [2,3]. In most of these processes the control function moves along certain trajectory. In this case an optimal control problem is said to be a movable optimal control problem [4,5]. Such problems are encountered in grinding in metallurgy, in action on the material with moving electron and laser beams, in controlling flying vehicles, in controlling diffusion processes, etc. The solution of such problems are one of the urgent issues.

The movable optimal control problem has begun to be studied by A.G. Butkovsky and A.M.Pustilnikov [6,7] in the 65-70 years of the XX century. In [13,16], optimal control problems for movable sources described by parabolic systems are considered and the stated

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problem is solved with the help of the maximum principle. In [8], an optimal control problem reducing the system from one state to another state is studied, and a method for finding approximate solution of the problem is given. Similar problems are researched and the existence of an optimal controller is proved in [9], [10], [11]. A rule for finding optimal controller is also given in these papers. To this end, a functional gradient is used [12].

In the present paper, a problem on reducing the system from the given initial state to the finite state is considered, here the controllers satisfy additional constraints as well. Unlike the existing papers, in the present paper, the scaler product in the subspace of the space $L_2(0,T)$ is not determined traditionally, it is determined appropriately and the considered problem is reduced to relevant L - moment problem. It should be noted that an optimal control problem with additional constraint for a hyperbolic type equation was first studied in [12]. Necessary and sufficient conditions for the existence of the obtained L - moment problem are proved.

2 Problem statement

Assume the the controlled system is described by the equation

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial y(x,t)}{\partial x}\right] + q(x)y(x,t) + u(t)\delta(x-\vartheta(t)), \tag{2.1}$$

the inital condition

$$y(x,0) = \varphi(x), \tag{2.2}$$

and boundary conditions

$$y(0,t) = 0, \quad y(l,t) = 0. \tag{2.3}$$

Here $p(x) \in C^1[0,l]$, $q(x) \in C[0,l]$ are the given functions. Functions $u(t)$ and $\vartheta(t)$ are control functions, the function $\delta(x)$ is Dirac's "delta" function and
satisfics the relation

$$\int\limits_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0).$$

In equation (2.1) the expression $u(t)\delta(x-\vartheta(t))$ indicates that there exists external heat source [2], [9]. Here $u(t)$ is the intensity of external heat source, $\vartheta(t)$ is the movement trajectory of external heat source.

The class of possible controllers is taken in the form $U = \{u(t) \in L_2(0,T) : \|u(t)\| \leq 1; \vartheta(t) \in L_2(0,T) : \ 0 \leq \vartheta(t) \leq l\}$.

An optimal control problem for the considered system is stated as follows: Find a controller $u(t), \vartheta(t) \in U$ from the class of possible controllers reducing system (2.1)-(2.3) from the initial state (2.2) to the finite system

$$y(x,T) = Q^*(x), \tag{2.4}$$

as soon as possible and the constraint

$$E(u;\vartheta) = \int\limits_0^T \int\limits_0^l y^2(x,t)dxdt + \alpha \int\limits_0^T u^2(t)dt \leq L, \quad (\alpha > 0) \tag{2.5}$$

be satisfied.
$Q^*(x)$ is a given continuous function and α and L are positive real numbers.

3 Solution of a mixed problem for every fixed controller

At first we determine the solution of mixed problem (2.1)-(2.3) for every fixed controller. To this end at first we look for the solution of the homogeneous equation

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial y(x,t)}{\partial x}\right] + q(x)y(x,t) \tag{3.1}$$

relevant to equation (2.1), satisfying boundary conditions (2.3) in the form of

$$y(x,t) = X(x)T(t), \tag{3.2}$$

then for the functions $X(x)$ and $T(t)$ we find the equations

$$\frac{d}{dx}\left[p(x)\frac{dX(x)}{dx}\right] + q(x)X(x) = -\lambda X(x), \tag{3.3}$$

$$T'(t) + \lambda T(t) = 0. \tag{3.4}$$

For the solution determined by formula (3.2) to satisfy boundary conditions (2.3)

$$X(0) = 0,\ X(l) = 0. \tag{3.5}$$

The following theorems are valid [10];

Theorem 3.1 *If* $p(x) \in C^1[0,l],\ q(x) \in C[0,l],\ p(x) > 0, q(x) \le 0$, *then the eigen values* $\{\lambda_n\}$*of spectral problem* (3.3), (3.5) *form a positive, increasing sequence with limit* $+\infty$*:*

$$0 < \lambda_1 < \lambda_2 < ... < \lambda_n < ...;\ \lim_{n\to\infty}\lambda_n = +\infty.$$

Theorem 3.2 *Within conditions of theorem 1, the system of eigen-functions of spectral problem* (3.3),(3.5) *forms an orthogonal system in the segment* $\{X_n(x)\}$ $[0,l]$.

As the system of eigen-functions of spectral problem (3.3), (3.5) *is orthogonal in the segment* $[0,l]$ *we can reduce it to the orthonormal system in* $[0,l]$. *Therefore, without loss of generality, assume that the system of eigen-functions of spectral problem* (3.3), (3.5) *is orthonormal.*

If in equation (3.4), we write $\lambda = \lambda_n$ ($\lambda_n$is the sequence of eigen values of spectral problem ((3.3), (3.5)), the solution of the obtained equation will be in the form $T_n(t) = e^{-\lambda_n t}$.

Thus, we can define the solution of mixed problem (3.1), (2.2), (2.3) as follows:

$$y^*(x,t) = \sum_{n=1}^{\infty}\varphi_n e^{-\lambda_n t}X_n(x), \tag{3.6}$$

where $\varphi_n = \frac{2}{l}\int\limits_0^l \varphi(x)X_n(x)dx$.

We now look for the solution of equation (2.1) satisfying boundary and initial conditions in the form

$$\bar{y}(x,t) = \sum_{n=1}^{\infty} y_n(t)X_n(x). \tag{3.7}$$

We expand the function$\delta(x - \vartheta(t))$ in Fourier series with respect to the orthonormal system $\{X_n(x)\}$, and get:

$$\delta(x-\vartheta(t)) = \sum_{n=1}^{\infty}\alpha_n(t)X_n(x),$$

$$\alpha_n(t) = \int\limits_0^l \delta(x - \vartheta(t))X_n(x)dx = X_n[\vartheta(t)],$$

$$\delta(x - \vartheta(t)) = \sum_{n=1}^{\infty} X_n[\vartheta(t)]X_n(x).$$

Taking into account expansion of the function $\delta(x - \vartheta(t))$ in Fourier series and formula (3.7) in equation (2.1), for determining $y_n(t)$ we get the problem:

$$\begin{cases} y_n'(t) = -\lambda_n y_n(t) + u(t)X_n[\vartheta(t)], \\ y_n(0) = 0. \end{cases} \tag{3.8}$$

It is known that the solution of problem (3.8) satisfying homogeneous initial condition will be in the form:

$$y_n(t) = \int\limits_0^t u(\tau)X_n[\vartheta(\tau)]e^{\lambda_n \tau}d\tau l^{-\lambda_n t}.$$

Hence we get

$$\bar{y}(x,t) = \sum_{n=1}^{\infty} \int\limits_0^t u(\tau)X_n[\vartheta(\tau)]e^{-\lambda_n(t-\tau)}d\tau\, X_n(x). \tag{3.9}$$

Thus, the formal solution of mixed problem (2.1)-(2.3) for each fixed controller is determined by the formula

$$y(x,t) = \sum_{n=1}^{\infty} \left[\varphi_n + \int\limits_0^t u(\tau)X_n[\vartheta(\tau)]e^{\lambda_n \tau}d\tau \right] e^{-\lambda_n t}X_n(x). \tag{3.10}$$

It should be noted that if $p(x) \in C^1[0,l]$, $q(x) \in C[0,l]$, $p(x) > 0$, $q(x) \leq 0$, $\varphi(x) \in L_2[0,l]$, the solution determined by formula (3.10) is a generalized solution of mixed problem (2.1)-(2.3). The function $y(x,t) \in \overset{0}{W}{}_2^1(0 \leq x \leq l,\, 0 \leq t \leq T)$ satisfying the integral identity

$$\int\limits_0^T \int\limits_0^l y(x,t)\frac{\partial \omega(x,t)}{\partial x}dxdt \equiv \int\limits_0^T \int\limits_0^l p(x)\frac{\partial y(x,t)}{\partial x} \cdot \frac{\partial \omega(x,t)}{\partial x}dxdt$$

$$- \int\limits_0^T \int\limits_0^l q(x)y(x,t)\omega(x,t)dxdt - \int\limits_0^T u(\tau)\omega[\vartheta(\tau),\tau]d\tau$$

and condition $y(x,0) = \varphi(x)$ for the function satisfying any condition $\omega(x,t) \in \overset{0}{W}{}_2^1(0 \leq x \leq l,\; 0 \leq t \leq T),\; \omega(x,T) = 0$ is called a generalized solution.

4 Study of an optimal control problem

Now using the formula for solving problem (2.1)-(2.3) we study an optimal control problem.
Taking into account formula (3.10), condition (2.4) takes the form

$$\sum_{n=1}^{\infty}\left[\varphi_n+\int_0^T u(\tau)X[\vartheta(\tau)]e^{\lambda_n\tau}d\tau\right]e^{-\lambda_n t}X_n(x)=Q^*(x),\ 0\le x\le l. \tag{4.1}$$

As $Q^*(x)$ is a continious function on the segment $[0,l]$, the expansion
$Q^*(x)=\sum_{n=1}^{\infty}Q_n^*X_n(x),\quad Q_n^*(x)=\int_0^l Q^*X_n(x)dx;\ n=1,2,...$ is valid.
Then we can write relation (4.1) in the following form:

$$\sum_{n=1}^{\infty}\left[\varphi_n+\int_0^T u(\tau)X_n[\vartheta(\tau)]e^{\lambda_n\tau}d\tau\right]e^{-\lambda_n T}X_n(x)=\sum_{n=1}^{\infty}Q_n^*X_n(x),$$

Hence, with respect to the orthonormal system, by the uniqueness of Fourier coefficients we get

$$\varphi_n e^{-\lambda_n T}+\int_0^T u(\tau)X_n[\vartheta(\tau)]e^{-\lambda_n(T-\tau)}d\tau=Q_n^*,$$

or

$$\int_0^T u(\tau)X_n[\vartheta(\tau)]e^{-\lambda_n(T-\tau)}d\tau=Q_n^*-\varphi_n e^{-\lambda_n T};\ n=1,2,\dots. \tag{4.2}$$

Using formula (3.10), we transform constraint (2.5) equivalently. For simplicity we assume $\varphi(x)\equiv 0$. Then formula (3.10) takes the form

$$y(x,t)=\sum_{n=1}^{\infty}\int_0^t u(\tau)X_n[\vartheta(\tau)]e^{\lambda_n\tau}d\tau e^{-\lambda_n t}X_n(x)$$

We transform constraint (2.5) in the following way:

$$E(u,\vartheta)=\int_0^T\int_0^l y^2(x,t)dxdt+\alpha\int_0^T u^2(t)dt$$

$$=\int_0^T\int_0^l\left[\sum_{n=1}^{\infty}\int_0^t u(\tau)X_n[\vartheta(\tau)]e^{-\lambda_n(t-\tau)}d\tau X_n(x)\right]^2dxdt$$

$$+\alpha\int_0^T u^2(\tau)d\tau=\int_0^T\sum_{n=1}^{\infty}\left[\int_0^t u(\tau)X_n[\vartheta(\tau)]e^{-2\lambda_n(t-\tau)}d\tau\right]^2dt+\alpha\int_0^T u^2(\tau)d\tau$$

$$=\int_0^T\sum_{n=1}^{\infty}\left[\int_0^t\int_0^t u(\tau)X_n[\vartheta(\tau)]e^{-2\lambda_n(t-\tau)}u(s)X_n[\vartheta(s)]e^{-2\lambda_n(t-s)}d\tau ds\right]d\tau$$

$$+\alpha\int_0^T u^2(\tau)d\tau=\int_0^T\left[\int_0^t\int_0^t\sum_{n=1}^{\infty}u(\tau)u(s)X_n[\vartheta(\tau)]X_n[\vartheta(s)]\right.$$

$$\left.\times e^{-2\lambda_n(2t-\tau-s)}d\tau ds\right]dt+\alpha\int_0^T u^2(\tau)d\tau.$$

Make the following denotations

$$H(t,s)=\begin{cases}(T-s)\sum\limits_{n=1}^{\infty}X_n(\vartheta(s))X_n[\vartheta(\tau)]e^{-2\lambda_n(2T-\tau-s)}, t\le s\\(T-t)\sum\limits_{n=1}^{\infty}X_n(\vartheta(s))X_n[\vartheta(\tau)]e^{-2\lambda_n(2T-\tau-s)}, t\ge s.\end{cases}\tag{4.3}$$

Then we can write constraint (2.5) in the form

$$E(u,\vartheta)=\int_0^T\left[\int_0^T H(t,s)u(s)ds+\alpha u(t)\right]u(t)dt\le L.\tag{4.4}$$

Denoting $h_n(\tau)=X_n[\vartheta(\tau)]e^{-\lambda_n(T-\tau)},\ \ i=1,2,...$ we can write equalities (4.2) in the form

$$\int_0^T h_n(\tau)u(\tau)d\tau=Q_n^*,\ n=1,2,...\tag{4.5}$$

Assume that the movement trajectory of the external heat source $u(t)$ is known. Then knowing the movement trajectory of exterinal heat source $u(t)$ it is required to choose the intensity of external heat source $u(t)$ being the solution of moment problem (4.5) and satisfying additional constraint (4.10).

We now determine the scaler product of elements in the space $L_2(0,T)$ by the formula

$$\langle\gamma_1(t),\gamma_2(t)\rangle=\int_0^T\left[\int_0^T H(t,s)\gamma_1(s)ds+\alpha\gamma_1(t)\right]\gamma_2(t)dt,\tag{4.6}$$

It is easy to see that the scaler product defined by equality (4.6) satisfies all axioms of scalar products.

Obviously, all set of functions $\gamma(t)\in L_2(0,T)$ whore norms are defined by the equality

$$\|\gamma(t)\|=\sqrt{\langle\gamma(t),\gamma(t)\rangle}<+\infty$$

coincides with the space $L_2(0,T)$.

Determine the functions $\rho_k(t)(k=1,2,...)$ as the solution of the system of equations

$$\int_0^T H(t,s)\rho_k(s)ds+\alpha\rho_k(t)=h_k(t),\ k=1,2,...\tag{4.7}$$

It is easy to see that $H(t,s)$ is a symmetric, continuous, positive kernel and is not an eigen value of the "-1" $H(t,s)$ kernel on any segment $[0,T]$ $(T>0)$. Then, according to the known theorem [15] for each $h_k(t)\in L_2(0,T)$ equation (4.7) has a unique solution $\rho_k(t)\in$

$L_2(0,T)$ $(k = 1, 2, ..., n)$. In addition, as the system $\{h_k(t)\}$ is linear independent, the system $\{\rho_k(t)\}$ is also linear independent.

Taking into account (4.7), we can write (4.5) in the following form:

$$\int_0^T \left[\int_0^T H(t,s)\rho_k(s)ds + \alpha\rho_k(t) \right] u(t)dt = Q_k^*, \; k = 1, 2, ... \tag{4.8}$$

Thus, the stated problem is equivalent to the following problem: Find such a controller $u(t) \in L_2(0,T)$ and minimal time T that for any n the following conditions be satisfied

$$\int_0^T \left[\int_0^T H(t,s)\rho_k(s)ds + \alpha\rho_k(t) \right] u(t)dt = Q_k^*, \; k = 1, 2, ..., n, \tag{4.9}$$

$$\int_0^T \left[\int_0^T H(t,s)u(s)ds + \alpha u(t) \right] u(t)dt \leq L. \tag{4.10}$$

Theorem 4.1 *When in space $L_2(0,T)$ a scaler product is determined by formula* (4.6)*, necessary and sufficient condition for the existence of the controller $u(t)$ whose norm does not exceed the number L and sequence of moments is $\{Q_k^*\}$, is the satisfaction of the inequality*

$$\left[\sum_{k=1}^n \xi_k Q_k^* \right]^2 \leq L^2 \cdot \int_0^T \left\{ \sum_{k,j=1}^n \xi_k \xi_j \left[\int_0^T H(t,s)\rho_k(s)ds + \alpha\rho_k(t) \right] \rho_j(t)dt \right\}$$

for all the collection of numbers $\xi_1, \xi_2, ..., \xi_n$.

Proof. Necessity. Assume that $u(t) \in L_2(0,T)$ is the solution of the moment problem (4.9), (4.10). Then, multiplying the both hand side of equality (4.9) by ξ_k and sum from 1 to n with respect to k, taking into account inequality (4.10) and using the Cauchy-Bunyakovsky inequality, we get the required inequality:

$$\left[\sum_{k=1}^n \xi_k Q_k^* \right]^2 = \left\{ \int_0^T \left[\int_0^T \sum_{k=1}^n \xi_k t(t,j)\rho_k(\xi)d\xi + \sum_{k=1}^n \xi_k \rho_k(t)u(t)dt \right] \right\}^2$$

$$= \left\{ \int_0^T \left[\int_0^T H(t,\xi) \sum_{k=1}^n \xi_k \rho_k(\xi)d\xi + \sum_{k=1}^n \xi_k \rho_k(t) \right] u(t)dt \right\}^2$$

$$\leq \langle u(t), u(t) \rangle . \; \langle \sum_{k=1}^n \xi_k \rho_k(t), \sum_{k=1}^n \xi_k \rho_k(t) \rangle$$

$$\leq l^2 \left\{ \int_0^T \left[\int_0^T H(t,\xi) \sum_{k=1}^n \xi_k \rho_k(t) + \sum_{k=1}^n \xi_k \rho_k(t) \right] \sum_{k=1}^n \xi_k \rho_k(t)dt \right\}$$

$$= l^2 \int_0^T \sum_{k,j=1}^n \xi_k \xi_j h_k(t) \rho_j(t)dt.$$

Sufficiency. Assume that $\{\rho_k(t)\}_1^n$ is the solution of the system of integral equations, $\{\xi_k^0\}^n$ is any solution of the problem

$$\min_{\sum\limits_{k=1}^{n}\xi_k Q_k^*}\left\{\int\limits_0^T\left[\int\limits_0^T H(t,\xi)\sum_{k=1}^{n}\xi_k\rho_k(\xi)d\xi+\sum_{k=1}^{n}\xi_k\rho_k(t)\right]\sum_{k=1}^{n}\xi_k\rho_k(t)\right\}\le\frac{1}{L^2},$$

and denote

$$\beta_n=\left\{\int\limits_0^T\left[\int\limits_0^T H(t,\xi)\sum_{k=1}^{n}\xi_k^0\rho_k(\xi)d\xi+\sum_{k=1}^{n}\xi_k^0\rho_k(t)\right]\sum_{k=1}^{n}\xi_k^0\rho_k(t)dt\right\}^{-\frac{1}{2}}\le L,$$

$$u_n(t)=\beta_n^2\sum_{k=1}^{n}\xi_k^0\rho_k(t).$$

It is easy to see that the function $u_n(t)$ satisfics equality (4.9) and inequality (4.10). From the expression $u_n(t)$ it is clear that $\langle u_n(t),u_n(t)\rangle\le L^2$. Therefore, from the sequence $\{u_n(t)\}$ we can seperate a subsequence weakly converging to any function $u(t)\in L_2(0,T)$ on $L_2(0,T)$.

Obviously, this limit function will satisfy equalities (4.9) and inequality (4.10).

Thus, theorem 3 is proved.

Corollary 4.1 *The problem stated in the paper is first reduced to l moment problem in relevant subspace in $L_2(0,T)$.*

Corollary 4.2 *The existence and uniqueness of the obtained L - moment problem is proved.*

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