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$\mathbb{R}$-boundedness of the operator-valued functions in weighted L_p-spaces

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Abstract. *In the present paper, some properties of convolution-differential operator equations with unbounded operator coefficients in Banach space-valued weighted L_p -spaces are investigated. In particular, we study the R- positivity of the corresponding operators. Finally, $R-$ boundednsess and uniformly boundedness of the family of operator-valued functions in weighted L_p- spaces are established.*

Keywords. convolution differential-operator equation, R - positivity, uniformly R - boundedness, operator-valued functions, weighted L_p- spaces, weighted multiplier condition.

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1 Introduction

The some properties for differential operator equations, especially for parabolic and elliptic type have been studied extensively e.g. in [1-9], [15-17], [19], [22] and the references therein. Moreover, convolution-differential equations $(CDEs)$ have been treated e.g. in [11-14], [18]. Convolution equations in vector valued spaces studied e.g. in [6], [10], [14], [18], [20]. In [8], [13] and [21] the convolution differential-operator equations $(CDOEs)$ were investigated in L_p- spaces. The main aim of the present paper is to establish R-boundedness of the operator-valued functions arising in the solution of the following $CDOE$

$$\sum_{k=0}^{l} a_k * \frac{d^k u}{dx^k} + A * u = f(x), \tag{1.1}$$

in E-valued weighted L_p space, where $A = A(x)$ is a some possibly unbounded operator in a Banach space $E,\ a_k = a_k(x)$ are complex valued functions on $(-\infty, +\infty)$. Particularly, we prove that the sets of corresponding realization operators are uniformly $R-$ bounded.

The notion of R-boundedness plays an important role in the study of maximal regularity of the CDOEs. $R-$boundedness of the family of the operator-valued functions arising in the solution of convolution differential-operator equations is necessary for maximal regularity of type L_p for (1.1).

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2 Notations and definitions

Denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$. Suppose that E_1 and E_2 are Banach spaces. By $\mathcal{L}(E_1, E_2)$ we denote the space of bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$,we write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.

Let $\gamma = \gamma(x)$, $x = (x_1, x_2, ..., x_n)$ be a positive measurable real-valued function on a measurable subset $\Omega \subset \mathbb{R}^n$. By the sumbol $L_{p,\gamma}(\Omega; E)$ we mean the space of all strongly E -valued functions on a $\Omega \subset \mathbb{R}^n$ with norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \int\limits_{\Omega} \left(\|f(x)\|_E^p \gamma(x)dx\right)^{1/p}, \ 1 \leq p < \infty,$$

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)} = ess \sup_{x\in\Omega} \left[\gamma(x) \|f(x)\|_E\right].$$

For $\gamma(x) \equiv 1$, the space $L_{p,y}(\Omega, E)$ will be denoted by $L_p = L_p(\Omega; E)$.

The weight function $\gamma(x)$ is said to satisfy an A_p condition, i.e., $\gamma(x) \in A_p$, $1 < p < \infty$ if there is a positive constant C such that

$$\sup_Q \left(\frac{1}{|Q|}\int\limits_Q \gamma(x)dx\right)\left(\frac{1}{|Q|}\int\limits_Q \gamma^{-\frac{1}{p-1}}(x)dx\right)^{p-1} \leq C,$$

for all compact sets $Q \subset \mathbb{R}^n$.

Suppose that

$$S_\varphi = \{\lambda;\ \lambda \in \mathbb{C},\quad |\arg \lambda| \leq \varphi\} \cup \{0\},\ \ 0 \leq \varphi < \pi.$$

A closed linear operator function $A = A(x)$, $x \in \mathbb{R}$ is said to be uniformly φ-positive in banach space E, if $D(A(x))$ is dense in E and does not depend on x and there is a positive constant M so that

$$\left\|(A(x) + \lambda I)^{-1}\right\|_{\mathcal{L}(E)} \leq M(1 + |\lambda|)^{-1},$$

for every $x \in \mathbb{R}$ and $\lambda \in S_\varphi$, $\varphi \in [0, \pi)$, where I is an identity operator in E. For a scalar λ, we sometimes write $A + \lambda$ or A_λ instead of $A + \lambda I$. It is known [19, §1.15.1] that there exists fractional powers A^θ of the positive operator A.

Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graph norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \left\|A^\theta u\right\|^p\right)^{\frac{1}{p}},\ \ 1 \leq p < \infty,\quad -\infty < \theta < \infty.$$

By $S = S(\mathbb{R}^n; E)$ we denote the Schwartz space of rapidly decreasing smooth E -valued functions on $\mathbb{R}^n$. For $E = \mathbb{C}$, we denote this space by the symbol $S = S(\mathbb{R}^n; \mathbb{C})$. $S'(\mathbb{R}^n; E)$ we denote the space of linear continuous mappings from S to E which is called the Schwartz space of E-valued distributions. Recall $S(\mathbb{R}^n; E)$ is norm dense in $L_{p,\gamma}(\mathbb{R}^n; E)$ when $1 \leq p < \infty$, $\gamma \in A_p$.

Let Ω be a domain in $\mathbb{R}^n$. $C(\Omega, E)$ and $C^m(\Omega; E)$ will denote the spaces of E -valued bounded, uniformly strongly continuous and m -times continuously differentiable functions on Ω, respectively. For $E = \mathbb{C}$ the space $C^{(m)}(\Omega, E)$ will be denoted by $C^{(m)}(\Omega)$.

Let F denote the Fourier transform. A function $\Psi \in L_\infty(\mathbb{R}^n; \mathcal{L}(E_1, E_2))$ is called a multiplier from $L_{p,\gamma}(\mathbb{R}^n; E_1)$ to $L_{p,\gamma}(\mathbb{R}^n; E_2)$ for $p \in (1, \infty)$ if the map $u \to Bu = F^{-1}\Psi(\xi)Fu$, $u \in S(\mathbb{R}^n; E_1)$ are well defined and extends to a bounded linear operator

$$B : L_{p,\gamma}(\mathbb{R}^n; E_1) \to L_{p,\gamma}(\mathbb{R}^n; E_2)$$

The collection of all Fourier multipliers from $L_{p,\gamma}(\mathbb{R}^n; E_1)$ to $L_{p,\gamma}(\mathbb{R}^n; E_2)$ will be denoted by $M_{p,\gamma}^{p,\gamma}(E_1, E_2)$. For $E_1 = E_2 = E$ it is simply denoted by $M_{p,\gamma}^{p,\gamma}(E)$. Let $M(h)$ denote a set of some parameters.

Consider the family $B_h = \{\Psi_h;\ \Psi_h \in M_{p,\gamma}^{p,\gamma}(E_1, E_2),\ h \in M(h)\}$ of multipliers from the collection $M_{p,\gamma}^{p,\gamma}(E_1, E_2)$. The multipliers Ψ_h are said to be uniformly bounded (UBM) with respect to h if there exists a positive constant M independent of $h \in M(h)$ such that

$$\left\|F^{-1}\Psi_h Fu\right\|_{L_{p,\gamma}(R^n;E_2)} \le M \left\|u\right\|_{L_{p,\gamma}(R^n;E_1)},$$

for all $h \in M(h)$ and $u \in S(\mathbb{R}^n; E_1)$.

The Banach space E is called UMD -space ([7], [8]) if the Hilbert operator of a function $f \in S(\mathbb{R}; E)$, is defined by $Hf = \frac{1}{\pi} PV\left(\frac{1}{t}\right) * f$, i.e.,

$$(Hf)(t) = \frac{1}{\pi} \lim_{\varepsilon\to 0} \int\limits_{|\tau|>\varepsilon} \frac{f(t-\tau)}{\tau} d\tau$$

is bounded in $L_p(\mathbb{R}; E)$, for $p \in (1, \infty)$ (see e.g. [6]). UMD spaces include e.g. L_p, l_p spaces, Hilbert spaces, Sobolev spaces and Lorentz spaces $L_{pq},\ p, q \in (1, \infty)$.

A family of operators $T \subset \mathcal{L}(E_1, E_2)$ is called R-bounded (see [6], [8], [21]) if there is a constant $C > 0$ such that for all $T_1, T_2, ..., T_n \in T$ and $u_1, u_2, ..., u_n \in E_1$, $n \in \mathbb{N}$

$$\int\limits_0^1 \left\|\sum_{j=1}^n r_j(y) T_j u_j\right\|_{E_2} dy \le C \int\limits_0^1 \left\|\sum_{j=1}^n r_j(y) u_j\right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1; 1\}$-valued random variables on $[0, 1]$, e.g. the Rademacher functions $r_j(t) = sign(sin(2^j \pi t))$. The smallest C for which the above estimate holds is called an R-bound of the collection T and denoted by $R(T)$.

A set $T_h \subset \mathcal{L}(E_1, E_2)$ depending on the parameter $h \in M(h)$ is called uniformly R-bounded with respect to h if there is a positive constant C, independent of $h \in M(h)$, such that for all $T_1(h), T_2(h), ..., T_n(h) \in T_h$ and $u_1, u_2, ...u_n \in E_1$, $n \in \mathbb{N}$

$$\int\limits_0^1 \left\|\sum_{j=1}^n r_j(y) T_j(h) u_j\right\|_{E_2} dy \le C \int\limits_0^1 \left\|\sum_{j=1}^n r_j(y) u_j\right\|_{E_1} dy$$

that implies

$$\sup_{h\in M(h)} R(T_h) \le C.$$

A Banach space E is said to be a space satisfying a weighted multiplier condition with respect to $p \in (1, \infty)$ and weighted function γ if, for any $\Psi \in L_\infty(\mathbb{R}, \mathcal{L}(E))$ the R-boundedness of the set

$$\left\{|\xi|^k D^k \Psi(\xi) : \xi \in \mathbb{R}\backslash \{0\},\ \ k = 0, 1\right\}$$

implies that Ψ is a Fourier multiplier in $L_{p,\gamma}(\mathbb{R};E)$, i.e. $\Psi \in M_{p,\gamma}^{p,\gamma}(E)$ for any $p \in (1,\infty)$. If $E=\mathbb{C}$ and $\gamma \in A_p, p \in (1,\infty)$ then $\Psi \in M_{p,\gamma}^{p,\gamma}(\mathbb{C})$.

Note that, if E is UMD space and $\gamma(x) \equiv 1$ then by virue of [6], [10] and [21] the space E satisfies the multiplier condition. The UMD spaces satisfy the uniformly multiplier condition.

It is well known (see [6], [8]) that any Hilbert space satisfies the multiplier condition. There are, however, Banach spaces which are not Hilbert spaces but satisfy the multiplier condition.

In a similar way as [7. lemma 2.3]:

Lemma 2.1. *If $\lambda \in S_{\varphi_1}$, $\mu \in S_{\varphi_2}$, where $\varphi_1+\varphi_2<\pi$. Then there is a positive constant M independent of λ and μ such that $|\lambda+\mu| \geq M(|\lambda|+|\mu|)$.*

A positive operator A is said to be R-positive in the Banach space E if there exists $\varphi \in [0,\pi)$ such that the set

$$\left\{\xi\,(A+\xi I)^{-1};\ \xi \in S_\varphi\right\}$$

is R-bounded.

The operator $A(x)$ is said to be an uniformly R-positive in a Banach space E if there exists $\varphi \in [0,\pi)$ such that the set $\left\{\xi\,(A(x)+\xi I)^{-1};\, \xi \in S_\varphi\right\}$ is uniformly $R-$bounded.

Note that, in a Hilbert space every bounded set is R-bounded. Therefore, in a Hilbert space, the notion of $R-$boundedness is a equivalent to boundedness of a family of operators and in a Hilbert spaces all positive operators are R-positive (see e.g. [6], [8]).

Let $A=A(x)$, $x \in \mathbb{R}$ be closed linear operator in E with domain definition $D(A)$ independent of x. Then the Fourier transformation of $A(x)$ is defined as

$$\left\langle \widehat{A}u,\varphi\right\rangle = \langle Au,\widehat{\varphi}\rangle\,,\ u \in S'(\mathbb{R};D(A)),\varphi \in S(\mathbb{R}).$$

(For details see e.g. [2, section 3]).

$A(x)$ is differentiable if there is the limit

$$\left(\frac{d}{dx}A\right)u = A'(x)u = \lim_{h\to 0}\frac{A(x+h)u-A(x)u}{h}, u \in D(A)$$

in the sense of $E-$norm.

Let $A=A(x), x \in \mathbb{R}$, be closed linear operator in E with domain definition $D(A)$ independent of x and $u \in L_p(\mathbb{R};E(A))$. Then define:

$$(A*u)(x) = \int\limits_{\mathbb{R}} A(x-y)u(y)dy.$$

3 $\mathbb{R}$ - boundedness of the operator-valued functions

Consider the following CDOE

$$(L+\lambda)u = \sum_{k=0}^{l} a_k * \frac{d^k u}{dx^k} + A_\lambda * u = f(x), \tag{3.1}$$

in $E-$valued weighted L_p-spaces, where $x \in \mathbb{R}$, $A=A(x)$ is a linear operator in a Banach space E, $a_k=a_k(x)$ are complex valued functions, λ is a complex parameter.

By applying the Fourier transform to equation (3.1) we obtain

$$\left[\sum_{k=o}^{l}\widehat{a}_k(\xi)(i\xi)^k+\widehat{A}(\xi)+\lambda\right]\hat{u}(\xi)=\hat{f}(\xi),$$

where $\widehat{a}_k(\xi)$,$\widehat{A}(\xi)$, $\hat{u}(\xi)$ and $\hat{f}(\xi)$ denote the Fourier transforms of $a_k(x)$, $A(x)$, $u(x)$ and $f(x)$ respectively.

Condition 3.1. *Let* $a_k\in L_1(R)$, $\widehat{A}(\xi)$ *is a uniformly* $\varphi-$ *positive operator in* $E,\varphi\in[0,\pi)$. *Moreover, assume*

$$L(\xi)=\sum_{k=0}^{l}\widehat{a}_k(\xi)(i\xi)^k\in S_{\varphi_1},\ \varphi_1<\pi-\varphi,\ |L(\xi)|\geq C\max_k|\widehat{a}_k(\xi)|\,|\xi|^l,\ \xi\in\mathbb{R}\backslash\{0\}.$$

It is easy to see that, since $L(\xi)\in S_{\varphi_1}$ for all $\xi\in\mathbb{R}\backslash\{0\}$ and $\widehat{A}(\xi)$ is uniformly $\varphi-$positive, the operator $\left(\widehat{A}(\xi)+L(\xi)+\lambda\right)$ is invertible in E. So, we obtain that the solution of the equation (3.1) can be represented in the form

$$u(x)=F^{-1}\left[\widehat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}\widehat{f}.$$

For the sake of simplicity, we denote the $\left[\widehat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}$ by $H(\xi,\lambda)$. Then we have $u(x)=F^{-1}H(\xi,\lambda)\hat{f}$.

For the maximal regularity properties of the problem (3.1) it is sufficient to show the uniformly boundedness and $R-$boundedness of the following operator-valued functions, respectively

$$A(\xi,\lambda)=\widehat{A}_\lambda(\xi)H(\xi,\lambda),$$

$$B(\xi,\lambda)=\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^kH(\xi,\lambda),\ \text{ where } \hat{A}_\lambda(\xi)=\hat{A}(\xi)+\lambda I. \tag{3.2}$$

Since, $\quad \|A_\lambda * u\|_{L_{p,\gamma}(\mathbb{R};E)}\leq C\left\|F^{-1}\left[A(\xi,\lambda)\widehat{f}\right]\right\|_{L_{p,\gamma}(\mathbb{R};E)}$

and $\quad \displaystyle\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\left\|a_k*\frac{d^ku}{dx^k}\right\|_{L_{p,\gamma}(\mathbb{R};E)}\leq C\left\|F^{-1}\left[B(\xi,\lambda)\widehat{f}\right]\right\|_{L_{p,\gamma}(\mathbb{R};E)}.$

Lemma 3.1. *Suppose the Condition 3.1 holds. Then the operator-valued functions* $A(\xi,\lambda)$ *and* $B(\xi,\lambda)$ *are uniformly bounded.*

Proof. By virtue of Lemma 2.1 for $L(\xi)\in S_{\varphi_1}$, $\lambda\in S_\varphi,\varphi_1+\varphi<\pi$, the from resolvent properties of positive operators and the from well-known Hausdorff-Youngs inequality imply the uniformly boundedness of the operator-valued functions $A(\xi,\lambda)$ and $B(\xi,\lambda)$ with respect to λ. This implies (see, [13], [18])

$$\|A(\xi,\lambda)\|_{\mathcal{L}(E)}\leq C,\ \ \|B(\xi,\lambda)\|_{\mathcal{L}(E)}\leq C.$$

Let us note that for the sake of simplicity we shall not change constants in every step.

Condition 3.2. *Suppose the following conditions hold:*

1)*Condition 3.1 is satisfied* ;

2)$\widehat{A}(\xi)$ *is uniformly R-positive in E;*

3)$\widehat{a}_k \in C^{(1)}(R)$, $|\xi\widehat{a}_k'(\xi)| \leq C,\ \widehat{A}\,'(\xi)\widehat{A}^{\,-1}(\xi) \in C(\mathbb{R};\mathcal{L}(E))$;

4)$R\left(\left\{\xi\widehat{A}\,'(\xi)(\hat{A}(\xi)+\xi)^{-1}; \xi \in S_\varphi\right\}\right) \leq C.$

Lemma 3.2. *Suppose the (1), (2) parts of Condition 3.2 hold, and* $\lambda \in S_\varphi$, $\varphi \in [0,\pi)$, $\xi \in \mathbb{R}\backslash\{0\}$ *. Then, the family of operator-functions* $\{A(\xi,\lambda)\} = \{A(\xi,\lambda); \xi \in \mathbb{R}\backslash\{0\}\}$ *and* $\{B(\xi,\lambda)\} = \{B(\xi,\lambda); \xi \in \mathbb{R}\backslash\{0\}\}$ *are uniformly* $R-$ *bounded.*

Proof. Taking into account $R-$positivity of $\hat{A}(\xi)$ we have the set $\hat{A}(\xi)\,H(\xi,\lambda)$ is $R-$bounded. By vitue of $I-(\lambda+L(\xi))\,H(\xi,\lambda) = \widehat{A}(\xi)H(\xi,\lambda)$ we obtain the $R-$boundedness of $\{(\lambda+L(\xi))\,H(\xi,\lambda)\}$.

On the other hand by Condition 3.1 and by Lemma 2.1 so that

$$|\lambda+L(\xi)|^{-1} \leq C\,|\lambda|^{-1}.$$

By using the additional and product properties of $R-$bounded operators [6, Proposition 3,4] and Kahane's contraction principle for family of $R-$bounded operators [6, Lemma 3.5] we have the $R-$boundedness family of operator-functions $\{\lambda H(\xi,\lambda)\}$. So, we proved the $R-$boundedness of collection $\{A(\xi,\lambda)\}$. Therefore, by Lemma 3.1 we obtain the uniformly $R-$boundedness of $A\{(\xi,\lambda)\}$ .This means that

$$\sup_\lambda R\{A(\xi,\lambda)\} \leq C. \tag{3.3}$$

Due to $R-$boundedness family of operator-functions $\{(\lambda+L(\xi))\,H(\xi,\lambda)\}$ for all $\xi_i \in \mathbb{R}\backslash\{0\}$, $(\lambda+L(\xi_i))\,H(\xi_i,\lambda)$, $u_i \in E$, $i=\overline{1,n}$, we have

$$\int_0^1 \left\|\sum_{i=1}^n r_i(y)\,(\lambda+L(\xi_i))\,H(\xi_i,\lambda)u_i\right\|_E dy \leq C\int_0^1 \left\|\sum_{i=1}^n r_i(y)u_i\right\|_E dy, \tag{3.4}$$

where $\{r_i(y)\}$ of is a sequence of independent symmetric $\{-1;1\}$ $-$valued random variables on $[0,1]$.

Now we prove the uniformly $R-$boundedness of the family of operator-functions $\{B(\xi,\lambda)\}$. It is known that

$$B(\xi,\lambda) = \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^k H(\xi,\lambda) \tag{3.5}$$

$$= \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^k\,(\lambda+L(\xi))^{-1}\,(\lambda+L(\xi))\,H(\xi,\lambda).$$

In view of Kahane's contraction principle, we get from (3.4) and (3.5)

$$\int_0^1 \left\|\sum_{i=1}^n r_i(y)B(\xi_i,\lambda)u_i\right\|_E dy \leq$$

$$\leq C\int_0^1 \left\|\sum_{i=1}^n r_i(y)\,(\lambda+L(\xi_i))\,H(\xi_i,\lambda)u_i\right\|_E dy \leq C\int_0^1 \left\|\sum_{i=1}^n r_i(y)u_i\right\|_E dy. \tag{3.6}$$

It implies that the uniformly R-boundedness the family of operator-functions $\{B(\xi,\lambda)\}$, i.e.

$$\sup_{\lambda} R\{B(\xi,\lambda)\} \le C. \tag{3.7}$$

Remark 3.1. The estimates (3.3), (3.7) it is easy to see that operator-valued functions. $A(\xi,\lambda)$ and $B(\xi,\lambda)$ are R-bounded and it's R-bounds are independent of λ.

Theorem 3.1. *Let all parts of Condition 3.2 be hold, and $\lambda \in S_\varphi$, $\varphi \in [0,\pi)$, $\xi \in \mathbb{R}\backslash\{0\}$. Then, the family of operator-valued functions*

$$\left\{\xi A'(\xi,\lambda)\right\} = \left\{\xi A'(\xi,\lambda); \xi \in \mathbb{R}\backslash\{0\}\right\} \text{ and } \left\{\xi B'(\xi,\lambda)\right\} = \left\{\xi B'(\xi,\lambda); \xi \in \mathbb{R}\backslash\{0\}\right\}$$

are uniformly R-bounded, i.e.

$$\sup_{\lambda} R\left\{\xi A'(\xi,\lambda)\right\} \le C \text{ and } \sup_{\lambda} R\left\{\xi B'(\xi,\lambda)\right\} \le C$$

Proof. Through this section $\frac{d}{d\xi}A(\xi,\lambda)$ and $\frac{d}{d\xi}B(\xi,\lambda)$ will be denoted as $A'(\xi,\lambda)$ and $B'(\xi,\lambda)$ respectively.

It is clear that

$$\xi A'(\xi,\lambda) = \xi\left[\widehat{A}_\lambda(\xi)H(\xi,\lambda)\right]' = \xi\left[\widehat{A}'_\lambda(\xi)H(\xi,\lambda) + \widehat{A}_\lambda(\xi)H'(\xi,\lambda)\right] =$$

$$\xi\left[\widehat{A}'(\xi)H(\xi,\lambda) - \widehat{A}_\lambda(\xi)H(\xi,\lambda)\left(\widehat{A}'(\xi) + L'(\xi)\right)H(\xi,\lambda)\right] = \tag{3.8}$$

$$\xi\widehat{A}'(\xi)H(\xi,\lambda) - \widehat{A}_\lambda(\xi)H(\xi,\lambda)\left[\xi\widehat{A}'(\xi)H(\xi,\lambda) + \xi L'(\xi)H(\xi,\lambda)\right]$$

By $R-$positivity of $\widehat{A}(\xi)$, in view the (4) part of Condition 3.2 and by Lemma 3.2 we obtain the sets $\left\{\widehat{A}_\lambda(\xi)H(\xi,\lambda)\right\}$ and $\left\{\xi\widehat{A}'(\xi)H(\xi,\lambda)\right\}$ are $R-$bounded.

Then by virtue (3.8) we get for a $R-$boundedness of the set $\{\xi A'(\xi,\lambda)\}$, it is sufficient to prove the $R-$boundedness of the set $\{\xi L'(\xi)H(\xi,\lambda)\}$. It is easy to see that

$$\xi L'(\xi)H(\xi,\lambda) = \sum_{k=0}^{l}\left(\xi\widehat{a}'_k(\xi)(i\xi)^k + k\widehat{a}_k(\xi)(i\xi)^k\right)H(\xi,\lambda)$$

$$= \sum_{k=0}^{l}\left(\xi\widehat{a}'_k(\xi)(i\xi)^k + k\widehat{a}_k(\xi)(i\xi)^k\right)|\,\lambda + L(\xi)|^{-1}\,|\,\lambda + L(\xi)|\,H(\xi,\lambda). \tag{3.9}$$

Due to boundedness of $\sum\limits_{k=0}^{l}\widehat{a}_k(\xi)$, in view of Condition 3.2, by virtue of Lemma 2.1 and well known inequality

$$y^k \le C\left(1 + y^l\right),\;\; y = |\lambda|^{-\frac{1}{l}}\,|\xi|\,,\;\; k \le l$$

we obtain that

$$\sum_{k=0}^{l}\left(\xi\widehat{a}'_k(\xi)(i\xi)^k + k\widehat{a}_k(\xi)(i\xi)^k\right)|\,\lambda + L(\xi)|^{-1}$$

$$\leq C\,|\,\lambda + L(\xi)|\,|\,\lambda + L(\xi)|^{-1} \leq C. \tag{3.10}$$

So, due to (3.9) and (3.10) we have the $R-$boundedness of the set $\{\xi L'(\xi)H(\xi,\lambda)\}$. Moreover, by using of additional and product properties of $R-$bounded operators [6, Proposition 3.4] and Kahane's contraction principle [6, Lemma 3.5] we have $R-$boundedness of $\{\xi A'(\xi,\lambda)\}$. Therefore, by Lemma 3.1 we obtain the uniformly $R-$boundedness of $\{\xi A'(\xi,\lambda)\}$, i.e.

$$\sup_{\lambda} R\left\{\xi A'(\xi,\lambda)\ \right\} \leq C.$$

Now, we prove the uniformly $R-$boundedness of the family operator-valued functions $\{\xi B'(\xi,\lambda)\}$. It is known that

$$\xi B^{'}(\xi,\lambda) = \xi\left[\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^k H(\xi,\lambda)\right]^{'}$$

$$= \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}}\left[\left(\xi\,\widehat{a}_k'(\xi)(i\xi)^k + k(i\xi)^k\widehat{a}_k(\xi)\right)H(\xi,\lambda)\right.$$

$$\left.-\xi\widehat{a}_k(\xi)(i\xi)^k H(\xi,\lambda)\left(\widehat{A}\,'(\xi) + \sum_{k=0}^{l}\left(\,\widehat{a}_k'(\xi)(i\xi)^k + ik(i\xi)^{k-1}\widehat{a}_k(\xi)\right)\right)H(\xi,\lambda)\right]$$

$$= \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}}\left(\xi\widehat{a}_k'(\xi)(i\xi)^k + k(i\xi)^k\widehat{a}_k(\xi)\right)H(\xi,\lambda)$$

$$-\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^k H(\xi,\lambda)\left[\xi\widehat{A}\,'(\xi)H(\xi,\lambda)\right.$$

$$\left.+\sum_{k=0}^{l}\left(\xi\widehat{a}_k'(\xi)(i\xi)^k + k(i\xi)^k\widehat{a}_k(\xi)\right)H(\xi,\lambda)\right]. \tag{3.11}$$

The $R-$boundedness of the all terms in the last expression (3.11) and by using arguments the proof of Lemma 3.2 in a similar way we obtain the $R-$boundedness of the family of operator-functions $\{\xi B'(\xi,\lambda)\}$. This means that

$$\sup_{\lambda} R\left\{\xi B'(\xi,\lambda)\ \right\} \leq C.$$

Hence, family of operator-valued functions $\{\xi A'(\xi,\lambda)\}$ and $\{\xi B'(\xi,\lambda)\}$ are uniformly

$R-$bounded.

By using these results the existence and uniqueness of maximal regular solution of the CDOEs is obtained in weighted L_p-spaces. Really, in application, the maximal regularity properties of the Cauchy problem for degenerate and nondegenerate parabolic equation in mixed L_p norms are obtained.

Example: Consider the Cauchy problem for convolution parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sum\limits_{k=0}^{l} a_k * \dfrac{\partial^k u}{\partial x^k} + A * u = f(t,x) \\ u(0,x) = 0, \;\; t \in \mathbb{R}^+, \;\; x \in \mathbb{R}. \end{cases}$$

For this example will be denote the space of all $\mathbf{p} = (p, p_1)-$summable $E-$valued functions with mixed norm through $L_{\mathbf{p},\gamma}\left(\mathbb{R}_+^2; E\right)$. So, $L_{\mathbf{p},\gamma}\left(\mathbb{R}_+^2; E\right)$ denote the space of all measurable $E-$valued functions defined on $\mathbb{R}_+^2$ with the norm

$$\|f\|_{L_{\mathbf{p},\gamma}\left(\mathbb{R}_+^2;E\right)} = \left(\int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}_+} \|f(t,x)\|_E^p \, \gamma(x) dx \right)^{\frac{p_1}{p}} dt \right)^{\frac{1}{p_1}} < \infty.$$

For the prove of uniformly $R-$positivity of the operator $Lu = \sum\limits_{k=0}^{l} a_k * \dfrac{\partial^k u}{\partial x^k} + A * u,$ by virtue of (3.1), we need to show the uniformly $R-$boundedness of the set $\left\{ \lambda \left(L + \lambda\right)^{-1}; \lambda \in S_\varphi \right\}$ (see [6], [13]). It is known that

$$\lambda \left(L + \lambda\right)^{-1} = F^{-1}\left[\lambda H(\xi, \lambda)\right] \widehat{f},$$

i.e., we need to show the uniformly $R-$boundedness of the set $\{F^{-1}\left[\lambda H(\xi,\lambda)\right] \widehat{f}; \lambda \in S_\varphi\}$. In a similar way the uniformly $R-$boundedness the Cauchy problem for degenerate parabolic convolution differential-operator equations is derived.

Result. Suppose the Condition 3.2 is satisfied and E is a Banach space satisfies a weighted multiplier condition. Then operator-valued functions $A(\xi, \lambda)$ and $B(\xi, \lambda)$ are UBM in $L_{p,\gamma}(\mathbb{R}; E)$. Moreover, if E is an UMD space, then operator-valued functions $A(\xi, \lambda)$ and $B(\xi, \lambda)$ are UBM in $L_{p,\gamma}(\mathbb{R}; E)$. On the other hand the operator L is a generator of analytic semigroup in $L_{p,\gamma}(\mathbb{R}; E)$.

Remark 3.2. Applying this results we establish the maximal regularity properties of the some problems for CDOEs, e.g. the boundary value problem for anisotropic elliptic convolution equation and infinite systems of degenerate elliptic integro-differential equations etc. As we mentioned before, on Hilbert space (and only on Hilbert space) boundedness and $R-$boundedness are the same. On the other hand, it turns out that many results known for Hilbert spaces can be carred over to L_p-spaces, if boundedness is replaced by $R-$boundedness (see [6], [8], [13], [21]).

References

1. Agarwal, R.P., Bohner, R., Shakhmurov, V.B.: *Maximal regular boundary value problems in Banach-valued weighted spaces*, Bound. Value Probl. **1**, 9–42 (2005).
2. Agarwal, R.P., O'regan, D., Shakhmurov, V.B.: *Separable anisotropic differential operators in weighted abstract spaces and applications*, J. Math. Anal. Appl. **338**, 970–983 (2008).
3. Amann, H.: *Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications*, Math. Nachr. **186**, 5–56 (1997).
4. Besov, O.V., Ilin, V.P., NikoIskii, S.M.: Integral representations of functions and embedding theorems, *Moscow* (1975).

5. Bourgain, J.: *Some remarks on Banach spaces in which martingale differences are unconditional*, Arkiv Math. **21**, 163–168 (1983).
6. Denk, R., Hieber, M., Prüss, J.: *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc. **166** (788) (2003).
7. Dore, G., Yakubov, S.: *Semigroup estimates and non coercive boundary value problems*, Semigroup Forum, **60**, 93–121 (2000).
8. Girardi, M., Weis, L.: *Operator-valued multiplier theorems on $L_p(X)$ and geometry of Banach spaces*, J. Funct. Anal. **204** (2), 320–354 (2003).
9. Guliev, V.S.: *To the theory of multipliers of Fourier integrals for functions with values in Banach spaces*, Tr. Mat. Inst. Steklova, **214** (17), 164–181 (1996).
10. Haller, R., Heck H., Noll A.: *Mikhlin's theorem for operator-valued Fourier multipliers in n variables*, Math. Nachr. **244**, 110–130 (2002).
11. Musaev, H.K., Shakhmurov, V.B.: *Regularity properties of degenerate convolution-elliptic equations*, Bound. Value Probl. **50** (2016), 1–19 (2016).
12. Prüss, J.: Evolutionary integral equations and applications, *Birkhauser, Basel* (1993).
13. Shakhmurov, V.B., Musaev, H.K.: *Separability properties of convolution-differential operator equations in weighted L_p spaces*, Appl. Comput. Math. **14** (2), 221–233 (2015).
14. Shakhmurov, V. B., Musaev, H.K.: *Maximal regular convolution differential equations in weighted Besov spaces*, Appl. Comput. Math. **16** (2), 190–200 (2017).
15. Shakhmurov, V.B., Ekincioglu, I.: *Linear and nonlinear convolution-elliptic equations*, Bound. Value Probl. **2013** (211), 1–21 (2013).
16. Shakhmurov, V.B.: *Maximal B-regular boundry value problems with parameters*, J. Math. Anal. Appl. **320**, 1–19 (2006).
17. Shakhmurov, V.B.: *Embedding and maximal regular differential operators in Banach-valued weighted spaces*, Acta Math. Sin. (Engl. Ser.) **5** (22), 1493–1508 (2006).
18. Shakhmurov, V.B., Shahmurov, R.: *Sectorial operators with convolution term*, Math. Inequal. Appl. 13(2), 387–404 (2010).
19. Triebel, H.: Interpolation theory. Function spaces. Differential operators, North-Holland, *Amsterdam* (1978).
20. Triebel, H.: *Spaces of distributions with weights, Multiplier in L_p spaces with weights*, Math. Nachr. **78**, 339–356 (1977).
21. Weis, L.: *Operator-valued Fourier multiplier theorems and maximal L_p regularity*, Math. Ann. **319**, 735–758 (2001).
22. Yakubov, S., Yakubov, Ya .: Differential-operator equations. Ordinary and Partial Differential equations, *Chapmen and Hall /CRC, Boca Raton*, (2000).