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Determining a spectrum of a boundary value problem

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Abstract. *The paper is devoted to one Steklov problem for a first order elliptic type equation with non-local and global addends in the boundary condition. The domain under consideration is bounded with a parabola and straightline. The spectral parameter is only in the boundary condition.*

Keywords. elliptic type equation of first order, nonlocal and global addends of boundary condition, necessary conditions, regularization and Fredholm property

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1 Introduction

We consider a homogeneous boundary value problem for the CauchyRiemann equation with spectral parameters only in the boundary condition. Some boundary value problem for CauchyRiemann equations were considered in [1]-[3], [5],[7]. Study of eigenvalues and eigenfunctions id directly connected with quantum mechanics problems [8],[9],[10]. Note that the research scheme of above problems is in the following.

Proceeding from the fundamental solution of the equation under consideration we determine the main relation consisting of two parts.

The first part determines the arbitrary solution of the equation under consideration, the second part consists of necessary conditions. These necessary conditions contain singular addends regularized in a peculiar way. After regularization, the obtained expressions together with the given boundary conditions give the sufficient condition of Fredholm property of the stated problems [4],[11],[12].

Then the approximate values of the spectrum are determined from these Fredholm integral equations of second kind.

For that, the kernel of the integral equation is replaced by a degenerated kernel and obtained integral equations is replaced by the system of algebraic equation

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2 Problem statement

Let us consider the following boundary value problem:

$$\frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1} = 0, \quad x = (x_1, x_2) \in D \subset R^2 \tag{2.1}$$

$$u(x_1, 1) + \alpha(x_1)u(x_1, x_1^2) = \lambda\int_{-1}^{1} K(x_1, t)u(t, 1)dt, \;\; x_1 \in [-1, 1], \tag{2.2}$$

where $i = \sqrt{-1}$, $D = \{x : x_1 \in (-1, 1), x_2 \in (x_1^2, 1)\}$, $\alpha(x_1)$ are the given continuous (complex valued) functions , λ is a spectral parameter, $K(x_1, t)$ is the given continuous kernel, It is seen from boundary condition (2) that kernel $K(x_1, t)$ was determined in the square $[-1, 1]\times[-1, 1]$. The goal of the paper is to reduce the given boundary value problem to such a second kind Fredholm equation (and the system of equations) that the kernel of these integrals do not contain singularity (there may a weak singularity) $u(x_1, x_2)$ is the sought for function.

2.1. Main relations As known, the fundamental solution of the equation is of the form (see p.205 [6]):

$$U(x - \xi) = \frac{1}{2\pi}\frac{1}{x_2 - \xi_2 + i(x_1 - \xi_1)} \tag{2.3}$$

Multiplying equation (2.1) by the fundamental solution (2.3), integrating with respect to the domain D, and applying the Ostrogradskii-Gauss formula, we get:

$$0 = \int_D \frac{\partial u(x)}{\partial x_2} U(x - \xi)dx + i\int_D \frac{\partial u(x)}{\partial x_1} U(x - \xi)dx$$

$$= \int_\Gamma u(x)U(x - \xi)[\cos(\nu, x_2) + i\cos(\nu, x_1)]dx$$

$$-\int_D u(x)\left[\frac{\partial U(x - \xi)}{\partial x_2} + i\frac{\partial U(x - \xi)}{\partial x_1}\right] dx, \tag{2.4}$$

where ν is an external normal to the boundary Γ of the domain D, Taking into account that (2.3) is the fundamental solution of the equation (2.1), from (2.4) we have:

$$\frac{1}{2\pi}\int_\Gamma \frac{u(x)}{x_2 - \xi_2 + i(x_1 - \xi_1)}[\cos(\nu, x_2) + i\cos(\nu, x_1)]dx = \begin{cases} u(\xi), \;\; \xi \in D \\ \frac{1}{2}u(\xi), \; \xi \in \Gamma \end{cases} \tag{2.5}$$

The obtained expression (2.5) is the main relation.

3 Necessary conditions

Separating the necessary conditions from the main relation (2.5), we have :

$$\frac{1}{2}u(\xi_1,\xi_1^2) = -\frac{1}{2\pi}\int_{-1}^{1}\frac{u(x_1,x_1^2)}{x_1^2-\xi_1^2+i(x_1-\xi_1)}(1-2x_1 i)dx_1+\frac{1}{2\pi}\int_{-1}^{1}\frac{u(x_1,1)}{1-\xi_1^2+i(x_1-\xi_1)}dx_1,$$

$$\frac{1}{2}u(\xi_1,1) = -\frac{1}{2\pi}\int_{-1}^{1}\frac{u(x_1,x_1^2)}{x_1^2-1+i(x_1-\xi_1)}(1-2x_1 i)dx_1+\frac{1}{2\pi i}\int_{-1}^{1}\frac{u(x_1,1)}{x_1-\xi_1}dx_1, \tag{3.1}$$

This establishes the following

Theorem 3.1 *Every analytic function determined in the domain D satisfies necessary conditions (3.1)*

As can be seen from necessary conditions (3.1), these conditions are singular. Separate the singularity that enters into these conditions. we can reduce the obtained necessary conditions (3.1) to the form:

$$u(\xi_1,\xi_1^2) = \frac{i}{\pi}\int_{-1}^{1}\frac{u(x_1,x_1^2)}{x_1-\xi_1}dx_1+\frac{i}{\pi}\int_{-1}^{1}\frac{u(x_1,x_1^2)}{x_1+\xi_1+i}dx_1+\frac{1}{\pi}\int_{-1}^{1}\frac{u(x_1,1)}{1-\xi_1^2+i(x_1-\xi_1)}dx_1,$$

$$u(\xi_1,1) = -\frac{i}{\pi}\int_{-1}^{1}\frac{u(x_1,1)}{x_1-\xi_1}dx_1-\frac{1}{\pi}\int_{-1}^{1}u(x_1,x_1^2)\frac{1-2x_1 i}{x_1^2-1+i(x_1-\xi_1)}dx_1. \tag{3.2}$$

3.1. Regularization of necessary conditions. Taking into account boundary condition (2.2) , by means of necessary conditions (3.2), we create the following linear combination:

$$\alpha(\xi_1)u(\xi_1,\xi_1^2)-u(\xi_1,1) = \frac{i}{\pi}\int_{-1}^{1}\frac{\alpha(x_1)u(x_1,x_1^2)+u(x_1,1)}{x_1-\xi_1}dx_1$$

$$+\frac{i}{\pi}\int_{-1}^{1}\frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}u(x_1,x_1^2)dx_1+\frac{\alpha(\xi_1)i}{\pi}\int_{-1}^{1}\frac{u(x_1,x_1^2)}{x_1+\xi_1+i}dx_1$$

$$+\frac{\alpha(\xi_1)i}{\pi}\int_{-1}^{1}\frac{u(x_1,1)}{1+\xi_1^2+i(x_1-\xi_1)}dx_1+\frac{1}{\pi}\int_{-1}^{1}u(x_1,x_1^2)\frac{1-2x_i i}{x_1^2-1+i(x_1-\xi_1)}dx_1$$

$$=\frac{i}{\pi}\int_{-1}^{1}\frac{dx_1}{x_1-\xi_1}\lambda\int_{-1}^{1}K(x_1,t)u(t,1)dt+\frac{i}{\pi}\int_{-1}^{1}\frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}u(x_1,x_1^2)dx_1$$

$$+\frac{\alpha(\xi_1)i}{\pi}\int_{-1}^{1}\frac{u(x_1,x_1^2)}{x_1+\xi_1+i}dx_1+\frac{\alpha(\xi_1)i}{\pi}\int_{-1}^{1}\frac{u(x_1,1)}{1-\xi_1^2+i(x_1-\xi_1)}dx_1$$

$$+\frac{1}{\pi}\int_{-1}^{1}u(x_1,x_1^2)\frac{1-2x_i i}{x_1^2-1+i(x_1-\xi_1)}dx_1 = \frac{\lambda i}{\pi}\int_{-1}^{1}u(t,1)dt\int_{-1}^{1}K(x_1,t)\frac{dx_1}{x_1-\xi_1}$$

$$+\frac{i}{\pi}\int_{-1}^{1}\frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}u(x_1,x_1^2)dx_1+\frac{\alpha(\xi_1)i}{\pi}\int_{-1}^{1}\frac{u(x_1,x_1^2)}{x_1+\xi_1+i}dx_1$$

$$+\frac{\alpha(\xi_1)i}{\pi}\int_{-1}^{1}\frac{u(x_1,1)}{1-\xi_1^2+i(x_1-\xi_1)}dx_1+\frac{1}{\pi}\int_{-1}^{1}u(x_1,x_1^2)\frac{1-2x_1i}{x_1^2-1+i(x_1-\xi_1)}dx_1. \quad (3.3)$$

This proves the following

Theorem 3.2 *If $\alpha(x_1)$belongs to some Hölder class, the kernel $K(x_1,t)$ is a continuous function, then the obtained expression (3.3) is regular.*

4 Fredholm property

Let us consider the boundary condition (2.2) writhen at the point ξ_1, and subtracting the regularized expression (3.3) from it, we get:

$$u(\xi_1,1)=\frac{\lambda}{2}\int_{-1}^{1}K(x_1,t)u(t,1)dt-\frac{\lambda i}{2\pi}\int_{-1}^{1}u(t,1)dt\int_{-1}^{1}K(x_1,t)\frac{dx_1}{x_1-\xi_1}$$

$$-\frac{i}{2\pi}\int_{-1}^{1}\frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}dx_1\times\left[\frac{\lambda\int_{-1}^{1}K(x_1,t)u(t,1)dt-u(x_1,1)}{\alpha(x_1)}\right]$$

$$-\frac{\alpha(\xi_1)i}{2\pi}\int_{-1}^{1}\frac{dx_1}{x_1+\xi_1+i}\cdot\left[\frac{\lambda\int_{-1}^{1}K(x_1,t)u(t,1)dt-u(x_1,1)}{\alpha(x_1)}\right]$$

$$-\frac{\alpha(\xi_1)i}{2\pi}\int_{-1}^{1}\frac{u(x_1,1)}{1-\xi_1^2+i(x_1-\xi_1)}dx_1-\frac{1}{2\pi}\int_{-1}^{1}\frac{1-2x_1i}{x_1^2-1+i(x_1-\xi_1)}dx_1$$

$$\times\left[\frac{\lambda\int_{-1}^{1}K(x_1,t)u(t,1)dt-u(x_1,1)}{\alpha(x_1)}\right]. \quad (4.1)$$

This establishes the following

Theorem 4.1 *Under the conditions of theorem 3.2., if $\alpha(x_1)\neq 0$, then the stated boundary value problem (2.1)-(2.2) is reduced to regular, homogeneous Fredhol integral equation of second kind (4.1) with respect to the functions $u(x_1,1)$.*

We represent the equation (4.1) in the form:

$$y(\xi_1)=\lambda\int_{-1}^{1}K_1(\xi_1,t)y(t)dt+\int_{-1}^{1}K_0(\xi_1,t)y(t)dt,\quad \xi_1\in[-1,1], \tag{4.2}$$

where

$$y(\xi_1)=u(\xi_1,1) \tag{4.3}$$

$$K_1(\xi_1,t)=\frac{1}{2}K(\xi_1,t)-\frac{i}{2\pi}\int_{-1}^{1}\frac{K(x_1,t)}{x_1-\xi_1}dx_1-\frac{i}{2\pi}\int_{-1}^{1}\frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}\frac{K(x_1,t)}{\alpha(x_1)}dx_1$$

$$-\frac{\alpha(\xi_1)i}{2\pi}\int_{-1}^{1}\frac{K(x_1,t)}{x_1+\xi_1+i}\cdot\frac{dx_1}{\alpha(x_1)}-\frac{1}{2\pi}\int_{-1}^{1}\frac{K(x_1,t)}{x_1^2-1+i(x_1-\xi_1)}\frac{dx_1}{\alpha(x_1)}, \tag{4.4}$$

$$K_0(\xi_1,t)=\frac{i}{2\pi}\int_{-1}^{1}\frac{\alpha(\xi_1)-\alpha(t)}{t-\xi_1}\frac{1}{\alpha(t)}+\frac{\alpha(\xi_1)i}{2\pi}\frac{1}{t+\xi_i+i}\cdot\frac{1}{\alpha(t)}$$

$$-\frac{\alpha(\xi_1)i}{2\pi}\frac{1}{1-\xi_1^2+i(t-\xi_1)}+\frac{1}{2\pi}\frac{1-2ti}{t^2-1+i(t-\xi_1)}\frac{1}{\alpha(t)}. \tag{4.5}$$

5 Special case

Let $K(x,t)$ be a degenerate kernel, i.e.

$$K(x_1,t)=\sum_{s=1}^{n}a_s(x_1)b_s(t) \tag{5.1}$$

where $n\in N$ is a fixed integer, $a_s(x_1)$ and $b_s(t)$ are given continuous functions. Then from (4.4) we get

$$K_1(\xi_1,t)=\frac{1}{2}\sum_{s=1}^{n}a_s(\xi_1)b_s(t)-\frac{i}{2\pi}\sum_{s=1}^{n}b_s(t)\int_{-1}^{1}\frac{a_s(x_1)}{x_1-\xi_1}dx_1$$

$$-\frac{i}{2\pi}\sum_{s=1}^{n}b_s(t)\int_{-1}^{1}\frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}\frac{a_s(x_1)}{\alpha(x_1)}dx_1-\frac{\alpha(\xi_1)i}{2\pi}\sum_{s=1}^{n}b_s(t)\int_{-1}^{1}\frac{a_s(x_1)}{x_1-\xi_1+i}\frac{dx_1}{\alpha(x_1)}$$

$$-\frac{1}{2\pi}\sum_{s=1}^{n}b_s(t)\int_{-1}^{1}\frac{a_s(x_1)}{x_1^2-1+i(x_1-\xi_1)}\frac{dx_1}{\alpha(x_1)}$$

$$=\sum_{s=1}^{n}\left[\frac{a_s(\xi_1)}{2}-\frac{i}{2\pi}\int_{-1}^{1}\frac{a_s(x_1)}{x_1-\xi_1}dx_1-\frac{i}{2\pi}\int_{-1}^{1}\frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}\frac{a_s(x_1)}{\alpha(x_1)}dx_1\right.$$

$$-\frac{\alpha(\xi_1)i}{2\pi}\int_{-1}^{1}\frac{a_s(x_1)}{x_1+\xi_1+i}\frac{dx_1}{\alpha(x_1)}-\frac{1}{2\pi}\int_{-1}^{1}\frac{a_s(x_1)}{x_1^2-1+i(x_1-\xi_1)}\frac{dx_1}{\alpha(x_1)}\Bigg]b_s(t) \quad (5.2)$$

is also a degenerate kernel. Thus, we represent equation (4.2) in the form:

$$y(\xi_1)-\lambda\sum_{s=1}^{n}A_s(\xi_1)\int_{-1}^{1}b_s(t)y(t)dt=f(\xi_1), \quad (5.3)$$

where

$$A_s(\xi_1)=\frac{a_s(\xi_1)}{2}-\frac{i}{2\pi}\int_{-1}^{1}\frac{a_s(x_1)}{x_1-\xi_1}dx_1-\frac{i}{2\pi}\int_{-1}^{1}\frac{\alpha(\xi_1)-\alpha(x_1)}{x_1-\xi_1}\frac{a_s(x_1)}{\alpha(x_1)}dx_1$$

$$-\frac{\alpha(\xi_1)i}{2\pi}\int_{-1}^{1}\frac{a_s(x_1)}{x_1+\xi_1+i}\frac{dx_1}{\alpha(x_1)}-\frac{1}{2\pi}\int_{-1}^{1}\frac{a_s(x_1)}{x_1^2-1+i(x_1-\xi_1)}\frac{dx_1}{\alpha(x_1)}, \quad (5.4)$$

$$f(\xi_1)=\int_{-1}^{1}K_0(\xi_1,t)y(t)dt. \quad (5.5)$$

Multiplying equation (5.3) by $b_m(\xi_1)$, integrating from -1 to 1, we get

$$\int_{-1}^{1}b_m(\xi_1)y(\xi_1)d\xi_1-\lambda\sum_{s=1}^{n}\int_{-1}^{1}b_m(\xi_1)A_s(\xi_1)d\xi_1\int_{-1}^{1}b_s(t)y(t)dt=\int_{-1}^{1}b_m(\xi_1)f(\xi_1)d\xi_1. \quad (5.6)$$

Accepting the denotation

$$\int_{-1}^{1}b_s(\xi_1)y(\xi_1)d\xi_1=C_s, \quad s=\overline{1,n}, \quad (5.7)$$

from (5.6) we get

$$C_m-\lambda\sum_{s=1}^{n}C_sB_{ms}=f_m,\ \ m=\overline{1,n}, \quad (5.8)$$

where

$$B_{ms}=\int_{-1}^{1}b_m(\xi_1)A_s(\xi_1)d\xi_1,\ m=\overline{1,n}, \quad (5.9)$$

$$f_m=\int_{-1}^{1}b_m(\xi_1)f(\xi_1)d\xi_1,\ m=\overline{1,n}. \quad (5.10)$$

From the algebraic system of linear equations (5.3), we have :[8]

$$\Delta(\lambda)=\begin{vmatrix}1-\lambda B_{11} & \lambda B_{12} & ... & -\lambda B_{1n}\\ -\lambda B_{21} & 1-\lambda B_{22} & ... & -\lambda B_{2n}\\ ... & ... & ... & ...\\ -\lambda B_{n1} & -\lambda B_{n2} & ... & 1-\lambda B_{nn}\end{vmatrix}\neq 0. \quad (5.11)$$

Then from the system (5.8) we get:

$$C_m = \frac{1}{\Delta(\lambda)} \begin{vmatrix} 1-\lambda B_{11} & -\lambda B_{12} & ... & -\lambda B_{1m-1} f_1 & \lambda B_{1m+1} & ... & -\lambda B_{1n} \\ -\lambda B_{21} & 1-\lambda B_{22} & ... & -\lambda B_{2m-1} f_2 & -\lambda B_{2m+1} & ... & -\lambda B_{2n} \\ ... & ... & ... & ... & ... & ... & ... \\ -\lambda B_{n1} & -\lambda B_{n2} & ... & -\lambda B_{nm-1} f_n & -\lambda B_{nm+1} & ... & 1-\lambda B_{nn} \end{vmatrix}_{m=1,n} . \tag{5.12}$$

Then from (5.3) we find:

$$y(\xi_1) - \lambda \sum_{s=1}^{n} A_s(\xi_1) \frac{1}{\Delta(\lambda)} \sum_{k=1}^{n} B^{ks} f_k = f(\xi_1), \tag{5.13}$$

where B^{ks} are cofactors of the elements standing in the k-th line and s- th column of the determinant in the right hand side of (5.12) as the coefficient $\frac{1}{\Delta(\lambda)}$.

Taking into account (5.5) and (5.10), from (5.13) we have:

$$y(\xi_1) = \lambda \sum_{s=1}^{n} A_s(\xi_1) \frac{1}{\Delta(\lambda)} \sum_{k=1}^{n} B^{ks} \int_{-1}^{1} b_k(\tau) d\tau$$

$$\times \int_{-1}^{1} K_0(\xi_1, t) y(t) dt + \int_{-1}^{1} K_0(\xi_1, t) y(t) dt, \tag{5.14}$$

or

$$y(\xi_1) = \int_{-1}^{1} K_0(\xi_1, t) \left[1 + \lambda \sum_{s=1}^{n} \frac{A_s(\xi_1)}{\Delta(\lambda)} \sum_{k=1}^{n} B^{ks} \int_{-1}^{1} b_k(\tau) d\tau \right] y(t) dt. \tag{5.15}$$

It we replace the kernel $K_0(\xi_1, t)$ by the approximate by degenerate kernel, then , from (5.15) we get an algebraic equation that determines eigenvalues of the stated problem

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