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Asymptotic behavior of eigenvalues of a boundary value problem for a second order elliptic differential-operator equation with a spectral parameter linearly occurring in the boundary conditions

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Abstract. *In the paper we study asymptotic behavior of eigenvalues of a boundary value problem for a second order elliptic differential operator equation in the case when one and the same spectral parameter participates linearly in the equation and in the boundary conditions, moreover the spectral parameter in the boundary conditions stands in front of the derivative of the desired function. Asymptotic formulas for the eigenvalues of the considered boundary value problems are found.*

Keywords. spectral parameter · Hilbert space · boundary-value problem · differential-operator equation · asymptotical behavior of eigenvalues.

Mathematics Subject Classification (2010): 35G25, 35P20, 47A75

1 Introduction

In the paper, we study asymptotic behavior of the eigenvalues of the following boundary value problem for a second order elliptic differential-operator equation in separable Hilbert space H:

$$-u''(x) + Au(x) = \lambda u(x),\ x \in (0,1)\,, \tag{1.1}$$

$$u(0) - \lambda u'(0) = 0,$$

$$u(1) + \lambda u'(1) = 0, \tag{1.2}$$

where λ is a spectral parameter; A is a linear, unbounded, self-adjoint, positive-definite operator in H, and the inverse operator A^{-1} is completely continuous in H. It is shown that the eigenvalues of boundary value problem (1.1), (1.2) are real and simple.

Later we show that problem (1.1), (1.2) have two series eigenvalue, and one which convergence to zero.

Asymptotic behavior of eigenvalues of boundary value problems for equation (1.1) with the boundary conditions of the form

$$u'(0) + \lambda u(0) = 0, u(1) = 0, \tag{1.3}$$

was studied in the papers [9], [13], while with the boundary conditions of the form

$$u'(0) + \lambda u(0) = 0, u'(1) - \lambda u(1) = 0, \tag{1.4}$$

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was studied in [1], where it is proved that the eigenvalues of boundary value problems (1.1), (1.3) and (1.1), (1.4) are discrete and have two series of eigenvalues:

$$\lambda_k \sim \sqrt{\mu_k},\ (k \to +\infty)\,,\ \lambda_{n,k} \sim \mu_k + n^2\pi^2,\ (k, n \to +\infty).$$

In [2] some spectral properties of boundary value problems are studied for equation (1.1) with the boundary conditions of the form

$$u'(1) = 0,\ u'\,(0) - d\lambda u'(0) = 0, \tag{1.5}$$

where $d > 0$ is some number, and asymptotic formulas for the eigenvalues are found. It is shown that boundary value problem (1.1), (1.5) has one series of eigenvalues that behaves as $\lambda_{k,n} \sim \mu_k + n^2\pi^2,\ (k, n \to +\infty)$. A similar issue was studied in [3] for equation (1.1) with the boundary conditions of the form

$$u'\,(0) + d\lambda^2 u(0) = 0,\ u(1) = 0, \tag{1.6}$$

where $d > 0$ is some number. Peculiarity of boundary value problems (1.1), (1.6) is that this problem has two series of eigenvalues one of which converges to zero, i.e. for boundary value problems (1.1), (1.6) the classic case is violated. In [4]-[6] asymptotic behavior of eigenvalues of boundary value problems are studied for a second order elliptic differential operator equation in the case when unlike the papers [1], [2], [9], [13] one and the same spectral parameter occurs quadratically in the equation and linearly in the boundary condition.

Some spectral properties of boundary values problems for a fourth order elliptic differential -operator equation with a spectral parameter in the one of boundary conditions were studied in [8].

Boundary value problems with a spectral parameter in the equation and in the boundary conditions for second order ordinary differential equations compared with similar boundary value problem for second order elliptic differential –operator equations were studied in different aspects and deeper. Note some of them. In [10] boundary value problem (1.1), (1.5) is considered in the case when $H := R$ is a real axis and $A = 0$, where in particular, asymptotic behavior of eigenvalues of the considered boundary value problem is studied.

In [11], [7] some spectral properties of boundary value problems are studied for second order ordinary differential equations in the case when a spectral parameter participates linearly in the equation, and in the boundary conditions participates as a linear function.

In [14] a spectral problem is considered for ordinary differential equations with a quadratic spectral parameter in the equation and in the boundary conditions, where it is proved that the spectrum of the considered boundary value problems is discrete and the system of root function is doubly complete in certain spaces.

2 Some properties of eigenvalues

Lemma 2.1 *The eigenvalues of boundary value problem (1.1), (1.2) are real.*

Proof. We denote the eigen elements of the operator A, corresponding to the eigenvalues μ_k,, by $\varphi_k,\ k \in N$. It is known that the system $\{\varphi_k\}$ forms a complete orthonormed basis in the space H. Then from the expansion $u(x) = \sum\limits_{k=1}^{\infty} (u(x),\, \varphi_k)_H\, \varphi_k$, for the Fourier coefficients $u_k(x) = (u(x),\, \varphi_k)_H$, we obtain the following spectral problem:

$$-u_k''\,(x) + \mu_k u_k(x) = \lambda u_k(x),\ \ x \in (0, 1)\,, \tag{2.1}$$

$$\begin{aligned} u_k\,(0) - \lambda u_k'(0) = 0, \\ u_k(1) + \lambda u_k'(1) = 0. \end{aligned} \tag{2.2}$$

Thus the study of the eigenvalues of boundary value problems (1.1), (1.2) is reduced to the study of eigenvalues of boundary value problem (2.1), (2.2), for different natural k. Spectrum of boundary problem (1.1), (1.2) consists of those λ under which problem (2.1), (2.2) has nontrivial solution $u_k(x)$ even if for one k.

The number $\lambda = \mu_k$ cannot be the eigenvalue of problem (2.1), (2.2) as in this case this problem has only a trivial solution. In what follows, the proof of lemma 2.1 follows from the similar lemma in the papers [11], [12] on reality of eigenvalues of boundary value problems (2.1), (2.2) of the form for every k.

Lemma 2.2 *The eigenvalues of boundary value problem* (1.1)*,* (1.2) *are simple.*

As was noted above, problem (1.1), (1.2) is reduced to the study of spectral problem (2.1), (2.2) for every k. A similar problem for boundary values problem of the form (2.1), (2.2) was studied in [11].

Therefore we omit the proof of Lemma 2.2.

Lemma 2.3 *The number $\lambda = 0$ is not an eigenvalue of boundary value problem* (1.1)*,* (1.2)*.*

Proof. It suffices to prove that boundary value problem (2.1), (2.2) for $\lambda = 0$, i.e., the problem

$$-u_k''\,(x) + \mu_k u_k(x) = 0, \;\; x \in (0,1)\,, \tag{2.3}$$

$$u_k(0) = u_k(1) = 0 \tag{2.4}$$

for every k, has only a trivial solution. The general solution of the equation (2.3) has the form

$$u_k(x) = c_1 e^{-x\sqrt{\mu_k}} + c_2 e^{-(1-x)\sqrt{\mu_k}}, \tag{2.5}$$

where c_i, $i = 1, 2$ are arbitrary constants. Taking into account (2.5) in (2.4), we will get a system with respect to c_i, whose determinant is of the form:

$$D = 1 - e^{-2\sqrt{\mu_k}}.$$

Obviously, for any k, $D \neq 0$. Hence it follows that for any k, the function $u_k(x)$ determined by formula (2.5) is identically equal to zero, i.e., $\lambda = 0$ is not an eigenvalue of boundary value problem (2.1) and (2.2), and by the same token, of boundary value problem (1.1), (1.2).Lemma 2.3 is proved.

3 Asymptotic formulas for eigenvalues

Theorem 3.1 *Let A-be a self-adjoint, positive-definite operator in H, and A^{-1} be completely continuous in H.*

Then boundary value problem (1.1)*,* (1.2) *hawe two series eigenvalues: $\lambda_k \to 0$ at $k \to \infty$; $\lambda_{k,n} = \mu_k + \gamma_n$, where $\mu_k = \mu_k(A) \to +\infty$ are eigenvalues of the operator A, $\gamma_n \sim n^2\pi^2$ at $n \to \infty$.*

Proof. The general solution of ordinary differential equation (2.1) has the form:

$$u_k(x,\lambda) = c_1 e^{-x\sqrt{\mu_k - \lambda}} + c_2 e^{-(1-x)\sqrt{\mu_k - \lambda}}, \tag{3.1}$$

where c_i, $(i = 1, 2)$ are arbitrary constants. Having substituted (3.1) in (3.2), we get a system with respect to c_i , $i = 1, 2$, whose determinant is of the form:

$$D\,(\lambda) = \left(1 + \lambda\sqrt{\mu_k - \lambda}\right)^2 - \left(1 - \lambda\sqrt{\mu_k - \lambda}\right)^2 \; e^{-2\sqrt{\mu_k - \lambda}}.$$

Thus, the eigenvalues of boundary value problem (2.1), (2.2) and by the same token, of boundary value problem (1.1), (1.2), are the zeros of the equation $D(\lambda) = 0$ (with respect to λ, $\lambda \neq \mu_k$), at least for one k:

$$\left(1+\lambda\sqrt{\mu_k-\lambda}\right)^2 - \left(1-\lambda\sqrt{\mu_k-\lambda}\right)^2 e^{-2\sqrt{\mu_k-\lambda}} = 0. \tag{3.2}$$

Equation (3.2) is decomposed into two equations

$$\left(1+\lambda\sqrt{\mu_k-\lambda}\right) - \left(1-\lambda\sqrt{\mu_k-\lambda}\right) e^{-\sqrt{\mu_k-\lambda}} = 0, \tag{3.3}$$

$$\left(1+\lambda\sqrt{\mu_k-\lambda}\right) + \left(1-\lambda\sqrt{\mu_k-\lambda}\right) e^{-\sqrt{\mu_k-\lambda}} = 0. \tag{3.4}$$

Thus, the eigenvalues of boundary value problem (2.1), (2.2) consist of those real $\lambda \neq \mu_k$, that at least for one k satisfy at least one of the equations, (3.3) or (3.4). We rewrite equations (3.3), (3.4) following from in the form

$$sh\left(\frac{1}{2}\sqrt{\mu_k-\lambda}\right) + \lambda\sqrt{\mu_k-\lambda}\ ch\left(\frac{1}{2}\sqrt{\mu_k-\lambda}\right) = 0, \tag{3.5}$$

$$ch\left(\frac{1}{2}\sqrt{\mu_k-\lambda}\right) + \lambda\sqrt{\mu_k-\lambda}\ sh\left(\frac{1}{2}\sqrt{\mu_k-\lambda}\right) = 0. \tag{3.6}$$

At first we study equation (3.5). We find those eigenvalues λ, for which $\lambda < \mu_k$.

We will be consider cases $\lambda < 0$ and $0 < \lambda < \mu_k$. We take $\sqrt{\mu_k - \lambda} = y$. Hence $\lambda = \mu_k - y^2$. If $\lambda < 0$, then $\sqrt{\mu_k} < y < +\infty$. Thus equation (3.5) have following form

$$sh\frac{y}{2} + y\left(\mu_k - y^2\right) ch\frac{y}{2} = 0, \sqrt{\mu_k} < y < +\infty. \tag{3.7}$$

Equation (3.7) equivalent to equation

$$y\left(y^2-\mu_k\right) cth\frac{y}{2} - 1 = 0, \sqrt{\mu_k} < y < +\infty. \tag{3.8}$$

We consider of function $f_k(y) = y\left(y^2-\mu_k\right) cth\frac{y}{2} - 1$, $\sqrt{\mu_k} < y < +\infty$. The derivatives this function $f_k'(y) = \frac{(3y^2-\mu_k)shy - y(y^2-\mu_k)}{2sh^2\frac{y}{2}} > 0$ at each k under all $y \in \left(\sqrt{\mu_k}, +\infty\right)$. Thus function $f_k(y)$ monotone increasing in interval $\left(\sqrt{\mu_k}, +\infty\right)$ under each k. Consequence that

$$f_k\left(\sqrt{\mu_k}\right) = \lim_{y\to\sqrt{\mu_k}+0} f_k(y) = \lim_{y\to\sqrt{\mu_k}+0} y\left(y^2-\mu_k\right) cth\frac{y}{2} - 1 = -1 < 0$$

and

$$f_k(+\infty) = \lim_{y\to+\infty} f_k(y) = \lim_{y\to+\infty}\left[y\left(y^2-\mu_k\right) cth\frac{y}{2} - 1\right] = +\infty,$$

in corollary that in interval $\left(\sqrt{\mu_k}, +\infty\right)$ equation (3.8) under each k have exactly one zero. Define this zero element y_k. We show y_k asymptotically behavior as $\sqrt{\mu_k}$. Let $\varepsilon > 0$ sufficiently small number. Then

$$f_k\left(\sqrt{\mu_k}+\varepsilon\right) = \left(\sqrt{\mu_k}+\varepsilon\right)\left(\left(\sqrt{\mu_k}+\varepsilon\right)^2 - \mu_k\right) cth\frac{\sqrt{\mu_k}+\varepsilon}{2} - 1 =$$

$$= \left(\sqrt{\mu_k}+\varepsilon\right)\left(2\varepsilon\sqrt{\mu_k}+\varepsilon^2\right) cth\frac{\sqrt{\mu_k}+\varepsilon}{2} - 1 \to +\infty,\ at\ \ k \to +\infty.$$

Consequence, beginning with some k, $f_k(\sqrt{\mu_k}+\varepsilon) > 0$. At another hand, for any $k, f_k(\sqrt{\mu_k}) < 0$. Thus, y_k is between $\sqrt{\mu_k}$ and $\sqrt{\mu_k}+\varepsilon$. According any $\varepsilon > 0$, we have equivalents: $y_k \sim \sqrt{\mu_k}$ at $k \to +\infty$. Hence and from equality $\sqrt{\mu_k - \lambda} = y$ for eigenvalue boundary value problem (2.1), (2.2) which satisfying $\lambda < 0$, following relation is obtained: $\lambda_k \to 0$ at $k \to \infty$.

Now the consider case $0 < \lambda < \mu_k$. Itis clearly that in this case equation (3.5) have following form

$$sh\frac{y}{2} + y\left(\mu_k - y^2\right) ch\frac{y}{2} = 0, \ \ 0 < y < \sqrt{\mu_k}. \tag{3.9}$$

Let us consider the function $\psi_k(y) = sh\frac{y}{2} + y\left(\mu_k - y^2\right) ch\frac{y}{2}, y \in (0, \sqrt{\mu_k})$. Obviously, for every fixed k, and for all $y \in \left(0, \sqrt{\mu_k}\right)$, $\psi_k(y) > 0$. Therefore, equation (3.9) has no solutions on the interval $\left(0, \sqrt{\mu_k}\right)$ for any k. Consequently, problem (1.1), (1.2) has no solutions satisfying the condition $0 < \lambda < \mu_k$.

Now find the eigenvalues λ, for which $\lambda > \mu_k$. To the equation (3.5) we put $\sqrt{\lambda - \mu_k} = z$, $(0 < z < +\infty)$, then it takes the form

$$\sin\frac{z}{2} + z(\mu_k + z^2)\cos\frac{z}{2} = 0, \quad z \in (0, +\infty). \tag{3.10}$$

Let $z \neq (2n-1)\pi, \ \ n \in N$. In this case, equation (3.10) is equivalent to the equation

$$tg\frac{z}{2} + z(\mu_k + z^2) = 0, \quad z \in (0, +\infty), \ z \neq (2n-1)\pi, \ \ n \in N. \tag{3.11}$$

Let us consider the function

$$\varphi_k(z) = tg\frac{z}{2} + z(\mu_k + z^2), \quad z \in (0, +\infty), z \neq (2n-1)\pi, \ \ n \in N.$$

Since in each interval $((2n-1)\pi, \ (2n+1)\pi), \ \ n \in N$, the function $\varphi_k(z)$ takes the values from $-\infty$ to $+\infty$, and its derivative $\varphi_k'(z) = \frac{1}{2\cos^2\frac{z}{2}} + 3z^2 + \mu_k > 0$, then in it for every k, the function $\varphi_k(z)$ has only one zero $z_{n,k}$: $(2n-1)\pi < z_{n,k} < (2n+1)\pi, \ \ n \in N$. For every $k \in N$ we find asymptotic formulas for $z_{n,k}$, as $n \to \infty$. From (3.11) we have

$$tg\frac{z}{2} = -z(\mu_k + z^2), \quad z \in (0, +\infty), \ z \neq (2n-1)\pi, \ \ n \in N.$$

Denote $q_k(z) = -z(\mu_k + z^2), \quad z \in (0, +\infty)$. Obviously for every k, $q_k(z) < 0$, $q_k'(z) = -(\mu_k + 3z^2) < 0, \quad q_k''(z) = -6z < 0$. So $q_k(z)$ is a negative, decreasing, strictly upwards convex function for every k, $\lim\limits_{z \to 0^+} q_k(z) = 0$, and $\lim\limits_{z \to +\infty} q_k(z) = -\infty$. Obviously, the points $z_{n,k}$ are the abscissas of the intersection point of the function $q_k(z)$ and the branches of the function $tg\frac{z}{2}$. By increasing n and k, the points $z_{n,k}$ will approach the points $(2n-1)\pi$, i.e., $z_{n,k} \sim (2n-1)\pi$. Hence and from the equality $\sqrt{\lambda - \mu_k} = z$ for the eigenvalues satisfying the condition $\lambda > \mu_k$, we get the asymptotic formula:

$$\lambda_{k,n}^{(1)} \sim \mu_k + (2n-1)^2\pi^2. \tag{3.12}$$

Similarly we can investigate to equation (3.6) to equation (3.5) and we can easy show that in this case as problem (2.1), (2.2) have two series eigenvalue and from one is convergences to zero, second series asymptotically behaves as following

$$\lambda_{k,n}^{(2)} \sim \mu_k + (2n)^2\pi^2. \tag{3.13}$$

From (3.12) and (3.13) it follows that for the eigenvalues of boundary value problems (1.1), (1.2) we have the asymptotic formula $\lambda_{k,n} \sim \mu_k + n^2\pi^2$.

Theorem 3.1 is proved.

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