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Fractional integral operator and its higher order commutators in generalized weighted Morrey spaces on Heisenberg group

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Abstract. *In this paper we study the boundedness of the fractional integral operator* I_α, $x0 < \alpha < Q$ *on Heisenberg group* $\mathbb{H}_n$ *in the generalized weighted Morrey spaces* $M_{p,\varphi}(\mathbb{H}_n, w)$*, where* Q *is the homogeneous dimension of* $\mathbb{H}_n$*. In the case* $b \in BMO(\mathbb{H}_n)$ *we obtain the boundedness of the* k*th-order commutator of fractional integral operator* $[b, I_\alpha]^k$ *on the generalized weighted Morrey spaces.*

Keywords. Heisenberg group, fractional integral operator, generalized weighted Morrey space, commutator, BMO.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35

1 Introduction

The classical Morrey spaces were introduced by Morrey [22] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [6,21,23] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [7,8,25]). Komori and Shirai [18] defined weighted Morrey spaces $L_{p,\kappa}(w)$. Guliyev [9] gave a concept of the generalized weighted Morrey spaces $M_{p,\varphi}(\mathbb{R}^n, w)$ which could be viewed as extension of both $M_{p,\varphi}(\mathbb{R}^n)$ and $L_{p,\kappa}(w)$. In [9], the boundedness of the classical operators and their commutators in spaces $M_{p,\varphi}(\mathbb{R}^n, w)$ was also studied, see also [13–17].

The spaces $M_{p,\varphi}(\mathbb{R}^n, w)$ defined by the norm

$$\|f\|_{M_{p,\varphi}(\mathbb{R}^n,w)} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-1/p}\, \|f\|_{L_p(B(x,r),w)},$$

where the function φ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w is a non-negative measurable function on $\mathbb{R}^n$.

Heisenberg groups, in discrete and continuous versions, appear in many parts of mathematics, including Fourier analysis, several complex variables, geometry, and topology. We state some basic results about Heisenberg group. More detailed information can be found

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in [2–4] and the references therein. Let $\mathbb{H}_n$ be the $2n+1$-dimensional Heisenberg group. That is, $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$, with multiplication

$$(z,t)\cdot(w,s) = (z+w, t+s+2Im(z\cdot\bar{w})),$$

where $z\cdot\bar{w} = \sum_{j=1}^{n} z_j \bar{w}_j$. The inverse element of $u=(z,t)$ is $u^{-1}=(-z,-t)$ and we write the identity of $\mathbb{H}_n$ as $0=(0,0)$. The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations on $\mathbb{H}_n$, for $r>0$, by $\delta_r(z,t) = (rz, r^2t)$. These dilations are group automorphisms and the Jacobian determinant is r^Q, where $Q=2n+2$ is the homogeneous dimension of $\mathbb{H}_n$. A homogeneous norm on $\mathbb{H}_n$ is given by

$$|(z,t)| = (|z|^2+|t|)^{1/2}.$$

With this norm, we define the Heisenberg ball centered at $u=(z,t)$ with radius r by $B(u,r) = \{v \in \mathbb{H}_n : |u^{-1}v| < r\}$, and we denote by $B(u,2r) = \{y \in \mathbb{H}_n : \; |u^{-1}2v| < r\}$ the open ball centered at u, with radius $2r$. The volume of the ball $B(u,r)$ is $C_Q r^Q$, where C_Q is the volume of the unit ball $B(0,1)$.

Using coordinates $u=(z,t)=(x+iy,t)$ for points in $\mathbb{H}_n$, the left-invariant vector fields X_j, Y_j and T on $\mathbb{H}_n$ equal to $\frac{\partial}{\partial x_j}$, $\frac{\partial}{\partial y_j}$ and $\frac{\partial}{\partial t}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \;\; Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \;\; T = \frac{\partial}{\partial t},$$

respectively. These $2n+1$ vector fields form a basis for the Lie algebra of $\mathbb{H}_n$ with commutation relations

$$[Y_j, X_j] = 4T$$

for $j=1,...,n$, and all other commutators equal to 0.

Let $f \in L_1^{\rm loc}(\mathbb{H}_n)$. The fractional integral operator I_α is defined by

$$I_\alpha f(u) = \int_{\mathbb{H}_n} \frac{f(v)\,dV(v)}{|u^{-1}v|^{Q-\alpha}}, \qquad 0<\alpha<Q,$$

where Q is the homogeneous dimension of the Heisenberg group $\mathbb{H}_n$ and $|B(u,r)|$ is the Haar measure of the $\mathbb{H}_n$- ball $B(u,r)$.

The operator I_α play an important role in real and harmonic analysis and applications (see, for example [2] and [3]).

In the present work, we study Spanne-Guliyev type boundedness of the operator I_α on the generalized weighted Morrey spaces, including weak versions. Also we study Spanne-Guliyev type boundedness of the higher order commutator operator $[b, I_\alpha]^k$ on the generalized weighted Morrey spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries and some lemmas

By a weight function, briefly weight, we mean a locally integrable function on $\mathbb{H}_n$ which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E)=\int_E w(x)dx$, and denote the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E.

If w is a weight function, we denote by $L_{p,w}(\mathbb{H}_n)$ the weighted Lebesgue space defined by finiteness of the norm

$$\|f\|_{L_{p,w}(\mathbb{H}_n)}=\Big(\int_{\mathbb{H}_n}|f(u)|^p w(u)dV(u)\Big)^{\frac{1}{p}}<\infty, \quad if \quad 1\le p<\infty$$

and

$$\|f\|_{L_{\infty,w}(\mathbb{H}_n)}=\operatorname*{ess\,sup}_{u\in\mathbb{H}_n}|f(u)|w(u), \quad if \quad p=\infty.$$

We define the generalized weighed Morrey spaces as follows.

Definition 2.1 *Let $1\le p<\infty$, φ be a positive measurable function on $\mathbb{H}_n\times(0,\infty)$ and w be non-negative measurable function on $\mathbb{H}_n$. We denote by $M_{p,\varphi}(\mathbb{H}_n,w)\equiv M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f\in L^{loc}_{p,w}(\mathbb{H}_n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)}=\sup_{u\in\mathbb{H}_n,r>0}\varphi(u,r)^{-1}w(B(u,r))^{-\frac{1}{p}}\|f\|_{L_{p,w}(B(u,r))},$$

where $L_{p,w}(B(u,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(u,r))}\equiv\|f\chi_{B(u,r)}\|_{L_{p,w}(\mathbb{H}_n)}=\Big(\int_{B(u,r)}|f(v)|^p w(v)dV(v)\Big)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f\in WL^{loc}_{p,w}(\mathbb{H}_n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)}=\sup_{u\in\mathbb{H}_n,r>0}\varphi(u,r)^{-1}w(B(u,r))^{-\frac{1}{p}}\|f\|_{WL_{p,w}(B(u,r))}<\infty,$$

where $WL_{p,w}(B(u,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(u,r))}\equiv\|f\chi_{B(u,r)}\|_{WL_{p,w}(\mathbb{H}_n)}=\sup_{t>0}t\Big(\int_{\{v\in B(u,r):|f(v)|>t\}}w(v)dV(v)\Big)^{\frac{1}{p}}.$$

We recall a weight function w is in the Muckenhoupt's class $A_p(\mathbb{H}_n)$, $1<p<\infty$ [19], if

$$\begin{aligned}[w]_{A_p}&:=\sup_B[w]_{A_p(B)}\\&=\sup_B\Big(\frac{1}{|B|}\int_B w(u)dV(u)\Big)\Big(\frac{1}{|B|}\int_B w(u)^{1-p'}dV(u)\Big)^{p-1}<\infty,\end{aligned} \tag{2.1}$$

where the supremum is taken with respect to all the balls B and $\frac{1}{p}+\frac{1}{p'}=1$. Note that, for all balls B Hölder's inequality is

$$[w]^{\frac{1}{p}}_{A_p(B)}=|B|^{-1}\|w\|^{\frac{1}{p}}_{L_1(B)}\|w^{-\frac{1}{p}}\|_{L_{p'}(B)}\ge 1. \tag{2.2}$$

For $p = 1$, $w \in A_1(\mathbb{H}_n)$ is defined by the condition $Mw(u) \leq Cw(u)$ with $[w]_{A_1} = \sup_{u\in\mathbb{H}_n} \frac{Mw(u)}{w(u)}$, and for $p = \infty$, $A_\infty(\mathbb{H}_n) = \cup_{1\leq p<\infty} A_p(\mathbb{H}_n)$ and $[w]_\infty = \inf_{1\leq p<\infty}[w]_{A_p}$.

A weight function w is in the Muckenhoupt-Wheeden class $A_{p,q}(\mathbb{H}_n)$, $1 < p, q < \infty$ [20], if

$$\begin{aligned}[w]_{A_{p,q}} &:= \sup_B [w]_{A_{p,q}(B)} \\ &= \sup_B \Big(\frac{1}{|B|}\int_B w(u)^q dV(u)\Big)^{1/q} \Big(\frac{1}{|B|}\int_B w(u)^{-p'} dV(u)\Big)^{1/p'} < \infty,\end{aligned}$$

where the supremum is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls B Hölder's inequality is

$$[w]_{A_{p,q}(B)} = |B|^{\frac{1}{p}-\frac{1}{q}-1} \|w\|_{L_q(B)} \|w^{-1}\|_{L_{p'}(B)} \geq 1. \tag{2.3}$$

A weight function w is in the class $A_{1,q}(\mathbb{H}_n)$, $1 < q < \infty$ [20], if

$$\begin{aligned}[w]_{A_{1,q}} &:= \sup_B [w]_{A_{1,q}(B)} \\ &= \sup_B \Big(\frac{1}{|B|}\int_B w(u)^q dV(u)\Big)^{\frac{1}{q}} \Big(\operatorname*{ess\,sup}_{u\in B} \frac{1}{w(u)}\Big) < \infty.\end{aligned} \tag{2.4}$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s)\, w(s)\, ds, \qquad H_w^* g(t) := \int_t^\infty \Big(1 + \frac{s}{t}\Big)\, g(s)\, w(s)\, ds, \qquad 0 < t < \infty.$$

where w is a weight. The following theorem was proved in [10].

Theorem 2.1 *[10] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w g(t) \leq C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)\, ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 2.2 *[9] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w^* g(t) \leq C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \ln\big(1 + \frac{s}{t}\big) \frac{w(s)\, ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

3 Fractional integral operator in the spaces $M_{p,\varphi}(\mathbb{H}_n, w)$

The following Guliyev weighted local estimates are valid (see [5,9]).

Theorem 3.1 *Let* $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, *and* $\omega \in A_{p,q}(\mathbb{H}_n)$. *Then, for* $1 < p < q < \frac{Q}{\alpha}$, *the inequality*

$$\|I_\alpha f\|_{L_{q,w^q}(B(u,r))} \lesssim (w^q(B(u,r)))^{\frac{1}{q}} \int_r^\infty \|f\|_{L_{p,w^p}(B(u,t))} (w^q(B(u,t)))^{-\frac{1}{q}} \frac{dt}{t}$$

holds for any boll $B(u,r)$ *and for all* $f \in L^{loc}_{p,w}(\mathbb{H}_n)$.

Moreover, for $p = 1$ *the inequality*

$$\|I_\alpha f\|_{WL_{q,w^q}(B(u,r))} \lesssim (w^q(B(u,r)))^{\frac{1}{q}} \int_r^\infty \|f\|_{L_{1,w}(B(u,t))} (w^q(B(u,t)))^{-\frac{1}{q}} \frac{dt}{t} \qquad (3.1)$$

holds for any boll $B(u,r)$ *and for all* $f \in L^{loc}_{1,w}(\mathbb{H}_n)$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, and $w \in A_{p,q}(\mathbb{H}_n)$. For arbitrary $u \in \mathbb{H}_n$, set $B = B(u,r)$, $2B = B(u,2r)$.

We present f as

$$f = f_1 + f_2, \quad f_1(v) = f(v)\chi_{2B}(v), \quad f_2(v) = f(v)\chi_{(2B)^\complement}(v), \quad r > 0, \qquad (3.2)$$

and have

$$\|I_\alpha f\|_{L_{q,w^q}(B)} \le \|I_\alpha f_1\|_{L_{q,w^q}(B)} + \|I_\alpha f_2\|_{L_{q,w^q}(B)}.$$

Since $f_1 \in L_{p,w^p(\mathbb{H}_n)}$, $I_\alpha f_1 \in L_{q,w^q}(\mathbb{H}_n)$ and from the boundedness of I_α from $L_{p,w^p}(\mathbb{H}_n)$ to $L_{q,w^q}(\mathbb{H}_n)$ (see [1,24]) it follows that

$$\|I_\alpha f_1\|_{L_{q,w^q}(B)} \le \|I_\alpha f_1\|_{L_{q,w^q}} \le C\|f_1\|_{L_{p,w^p}} = C\|f\|_{L_{p,w^p}(2B)},$$

where the constant $C > 0$ does not depend on f.

It is clear that $z \in B$, $v \in (2B)^\complement$ implies $\frac{1}{2}|u^{-1}v| \le |z^{-1}v| \le \frac{3}{2}|u^{-1}v|$. We get

$$|I_\alpha f_2(z)| \le 2^{Q-\alpha} \int_{(2B)^\complement} \frac{|f(v)|}{|u^{-1}v|^{Q-\alpha}} dV(v).$$

By the Fubini's theorem we have

$$\begin{aligned}\int_{(2B)^\complement} \frac{|f(v)|}{|u^{-1}v|^{Q-\alpha}} dV(v) &\approx \int_{(2B)^\complement} |f(v)| \Big(\int_{|u^{-1}v|}^\infty \frac{dt}{t^{Q+1-\alpha}} \Big) dV(v) \\ &= \int_{2r}^\infty \Big(\int_{2r \le |u^{-1}v| < t} |f(v)| dV(v) \Big) \frac{1}{t^{Q+1-\alpha}} dt \\ &\le \int_{2r}^\infty \Big(\int_{B(u,t)} |f(v)| dV(v) \Big) \frac{dt}{t^{Q+1-\alpha}}.\end{aligned}$$

By applying Hölder's inequality, we get

$$\begin{aligned}\int_{(2B)^\complement} \frac{|f(v)|}{|u^{-1}v|^{Q-\alpha}} dV(v) &\lesssim \int_{2r}^\infty \|f\|_{L_{p,\omega^p}(B(u,t))} \|w^{-1}\|_{L_{p'}(B(u,t))} \frac{dt}{t^{Q+1-\alpha}} \\ &\lesssim \int_{2r}^\infty \|f\|_{L_{p,\omega^p}(B(u,t))} (w^q(B(u,t)))^{-\frac{1}{q}} \frac{dt}{t}.\end{aligned}$$

Moreover, for all $p \in [1, \infty)$,

$$\|I_\alpha f_2\|_{L_{q,w^q}(B)} \lesssim w^q(B)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,\omega^p}(B(u,t))} (w^q(B(u,t)))^{-\frac{1}{q}} \frac{dt}{t}. \tag{3.3}$$

Thus

$$\begin{aligned} \|I_\alpha f\|_{L_{q,w^q}(B)} \lesssim & \|f\|_{p,w^p}(2B) \\ & + w^q(B(u,t))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,\omega^p}(B(u,t))} (w^q(B(u,t)))^{-\frac{1}{q}} \frac{dt}{t}. \end{aligned}$$

On the other hand, since, $w \in A_{p,q}(\mathbb{H}_n)$, by the Hölder's inequality

$$[w]_{A_{p,q}} \ge |B|^{\frac{1}{p}-\frac{1}{q}-1} w^q(B)^{\frac{1}{q}} \|w^{-1}\|_{L_{p'}(B)} = |B|^{\frac{\alpha}{Q}-1} w^q(B)^{\frac{1}{q}} \|w^{-1}\|_{L_{p'}(B)} \ge 1.$$

Then

$$\begin{aligned} \|f\|_{L_{p,w^p}(2B)} & \approx |B|^{1-\frac{\alpha}{Q}} \|f\|_{L_{p,w^p}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{Q+1-\alpha}} \\ & \le |B|^{1-\frac{\alpha}{Q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(u,t))} \frac{dt}{t^{Q+1-\alpha}} \\ & \lesssim w^q(B)^{\frac{1}{q}} \|w^{-1}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(u,t))} \frac{dt}{t^{Q+1-\alpha}} \\ & \lesssim w^q(B)^{\frac{1}{q}} \int_{r}^{\infty} \|f\|_{L_{p,w^p}(B(u,t))} \|w^{-1}\|_{L_{p'}(B(u,t))} \frac{dt}{t^{Q+1-\alpha}} \\ & \lesssim w^q(B)^{\frac{1}{q}} \int_{r}^{\infty} \|f\|_{L_{p,w^p}(B(u,t))} \Big(w^q(B(u,t))\Big)^{-\frac{1}{q}} \frac{dt}{t}. \end{aligned} \tag{3.4}$$

Thus

$$\|I_\alpha f\|_{L_{q,w^q}(B)} \lesssim w^q(B)^{\frac{1}{q}} \int_{r}^{\infty} \|f\|_{L_{p,w^p}(B(u,t))} \Big(w^q(B(u,t))\Big)^{-\frac{1}{q}} \frac{dt}{t}.$$

Let $p = 1$. Since, $w \in A_{1,q}(\mathbb{H}_n)$, by the Hölder's inequality

$$[w]_{A_{p,q}} \ge |B|^{\frac{1}{p}-\frac{1}{q}-1} w^q(B)^{\frac{1}{q}} \|w^{-1}\|_{L_\infty(B)} = |B|^{\frac{\alpha}{Q}-1} w^q(B)^{\frac{1}{q}} \|w^{-1}\|_{L_\infty(B)} \ge 1.$$

Then from the boundedness of I_α from $L_{1,w^p}(\mathbb{H}_n)$ to $WL_{q,w^q}(\mathbb{H}_n)$ (see [1,24]) it follows that

$$\begin{aligned} \|I_\alpha f_1\|_{WL_{q,w^q}(B)} & \le \|I_\alpha f_1\|_{WL_{q,w^q}} \lesssim \|f_1\|_{L_{1,w}} = \|f\|_{L_{1,w}(2B)} \\ & \lesssim |B|^{1-\frac{\alpha}{Q}} \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(u,t))} \frac{dt}{t^{Q+1-\alpha}} \qquad (3.5) \\ & \lesssim w^q(B)^{\frac{1}{q}} \|w^{-1}\|_{L_\infty(B)} \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(u,t))} \frac{dt}{t^{Q+1-\alpha}} \\ & \lesssim w^q(B)^{\frac{1}{q}} \int_{r}^{\infty} \|f\|_{L_{1,w}(B(u,t))} \|w^{-1}\|_{L_\infty(B(u,t))} \frac{dt}{t^{Q+1-\alpha}} \\ & \lesssim w^q(B)^{\frac{1}{q}} \int_{r}^{\infty} \|f\|_{L_{1,w}(B(u,t))} \Big(w^q(B(u,t))\Big)^{-\frac{1}{q}} \frac{dt}{t}. \end{aligned}$$

Since

$$\|I_\alpha f_2\|_{WL_{q,w^q}(B)} \le \|I_\alpha f_2\|_{L_{q,w^q}(B)},$$

then

$$\|I_\alpha f_2\|_{WL_{q,w^q}(B)} \lesssim w^q(B)^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{1,\omega}(B(u,t))}(w^q(B(u,t)))^{-\frac{1}{q}} \frac{dt}{t}. \qquad (3.6)$$

Thus, from (3.5) and (3.6) it follows that

$$\|I_\alpha f\|_{WL_{q,w^q}(B)} \lesssim w^q(B)^{\frac{1}{q}} \int_{r}^{\infty} \|f\|_{L_{p,w^p}(B(u,t))}\Big(w^q(B(u,t))\Big)^{-\frac{1}{q}} \frac{dt}{t}.$$

For the operator I_α the following Spanne-Guliyev type result on the space $M_{p,\varphi}(w)$ is valid.

Theorem 3.2 *Let* $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $w \in A_{p,q}(\mathbb{H}_n)$, *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(u,s)(w^p B((u,s)))^{1/p}}{(w^q(B(u,t)))^{1/q}} \frac{dt}{t} \le C\varphi_2(u,r) \qquad (3.7)$$

where C does not depend on u and r. Then the operator I_α is bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{q,\varphi_2}(w^q)$ for $p = 1$. Moreover, for $p > 1$

$$\|I_\alpha f\|_{M_{q,\varphi_2}(w^q)} \lesssim \|f\|_{M_{p,\varphi_1}(w^p)},$$

and for $p = 1$

$$\|I_\alpha f\|_{WM_{q,\varphi_2}(w^q)} \lesssim \|f\|_{M_{1,\varphi_1}(w)}.$$

Proof. Using the Theorem 2.1 and the Theorem 3.1 for $p > 1$ we get

$$\begin{aligned} \|I_\alpha f\|_{M_{q,\varphi_2}(w^q)} &= \sup_{u\in\mathbb{H}_n, r>0} \varphi_2(u,r)^{-1} w(B(u,r))^{-\frac{1}{q}} \|I_\alpha f\|_{L_{q,w^q}(B(u,r))} \\ &\lesssim \sup_{u\in\mathbb{H}_n, r>0} \varphi_2(u,r)^{-1} \int_r^{\infty} \|f\|_{L_{p,w^p}(B(u,t))}(w^q(B(u,t)))^{-\frac{1}{q}} \frac{dt}{t} \\ &\lesssim \sup_{u\in\mathbb{H}_n, r>0} \varphi_1(u,r)^{-1}(w^p(B(u,r)))^{-\frac{1}{p}} \|f\|_{L_{p,w^p}(B(u,r))} = \|f\|_{M_{p,\varphi_1}(w^p)} \end{aligned}$$

and for $p = 1$

$$\begin{aligned} \|I_\alpha f\|_{WM_{q,\varphi_2}(w^q)} &= \sup_{u\in\mathbb{H}_n, r>0} \varphi(u,r)^{-1} w(B(u,r))^{-\frac{1}{q}} \|I_\alpha f\|_{L_{1,w}(B(u,r))} \\ &\lesssim \sup_{u\in\mathbb{H}_n, r>0} \varphi_2(u,r)^{-1} \int_r^{\infty} \|f\|_{L_{1,w}(B(u,t))}(w^q(B(u,t)))^{-\frac{1}{q}} \frac{dt}{t} \\ &\lesssim \sup_{u\in\mathbb{H}_n, r>0} \varphi_2(u,r)^{-1} w^p(B(u,r)) \|f\|_{L_{1,w}(B(u,r))} = \|f\|_{M_{1,\varphi_1}(w)}. \end{aligned}$$

Remark 3.1 Note that, in the case $w \equiv 1$, Theorems 3.1 and 3.2 were proved in [11], see also [5, 12].

4 Higher order commutators of fractional integral operators in the spaces $M_{p,\varphi}(\mathbb{H}_n, w)$

Given a function b locally integrable on $\mathbb{R}^n$ and the operator I_α, we consider the linear commutator $[b, I_\alpha]$ defined by setting, for smooth, compactly supported functions f,

$$[b, I_\alpha](f) = bI_\alpha(f) - I_\alpha(bf).$$

We recall the definition of the space of $BMO(\mathbb{H}_n)$.

Definition 4.1 *Suppose that $b \in L_1^{\rm loc}(\mathbb{H}_n)$, and let*

$$\|b\|_* = \sup_{u\in\mathbb{H}_n, r>0} \frac{1}{|B(u,r)|} \int_{B(u,r)} |b(v) - b_{B(u,r)}| dV(v) < \infty,$$

where

$$b_{B(u,r)} = \frac{1}{|B(u,r)|} \int_{B(u,r)} b(v) dV(v).$$

Define

$$BMO(\mathbb{H}_n) = \{b \in L_1^{\rm loc}(\mathbb{H}_n) \ : \ \|b\|_* < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{H}_n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

Lemma 4.1 *[20] Let $w \in A_\infty$. Then the norm $\|\cdot\|_*$ is equivalent to the norm*

$$\|b\|_{*,w} = \sup_{u\in\mathbb{H}_n, r>0} \frac{1}{w(B(u,r))} \int_{B(u,r)} |b(v) - b_{B(u,r),w}|\, w(y)\, dV(v),$$

where

$$b_{B(u,r),w} = \frac{1}{w(B(u,r))} \int_{B(u,r)} b(v)\, w(v)\, dV(v).$$

The following lemma is proved in [9].

Lemma 4.2

1 *Let $w \in A_\infty$ and $b \in BMO(\mathbb{H}_n)$. Let also $1 \le p < \infty$, $u \in \mathbb{H}_n$, $k > 0$ and $r_1, r_2 > 0$. Then,*

$$\Big(\frac{1}{w(B(u,r_1))} \int_{B(u,r_1)} |b(v) - b_{B(u,r_2),w}|^{kp} w(v) dV(v)\Big)^{\frac{1}{p}} \le C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big)^k \|b\|_*^k,$$

where $C > 0$ is independent of f, w, u, r_1 and r_2.

2 *Let $w \in A_p$ and $b \in BMO(\mathbb{H}_n)$. Let also $1 < p < \infty$, $u \in \mathbb{H}_n$, $k > 0$ and $r_1, r_2 > 0$. Then,*

$$\Big(\frac{1}{w^{-p'}(B(u,r_1))} \int_{B(u,r_1)} |b(v) - b_{B(u,r_2),w}|^{kp'}\, w(v)^{-p'} dV(v)\Big)^{\frac{1}{p'}} \le C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big)^k \|b\|_*^k,$$

where $C > 0$ is independent of b, w, u, r_1 and r_2.

The following lemma is valid.

Remark 4.1 [3,26] (1) Let $b \in BMO(\mathbb{H}_n)$. Then

$$\|b\|_* \approx \sup_{u\in\mathbb{H}_n, r>0} \left(\frac{1}{|B(u,r)|} \int_{B(u,r)} |b(v) - b_{B(u,r)}|^p dV(v) \right)^{\frac{1}{p}} \tag{4.1}$$

for $1 < p < \infty$.

(2) Let $b \in BMO(\mathbb{H}_n)$. Then there is a constant $C > 0$ such that

$$\left| b_{B(u,r)} - b_{B(u,\tau)} \right| \le C\|b\|_* \log \frac{\tau}{r} \ \text{ for } \ 0 < 2r < \tau, \tag{4.2}$$

where C is independent of f, u, r and τ.

For the$k$th order commutator of the fractional integral operator $[b, I_\alpha]^k$ (see [12])

$$[b, I_\alpha]^k f(u) = \int_{\mathbb{H}_n} (b(u) - b(v))^k \, \frac{|f(v)|}{|u^{-1}v|^{Q-\alpha}} \, dV(v)$$

the following Guliyev weighted local estimates are valid (see [9,12]).

Theorem 4.1 *Let* $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $b \in BMO(\mathbb{H}_n)$, *and* $w \in A_{p,q}(\mathbb{H}_n)$. *Then the inequality*

$$\begin{aligned} &\|[b, I_\alpha]^k f\|_{L_{q,w^q}(B(u,r))} \\ &\lesssim \|b\|_*^k \, w^q(B(u,r))^{\frac{1}{q}} \int_r^\infty \ln^k \left(e + \frac{t}{r} \right) \|f\|_{L_{p,w^p}(B(u,t))} w^q(B(u,t))^{-\frac{1}{q}} \frac{dt}{t} \end{aligned}$$

holds for any ball $B(u,r)$ *and for all* $f \in L^{loc}_{p,w^p}(\mathbb{H}_n)$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $b \in BMO(\mathbb{H}_n)$ and $w \in A_{p,q}(\mathbb{H}_n)$. For arbitrary $u \in \mathbb{H}_n$, set $B = B(u,r)$ for the ball centered at u and of radius r. We present f as $f = f_1 + f_2$ with $f = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^\complement}$. Hence,

$$\|[b, I_\alpha]^k f\|_{L_{q,w^q}(B)} \le \|[b, I_\alpha]^k f_1\|_{L_{q,w^q}(B)} + \|[b, I_\alpha]^k f_2\|_{L_{q,w^q}(B)}.$$

From the boundedness of $[b, I_\alpha]^k$ from $L_{p,w^p}(\mathbb{H}_n)$ to $L_{q,w^q}(\mathbb{H}_n)$ (see [1,24]) it follows that:

$$|[b, I_\alpha]^k f_1\|_{L_{q,w^q}(B)} \le |[b, I_\alpha]^k f_1\|_{L_{q,w^q}(\mathbb{H}_n)} \lesssim \|b\|_*^k \, \|f_1\|_{L_{p,w^p}(\mathbb{H}_n)} = \|b\|_*^k \, \|f\|_{L_{p,w^p}(2B)}.$$

For $z \in B$ we have

$$|[b, I_\alpha]^k f_2(z)| \lesssim \int_{\mathbb{H}_n} \frac{|b(v) - b(z)|^k}{|z^{-1}v|^{Q-\alpha}} |f_2(v)| dV(v) \approx \int_{(2B)^\complement} \frac{|b(v) - b(z)|^k}{|u^{-1}v|^{Q-\alpha}} |f(v)| dV(v),$$

because for $z \in B$ and $v \in (2B)^\complement$ the following inequalities $\frac{1}{2}|u^{-1}v| \le |z^{-1}v| \le \frac{3}{2}|u^{-1}v|$ holds.

Then,

$$\begin{aligned} \|[b, I_\alpha]^k f_2\|_{L_{q,w^q}(B)} &\lesssim \Big(\int_B \Big(\int_{(2B)^\complement} \frac{|b(v) - b(z)|^k}{|u^{-1}v|^{Q-\alpha}} |f(v)| dV(v) \Big)^q w^q(z) dV(z) \Big)^{\frac{1}{q}} \\ &\lesssim \Big(\int_B \Big(\int_{(2B)^\complement} \frac{|b(v) - b_{B,w}|^k}{|u^{-1}v|^{Q-\alpha}} |f(v)| dV(v) \Big)^q w^q(z) dV(z) \Big)^{\frac{1}{q}} \\ + \Big(\int_B \Big(\int_{(2B)^\complement} &\frac{|b(z) - b_{B,w}|^k}{|u^{-1}v|^{Q-\alpha}} |f(v)| dV(v) \Big)^q w^q(z) dV(z) \Big)^{\frac{1}{q}} = I_1 + I_2. \end{aligned}$$

Using Fubini's theorem let us estimate I_1 as follows

$$I_1 = (w^q(B))^{\frac{1}{q}} \int_{(2B)^{\complement}} \frac{|b(v)-b_{B,w}|^k}{|u^{-1}v|^{Q-\alpha}} |f(v)| dV(v)$$
$$\approx (w^q(B))^{\frac{1}{q}} \int_{(2B)^{\complement}} |b(v)-b_{B,w}|^k |f(v)| \int_{|u^{-1}v|}^{\infty} \frac{dt}{t^{Q-\alpha+1}} dV(v)$$
$$\lesssim (w^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \int_{B(u,t)} |b(v)-b_{B,w}|^k |f(v)| dV(v) \frac{dt}{t^{Q-\alpha+1}}.$$

Applying Fubini's theorem, Hölder's inequality and the second part of Lemma 4.2 we get

$$I_1 \lesssim (w^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \Big(\int_{B(u,t)} |b(v)-b_{B(u,r),w}|^{kp'} w(v)^{-p'} dV(v) \Big)^{\frac{1}{p'}} \|f\|_{L_{p,w^p}(B(u,t))} \frac{dt}{t^{Q-\alpha+1}}$$
$$\lesssim \|b\|_*^k (w^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \Big(1+\ln^k \frac{t}{r}\Big) \|w^{-1}\|_{L_{p'}(B(u,t))} \|f\|_{L_{p,w^p}(B(u,t))} \frac{dt}{t^{Q-\alpha+1}}$$
$$\lesssim [w]_{A_{p,q}} \|b\|_*^k (w^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \ln^k \Big(e+\frac{t}{r}\Big) \|f\|_{L_{p,w^p}(B(u,t))} (w^q(B(u,t)))^{-\frac{1}{q}} \frac{dt}{t}.$$

In order to estimate I_2 we get

$$I_2 = \Big(\int_B |b(z)-b_{B,w}|^{kq} w^q(z) dV(z) \Big)^{1/q} \int_{(2B)^{\complement}} \frac{|f(v)|}{|u^{-1}v|^{Q-\alpha}}\, dV(v).$$

According to the second part of Lemma 4.2, we get

$$I_2 \lesssim \|b\|_*^k \, (w^q(B))^{\frac{1}{q}} \int_{(2B)^{\complement}} \frac{|f(v)|}{|u^{-1}v|^{Q-\alpha}} dV(v).$$

Applying Fubini's theorem and Hölder's inequality gives

$$\int_{(2B)^{\complement}} \frac{|f(v)|}{|u^{-1}v|^{Q-\alpha}}\, dV(v) \lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(u,t))} \, \|w^{-1}\|_{L_{p'}(B(u,t))} \, \frac{dt}{t^{Q-\alpha+1}}$$
$$\lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(u,t))} \, (w^q(B(u,t)))^{-\frac{1}{q}} \, \frac{dt}{t}. \qquad (4.3)$$

So, by (4.3)

$$I_2 \lesssim \|b\|_*^k w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(u,t))} (w^q(B(u,t)))^{-\frac{1}{q}} \, \frac{dt}{t}.$$

Summing I_1 and I_2, for all $p \in (1,\infty)$ we get

$$\|[b,I_\alpha]^k f_2\|_{L_{p,w}(B)} \lesssim \|b\|_*^k \, (w^q(B))^{\frac{1}{q}}$$
$$\times \int_{2r}^{\infty} \ln^k \Big(e+\frac{t}{r}\Big) \, \|f\|_{L_{p,w^p}(B(u,t))} (w^q(B(u,t)))^{-\frac{1}{q}} \frac{dt}{t}. \qquad (4.4)$$

Finally,

$$\|[b,I_\alpha]^k f\|_{L_{p,w}(B)} = \|b\|_*^k \|f\|_{L_{p,w^p}(2B)}$$
$$+ \|b\|_*^k (w^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \ln^k \Big(e+\frac{t}{r}\Big) \|f\|_{L_{p,w^p}(B(u,t))} w^q(B(u,t))^{-\frac{1}{q}} \frac{dt}{t}$$

and the statement of Theorem 4.1 follows by (3.4).

For the operator $[b, I_\alpha]^k$ the following Spanne-Guliyev type result on the space $M_{p,\varphi}(w)$ is valid.

Theorem 4.2 *Let* $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $w \in A_{p,q}(\mathbb{H}_n)$, $b \in BMO(\mathbb{H}_n)$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \ln^k\Big(e+\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(u,s)(w^p(B(u,s)))^{1/p}}{(w(B(u,t)))^{1/q}}\frac{dt}{t} \le C\varphi_2(u,r), \tag{4.5}$$

where C *does not depend on* u *and* r. *Then the operator* $[b, I_\alpha]^k$ *is bounded from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$. *Moreover,*

$$\|[b, I_\alpha]^k f\|_{M_{q,\varphi_2}(w^q)} \lesssim \|b\|_*^k \|f\|_{M_{p,\varphi_1}(w^p)}.$$

Proof. Using the Theorem 2.2 and the Theorem 4.1 we have

$$\begin{aligned}\|[b, I_\alpha]^k f\|_{M_{q,\varphi_2}(w^q)} &\lesssim \|b\|_*^k \sup_{u\in\mathbb{H}_n, r>0} \varphi_2(u,r)^{-1} \\ &\times \int_r^\infty \ln^k\Big(e+\frac{t}{r}\Big)\|f\|_{L_{p,w^p}(B(u,t))}(w^q(B(u,t)))^{-\frac{1}{q}}\frac{dt}{t} \\ &\lesssim \|b\|_*^k \sup_{u\in\mathbb{H}_n, r>0} \varphi_1(u,r)^{-1}(w^p(B(u,r)))^{-\frac{1}{p}}\|f\|_{L_{p,w^p}(B(u,r))} \\ &= \|b\|_*^k \|f\|_{M_{p,\varphi_1}(w^p)}.\end{aligned}$$

Remark 4.2 Note that, in the case $w \equiv 1$ and $k = 1$, Theorems 4.1 and 4.2 were proved in [12].

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