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On Wiman-Valiron type estimations for parabolic equations

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Abstract. *In the paper we established Wiman-Valiron type estimations for parabolic equations of the form*

$$u'(t) + A(t)u(t) = 0$$

in Hilbert space, where $A(t)$ is a uniformly positive-definite self-adjoint operator with a discrete spectrum. Imposing asymptotic character conditions on the distribution function $N(\lambda)$ of the eigen-values of the operator $A(t)$, we derive estimations for the norm of the solution of the equation that in particular characterize the behavior of the solution as $t \to 0$ depending on the behavior of the Fourier coefficients of initial data.

Keywords. Differential equations · asymptotics · discreteness · entire function · logarithmic measure · differential inequality.

Mathematics Subject Classification (2010):

1 Introduction

Let $f(z) = \sum\limits_{0}^{\infty} a_n z^n$ be an entire function,

$$M(r) = \max_{|z|=r} |f(z)| \, ; \mu(r) = \max_{n} |a_n| \, r^n$$

be a maximum of the modulus and maximum term of the function $f(z)$ in the circle of radius r. As known, the inequality $\mu(r) \leq M(r)$ always holds. But it is very important to estimate $M(r)$ through $\mu(r)$ from above Wiman [8] and Valiron [7] fist established the classical result: The following inequality is valid

$$M(r) \leq \mu(r)(\log \mu(r))^{\frac{1}{2}+\varepsilon}, \varepsilon > 0. \tag{1.1}$$

And this inequality can violate only on some set $E \subset (0, \infty)$ of finite logarithmic measure, $\int\limits_E dr/r < \infty$.

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In 1963,the American mathematician Rosenbloom [3] established more accurate and overall result: for some class of increasing functions $\varphi(y), y > 0$, satisfying the condition of the form

$$\int^{\infty}\left(\int^{y}\varphi(t)dt\right)^{-\frac{1}{2}}dy < \infty, \tag{1.2}$$

the following estimation

$$\frac{M(r)}{\sqrt{\varphi\left(\log M(r)\right)}} \leq c\mu(r), \quad c = const > 0 \tag{1.3}$$

is valid.

Hence, in particular for the function $\varphi(y) = y^k$,the Wiman-Valiron result (1.1) is obtained. In the monograph of Suleymanov [4] , published in M.V.Lomonosov MSU publishing house in 2012, theory of Wiman-Valiron-Rosenbloom type estimations was constructed for solving evolution equations of type

$$u'(t) \pm A(t)u(t) = 0, \tag{1.4}$$

in Hilbert space, where $A(t)$ in particular, is a self-adjoint positive operator with a discrete spectrum, where the role of the functions $M(r)$ and $\mu(r)$ are played by the functions

$$M(t) = \|u(t)\|\,, \mu(t) = \max_{k}\left|(u(t), \varphi_k(t))\right|,$$

where $\{\varphi_k(t)\}_{k\geq 1}$ is a complete orthonormed system of eigen-functions of the operator $A(t)$. These estimations, in particular, characterize the behavior of the solution of equation (1.4) as $t \to 0$ or $t \to \infty$ depending on the behavior of Fourier coefficients of initial data.

2 Formulation and proof of the main result.

In the present paper we establish the Rosenbloom type estimations (1.3) for parabolic equations of the form

$$u'(t) + A(t)u(t) = 0 \tag{2.1}$$

in Hilbert space and study the behavior of the solution as $t \to 0$. Equation (2.1), in particular, contains the heat equation $(A(t) = -\Delta_x)$ in the domain $\Omega \subset R^n$ with a smooth boundary.

Let $u(t)$ be the solution of equation (2.1). Denote

$$\mu(t) = \max_{k}\left|(u(t), \varphi_k(t))\right|; g(t) = \frac{1}{2}\log\left(u(t), u(t)\right).$$

Let $N(\lambda)$ be the number of eigen-values λ_k of the operator $A(t)$ such that $\lambda_k \leq \lambda, \lambda > 0$. The following lemma was proved in Suleymanovs above mentioned monograph ([4], p.88):

The key lemma.*The following nonlinear differential equation*

$$e^{2g(t)} \leq \mu^2(t)\cdot \Delta N\left(g'(t), g''(t)\right),$$

where

$$\Delta N(a,b) = N\left(a + c\sqrt{b + k(t)a}\right) - N\left(a - c\sqrt{b + k(t)a}\right),$$

$$k(t) > 0, k(t) \in L_1(0,\infty), \quad c = const > 0$$

is valid. Note that for proving the lemma, in Suleymanovs monograph a special method based on theory of probability was offered.

In the following theorem we impose a condition on the asymptotic behavior of the distribution function $N(\lambda)$ of eigen values of the operator $A(t)$, establish Wiman-Valiron type estimations for solving equation (4)

Theorem 2.1 *Let the function satisfy the following conditions*
1.$N(\lambda) \leq c\lambda^{s+1}, c > 0, s \geq -1, \lambda \geq 1.$
2. For $\lambda > \delta > 0, \lambda \to \infty.$

$$N(\lambda, \delta) \equiv N(\lambda + \delta) - N(\lambda - \delta) = c\delta \left(1 + \lambda^{v}\right) \lambda^{s}, 0 < v < 1.$$

Let the function $\varphi(y) > 0, y > 0,$ *do not decrease and the following integral a be finite:*

$$\int^{\infty} \left(\int^{y} \varphi(t) dt \right)^{-\frac{1}{2(s+1)}} dy < \infty. \qquad (\varphi)$$

Then, maybe out of some set of finite logarithmic measure, for solving equation (2.1) the following Wiman-Valiron-Rosenbloom type estimation is valid:

$$\frac{\|u(t)\|}{\sqrt[4]{\varphi\left(t^{-\beta} \log \|u(t)\|\right)}} \leq \mu(t) t^{-\gamma}, \qquad (B - B)$$

where $\gamma >, 0 < \beta < 1, t \to 0$ *(* γ *is calculated specifically).*

Proof. We immediately note that the functions for which the asymptotic formulas of the form

$$N(\lambda) = c\lambda^{n/m} + O\left(\lambda^{\frac{n-1}{m}}\right), \lambda \to \infty \qquad (2.2)$$

are true satisfy conditions 1 and 2 on the function $N(\lambda)$.

Such formulas were established for self-adjoint positive elliptic operators of order in bounded domain $\Omega \subset R^{n}$ in the papers of Weyl, Courant, Hörmander, Agmon, Seeley, Shubin, Ivriy, Birman, Solomyak, Vasilyev, Clark, Kostyuchenko, Levitan, Mikhailets, Safarov and others.

For example, in the paper of Safarov and Netrusov ([2], 2005) the unimprovable asymptotic formula

$$N(\lambda) = \lambda^{n} + O\left(\lambda^{n-\alpha}\right), \quad 0 < \alpha < 1. \qquad (2.3)$$

was established for the Laplace-Dirichlet operator $(-\Delta_{D})$ in domain $\Omega \subset R^{n}$ with a smooth boundary for the function $N(\lambda)$.

For formula (2.2) condition 2 of the theorem is fulfilled for while for $s = \frac{n}{m} - 1, v = \frac{1}{m}$, formula (2.3) it is fulfilled for $s = n - 1 - \alpha, \nu = \alpha$.

The functions

$$\varphi(y) = y^{2s+1+\varepsilon}, or \;\; \varphi(y) = y^{2s+1} (\log y)^{2(s+1)+\varepsilon}$$

satisfy the condition (φ). Having calculated the derivatives g', g'', we find

$$g' < 0, g'' > 0, g'' - kg' > 0.$$

Introduce a new variable:

$$\xi(t) = \int_{0}^{t} \exp\left(\int_{0}^{y} k(\tau) d\tau \right) dy.$$

Hence we determine the inverse function $t = t(\xi)$. For simplicity we consider that $k(t) = k = const > 0$. Assume:

$$\tilde{g}(\xi) = g(t(\xi)) = g(t).$$

We find:

$$\tilde{g}(\xi') = g'(t)e^{-kt},$$

$$\sqrt{g' + kg''} = e^{kt}\sqrt{\tilde{g}''(\xi)}.$$

We determine $\lambda(t)$ and $\delta(t)$ as follows:

$$\lambda(t) = \left[|g'| - \frac{1}{2}\sqrt{g'' + k|g'|}\right] = e^{kt}\left[|\tilde{g}'| - \frac{1}{2}\sqrt{\tilde{g}''}\right]$$

$$\delta(t) = \sqrt{g'' + k|g'|} = e^{kt}\sqrt{\tilde{g}''}.$$

Then we get:

$$\lambda(t) = e^{kt}\left[|\tilde{g}'| - \frac{3}{2}\sqrt{\tilde{g}''} + \sqrt{\tilde{g}''}\right] = e^{kt}\left[|\tilde{g}'| - \frac{3}{2}\sqrt{\tilde{g}''}\right] + \delta(t).$$

We consider the two cases:

$$1^0)\ |\tilde{g}'| - \frac{3}{2}\sqrt{\tilde{g}''} > 0;\ 2^0)\ |\tilde{g}'| - \frac{3}{2}\sqrt{\tilde{g}''} \le 0.$$

Let condition 1^0) be fulfilled. Then it is clear that $\lambda(t) > \delta(t)$ and $\lambda < |\tilde{g'}|$ (since $\lambda \le 1$ then $|\tilde{g'}| > 1$). Then from $(*)$ we get:

$$\varDelta N(\lambda, \delta) \le c\sqrt{\tilde{g}''}\left(1 + |\tilde{g}'|^v\right)|\tilde{g}'|^s \le 2c\sqrt{\tilde{g}''}\,|\tilde{g}'|^{s+v}. \qquad **$$

Lemma 2.1 *Under the theorem conditions, the set*

$$E_1 = \left\{\xi > 0 : \tilde{g}''(\xi)\left|\tilde{g}'(\xi)\right|^{2s} > \bar{\xi}^{\frac{\alpha}{2}}\sqrt{\varphi\left(\xi^{-\beta}\tilde{g}(\xi)\right)}\right\}$$

has a finite measure. The proof of this lemma is similar to the proof of the similar lemma from Suleymanovs monograph ([4], p.74) with regard to the parameter β and s was changed by $s + v$. Thus,$(**)$ by out E_1 of the following inequality holds:

$$\varDelta N\left(\tilde{g}', \tilde{g}''\right) \le c\xi^{-\frac{\alpha}{2}}\sqrt{\varphi\left(\bar{\xi}^{\beta}\tilde{g}(\xi)\right)}.$$

Passing in the right hand side, to the variable t we get that out of the set E of finite logarithmic measure, the following inequality is fulfilled:

$$\varDelta N\left(g', g''\right) \le c\bar{t}^{-\frac{\alpha}{2}}\sqrt{\varphi\left(\bar{t}^{\beta}g(t)\right)}.$$

Then by virtue of the key lemma we get

$$e^{2g(t)} \le \mu^2(t)t^{-\frac{\alpha}{2}}\sqrt{\varphi\left(t^{-\beta}g(t)\right)}$$

or the soame,

$$\|u(t)\| \le \mu(t)t^{-\frac{\alpha}{4}}\sqrt[4]{\varphi\left(t^{-\beta}\log\|u(t)\|\right)}.$$

The theorem is proved.

Corollary. Assume $\varphi(y) = y^{2s+1+\varepsilon}$ We get:

$$\|u(t)\| \leq \mu(t)t^{-\gamma} \left(\log \|u(t)\|\right)^{k},$$

where

$$\gamma = \frac{\alpha + \beta\,(2k+1+\varepsilon)}{4}, \quad k = \frac{2s+1+\varepsilon}{4}.$$

The last estimation as $t \to 0$ is equivalent to the estimation

$$\|u(t)\| \leq \mu(t)t^{-\gamma} \left(\log t^{-\gamma}\mu(t)\right)^{k}.$$

Remark. Assuming here $\bar{\mu}(t) = \bar{t}^{\gamma}\mu(t), k = \frac{1}{2} + \varepsilon$, we get the exact form of the Wiman-Valiron estimation in theory of entire functions:

$$\|u(t)\| \leq \tilde{\mu}(t) \left(\log \tilde{\mu}(t)\right)^{\frac{1}{2}+\varepsilon}, \quad t \to +0.$$

Case 2^0) of the theorem is studied in the same way.

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