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Stability in linear delay Levin-Nohel difference equations

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Abstract. *In this paper we use the contraction mapping theorem to obtain asymptotic stability results about the zero solution for the following linear delay Levin-Nohel difference equation*

$$\Delta x(t) + \sum_{s=t-r(t)}^{t-1} a(t,s)x(s) + b(t)x(t-h(t)) = 0.$$

An asymptotic stability theorem with a necessary and sufficient condition is proved. In addition, the case of the equation with several delays is studied.

Keywords. Fixed points, delay Levin-Nohel difference equations, stability.

Mathematics Subject Classification (2010): 39A30, 39A70.

1 Introduction

Certainly, the Lyapunov direct method has been, for more than 100 years, the efficient tool for the study of stability properties of ordinary, functional, partial differential and difference equations. Nevertheless, the application of this method to problems of stability in differential and difference equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded terms ([12],[13],[18]-[21],[28]). Recently, Burton, Furumochi, Zhang, Raffoul, Islam, Yankson and others have noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [1]-[16],[22],[24]-[27],[30]-[32]). The fixed point theory does not only solve the problem on stability but has a significant advantage over Lyapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise (see [12]).

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In paper, we consider the following linear delay Levin-Nohel difference equation with variable delays

$$\Delta x(t)+\sum_{s=t-r(t)}^{t-1} a(t,s)x(s)+b(t)x(t-h(t))=0,\ t\geq t_0, \tag{1.1}$$

with an assumed initial condition

$$x\left(t\right)=\phi\left(t\right),\ t\in\left[m\left(t_0\right),t_0\right]\cap\mathbb{Z},$$

where $\phi:\left[m\left(t_0\right),t_0\right]\cap\mathbb{Z}\rightarrow\mathbb{R}$ is a bounded sequence and for $t_0\geq 0$

$$m(t_0)=\min\Big\{\inf\big\{s-r(s):s\geq t_0\big\},\inf\big\{s-h(s):s\geq t_0\big\}\Big\}.$$

Here $\triangle$ denotes the forward difference operator $\triangle x\left(t\right)=x\left(t+1\right)-x\left(t\right)$ for any sequence $\left\{x\left(t\right),\ t\in\mathbb{Z}^{+}\right\}$. Throughout this paper, we assume that $b:[0,\infty)\cap\mathbb{Z}\rightarrow\mathbb{R}$, $a:[0,\infty)\cap\mathbb{Z}\times\left[m\left(0\right),\infty\right)\cap\mathbb{Z}\rightarrow\mathbb{R}$ and $r,h\in[0,\infty)\cap\mathbb{Z}\rightarrow\mathbb{Z}^{+}$ with $t-r\left(t\right)\rightarrow\infty$ and $t-h\left(t\right)\rightarrow\infty$ as $t\rightarrow\infty$.

Equation (1.1) can be viewed as a discrete analogue of the linear delay Levin-Nohel integro-differential equation

$$x'\left(t\right)+\int_{t-r(t)}^{t} a(t,s)x(s)ds+b(t)x(t-h(t))=0,\ t\geq t_0. \tag{1.2}$$

In [16], Dung investigated (1.2) and obtained the asymptotic stability theorem with a necessary and sufficient condition.

Our purpose here is to use the contraction mapping theorem to show the asymptotic stability of the zero solution for Eq. (1.1). An asymptotic stability theorem with a necessary and sufficient condition is proved. In addition, a study of the general form of (1.1) with several delays is given. For details on contraction mapping principle we refer the reader to [29] and for more on the calculus of difference equations, we refer the reader to [17] and [23].

2 Main results

For the convenience of the reader, let us recall the definition of asymptotic stability. For each t_0, we denote $C(t_0)$ the space of bounded sequences on $[m(t_0),t_0]\cap\mathbb{Z}$ with the supremum norm $\|.\|_{t_0}$. For each $(t_0,\phi)\in[0,\infty)\cap\mathbb{Z}\times C(t_0)$, we denote by $x(t)=x(t,t_0,\phi)$ the unique solution of Eq. (1.1).

Definition 2.1 *The zero solution of Eq. (1.1) is called*

(i) *stable if for each $\varepsilon>0$ there exists a $\delta>0$ such that $|x(t,t_0,\phi)|<\varepsilon$ for all $t\geq t_0$ if $\|\phi\|_{t_0}<\delta$,*

(ii) *asymptotically stable if it is stable and $\lim\limits_{t\rightarrow\infty}|x(t,t_0,\phi)|=0$.*

In order to be able to construct a new fixed mapping, we transform the Levin-Nohel difference equation into an equivalent equation. For this, we use the variation of parameter formula. In the process, for any sequence x, we denote

$$\sum_{k=a}^{b} x\left(k\right)=0 \text{ and } \prod_{k=a}^{b} x\left(k\right)=1 \text{ for any } a>b.$$

Lemma 2.1 *Suppose that*

$$A(z) \neq 1,\ \forall z \in [t_0, \infty) \cap \mathbb{Z}, \tag{2.1}$$

where

$$A(z) = \sum_{s=z-r(z)}^{z-1} a(z,s) + b(z). \tag{2.2}$$

Then x is a solution of equation (1.1) if and only if

$$\begin{aligned} x(t) = \phi(t_0) \prod_{z=t_0}^{t-1} (1 - A(z)) - \sum_{s=t_0}^{t-1} L_x(s) \prod_{z=s+1}^{t-1} (1 - A(z)) \\ - \sum_{s=t_0}^{t-1} N_x(s) \prod_{z=s+1}^{t-1} (1 - A(z)),\ t \geq t_0, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} L_x(t) &= \sum_{s=t-r(t)}^{t-1} a(t,s) \sum_{u=s}^{t-1} \left(\sum_{v=u-r(u)}^{u-1} a(u,v)x(v) + b(u)x(u-h(u)) \right), \\ N_x(t) &= b(t) \sum_{u=t-h(t)}^{t-1} \left(\sum_{v=u-r(u)}^{u-1} a(u,v)x(v) + b(u)x(u-h(u)) \right). \end{aligned} \tag{2.4}$$

Proof. Obviously, we have

$$x(s) = x(t) - \sum_{u=s}^{t-1} \Delta x(u),\ x(t-h(t)) = x(t) - \sum_{u=t-h(t)}^{t-1} \Delta x(u).$$

Inserting these relations into (1.1), we get

$$\begin{aligned} \Delta x(t) + \sum_{s=t-r(t)}^{t-1} a(t,s) \left(x(t) - \sum_{u=s}^{t-1} \Delta x(u) \right) + b(t)x(t) \\ - b(t) \sum_{u=t-h(t)}^{t-1} \Delta x(u) = 0,\ t \geq t_0, \end{aligned}$$

or equivalently

$$\begin{aligned} \Delta x(t) + x(t) \left(\sum_{s=t-r(t)}^{t-1} a(t,s) + b(t) \right) - \sum_{s=t-r(t)}^{t-1} a(t,s) \left(\sum_{u=s}^{t-1} \Delta x(u) \right) \\ - b(t) \sum_{u=t-h(t)}^{t-1} \Delta x(u) = 0,\ t \geq t_0. \end{aligned}$$

After substituting Δx from (1.1), we obtain

$$\begin{aligned}
&\Delta x(t)+x(t)\left(\sum_{s=t-r(t)}^{t-1}a(t,s)+b(t)\right)\\
&+\sum_{s=t-r(t)}^{t-1}a(t,s)\sum_{u=s}^{t-1}\left(\sum_{v=u-r(u)}^{u-1}a(u,v)x(v)+b(u)x(u-h(u))\right)\\
&+b(t)\sum_{u=t-h(t)}^{t-1}\left(\sum_{v=u-r(u)}^{u-1}a(u,v)x(v)+b(u)x(u-h(u))\right)=0,\ t\geq t_0. \qquad (2.5)
\end{aligned}$$

Then

$$\Delta x(t)+A(t)x(t)+L_x(t)+N_x(t)=0,\ t\geq t_0,$$

where A, L_x and N_x are given by (2.2) and (2.4), respectively. By the variation of constants formula, we get

$$\begin{aligned}
x(t)=\phi(t_0)\prod_{z=t_0}^{t-1}(1-A(z))-\sum_{s=t_0}^{t-1}L_x(s)\prod_{z=s+1}^{t-1}(1-A(z))\\
-\sum_{s=t_0}^{t-1}N_x(s)\prod_{z=s+1}^{t-1}(1-A(z)),\ t\geq t_0. \qquad (2.6)
\end{aligned}$$

Since each step is reversible, the converse follows easily. This completes the proof.

Theorem 2.1 *Suppose that the following two conditions hold*

$$|A(z)|<1,\ \forall z\in[t_0,\infty)\cap\mathbb{Z},\ \lim_{t\to\infty}\prod_{z=0}^{t-1}(1-A(z))\ \textit{exists}, \qquad (2.7)$$

$$\sup_{t\geq 0}\sum_{s=0}^{t-1}\omega(s)\prod_{z=s+1}^{t-1}(1-A(z))=\alpha<1, \qquad (2.8)$$

where

$$\begin{aligned}
\omega(s)=\sum_{w=s-r(s)}^{s-1}|a(s,w)|\sum_{u=w}^{s-1}\left(\sum_{v=u-r(u)}^{u-1}|a(u,v)|+|b(u)|\right)\\
+|b(s)|\sum_{u=s-h(s)}^{s-1}\left(\sum_{v=u-r(u)}^{u-1}|a(u,v)|+|b(u)|\right).
\end{aligned}$$

Then the zero solution of (1.1) is asymptotically stable if and only if

$$\prod_{z=0}^{t-1}(1-A(z))\to 0\ \textit{as}\ t\to\infty. \qquad (2.9)$$

Proof. Sufficient condition. Suppose that (2.9) holds. Denoted by C the space of bounded sequences $x:[m(t_0),\infty)\cap\mathbb{Z}\to\mathbb{R}$ such that $x(t)=\phi(t)$, $t\in[m(t_0),t_0]\cap\mathbb{Z}$. It is known that C is a complete metric space endowed with a metric $\|x\|=\sup_{t\geq m(t_0)}|x(t)|$. Define the operator P on C by $(Px)(t)=\phi(t)$, $t\in[m(t_0),t_0]\cap\mathbb{Z}$ and

$$\begin{aligned}(Px)(t)=\phi(t_0)\prod_{z=t_0}^{t-1}(1-A(z))-\sum_{s=t_0}^{t-1}L_x(s)\prod_{z=s+1}^{t-1}(1-A(z))\\ -\sum_{s=t_0}^{t-1}N_x(s)\prod_{z=s+1}^{t-1}(1-A(z)),\ t\geq t_0.\end{aligned}$$

Obviously, Px is continuous for each $x\in C$. Moreover, it is a contraction operator. Indeed, let $x,y\in C$

$$\begin{aligned}&|(Px)(t)-(Py)(t)|\\ &\leq\sum_{s=t_0}^{t-1}|L_x(s)-L_y(s)|\prod_{z=s+1}^{t-1}(1-A(z))\\ &+\sum_{s=t_0}^{t-1}|N_x(s)-N_y(s)|\prod_{z=s+1}^{t-1}(1-A(z)).\end{aligned}$$

Since $x(t)=y(t)=\phi(t)$ for all $t\in[m(t_0),t_0]\cap\mathbb{Z}$, this implies that

$$\begin{aligned}&|L_x(s)-L_y(s)|\\ &\leq\left(\sum_{w=s-r(s)}^{s-1}|a(s,w)|\sum_{u=w}^{s-1}\left(\sum_{v=u-r(u)}^{u-1}|a(u,v)|+|b(u)|\right)\right)\|x-y\|,\end{aligned}$$

and

$$\begin{aligned}&|N_x(s)-N_y(s)|\\ &\leq\left(|b(s)|\sum_{u=s-h(s)}^{s-1}\left(\sum_{v=u-r(u)}^{u-1}|a(u,v)|+|b(u)|\right)\right)\|x-y\|.\end{aligned}$$

Consequently, it holds for all $t\geq t_0$ that

$$\begin{aligned}&|(Px)(t)-(Py)(t)|\\ &\leq\left[\sum_{s=t_0}^{t-1}\sum_{w=s-r(s)}^{s-1}|a(s,w)|\sum_{u=w}^{s-1}\left(\sum_{v=u-r(u)}^{u-1}|a(u,v)|+|b(u)|\right)\prod_{z=s+1}^{t-1}(1-A(z))\right.\\ &\left.+\sum_{s=t_0}^{t-1}|b(s)|\sum_{u=s-h(s)}^{s-1}\left(\sum_{v=u-r(u)}^{u-1}|a(u,v)|+|b(u)|\right)\prod_{z=s+1}^{t-1}(1-A(z))\right]\|x-y\|.\end{aligned}$$

Hence, it follows from (2.8) that

$$|(Px)(t)-(Py)(t)|\leq\alpha\|x-y\|,\ t\geq t_0.$$

Thus P is a contraction operator on C.

We now consider a closed subspace S of C that is defined by

$$S = \{x \in C : |x(t)| \to 0 \text{ as } t \to \infty\}.$$

We will show that $P(S) \subset S$. To do this, we need to point out that for each $x \in S$, $|(Px)(t)| \to 0$ as $t \to \infty$. Let $x \in S$, by the definition of P we have

$$\begin{aligned}
|(Px)(t)| &\leq \left|\phi(t_0) \prod_{z=t_0}^{t-1} (1 - A(z))\right| \\
&\quad + \left|\sum_{s=t_0}^{t-1} L_x(s) \prod_{z=s+1}^{t-1} (1 - A(z))\right| + \left|\sum_{s=t_0}^{t-1} N_x(s) \prod_{z=s+1}^{t-1} (1 - A(z))\right| \\
&= I_1 + I_2 + I_3, \ t \geq t_0.
\end{aligned}$$

The first term I_1 tends to 0 by (2.9). For any $T \in (t_0, t) \cap \mathbb{Z}$, we have the following estimate for the second term

$$\begin{aligned}
I_2 &\leq \left|\sum_{s=t_0}^{T-1} L_x(s) \prod_{z=s+1}^{t-1} (1 - A(z))\right| + \left|\sum_{s=T}^{t-1} L_x(s) \prod_{z=s+1}^{t-1} (1 - A(z))\right| \\
&\leq \sum_{s=t_0}^{T-1} \sum_{w=s-r(s)}^{s-1} |a(s,w)| \sum_{u=w}^{s-1} \left(\sum_{v=u-r(u)}^{u-1} |a(u,v)| \, \|x\| + |b(u)| \, \|\phi\|_{t_0} \right) \\
&\times \prod_{z=s+1}^{t-1} (1 - A(z)) \\
&+ \sum_{s=T}^{t-1} \sum_{w=s-r(s)}^{s-1} |a(s,w)| \sum_{u=w}^{s-1} \left(\sum_{v=u-r(u)}^{u-1} |a(u,v)| \, |x(v)| + |b(u)| \, |x(u - h(u))| \right) \\
&\times \prod_{z=s+1}^{t-1} (1 - A(z)) \\
&\leq \sum_{s=t_0}^{T-1} \sum_{w=s-r(s)}^{s-1} |a(s,w)| \sum_{u=w}^{s-1} \left(\sum_{v=u-r(u)}^{u-1} |a(u,v)| + |b(u)| \right) \\
&\times \prod_{z=s+1}^{t-1} (1 - A(z)) \left(\|x\| + \|\phi\|_{t_0}\right) \\
&+ \sum_{s=T}^{t-1} \sum_{w=s-r(s)}^{s-1} |a(s,w)| \sum_{u=w}^{s-1} \left(\sum_{v=u-r(u)}^{u-1} |a(u,v)| \, |x(v)| + |b(u)| \, |x(u - h(u))| \right) \\
&\times \prod_{z=s+1}^{t-1} (1 - A(z)) \\
&= I_{21} + I_{22}.
\end{aligned}$$

Since $t - r(t) \to \infty$ as $t \to \infty$, this implies that $u - r(u) \to \infty$ as $T \to \infty$. Thus, from the fact $|x(v)| \to 0, \ v \to \infty$ we can infer that for any $\varepsilon > 0$ there exists $T_1 = T > t_0$ such that

$$I_{22} < \frac{\varepsilon}{2\alpha} \sum_{s=T_1}^{t-1} \sum_{w=s-r(s)}^{s-1} |a(s,w)| \sum_{u=w}^{s-1} \left(\sum_{v=u-r(u)}^{u-1} |a(u,v)| + |b(u)| \right) \prod_{z=s+1}^{t-1} (1 - A(z)),$$

and hence, $I_{22} < \frac{\varepsilon}{2}$ for all $t \geq T_1$. On the other hand, $\|x\| < \infty$ because $x \in S$. This combined with (2.9) yields $I_{21} \to 0$ as $t \to \infty$. As a consequence, there exists $T_2 \geq T_1$ such that $I_{21} < \frac{\varepsilon}{2}$ for all $t \geq T_2$. Thus, $I_2 < \varepsilon$ for all $t \geq T_2$, that is, $I_2 \to 0$ as $t \to \infty$. Similarly, $I_3 \to 0$ as $t \to \infty$. So $P(S) \subset S$.

By the Contraction Mapping Principle, P has a unique fixed point x in S which is a solution of (1.1) with $x(t) = \phi(t)$ on $[m(t_0), t_0] \cap \mathbb{Z}$ and $x(t) = x(t, t_0, \phi) \to 0$ as $t \to \infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (1.1) is stable. By condition (2.7), we can define

$$K = \sup_{t \geq 0} \prod_{z=0}^{t-1} (1 - A(z)) < \infty. \tag{2.10}$$

Using the formula (2.3) and condition (2.8), we can obtain

$$|x(t)| \leq K \|\phi\|_{t_0} \prod_{z=0}^{t_0-1} (1 - A(z))^{-1} + \alpha(\|x\| + \|\phi\|_{t_0}),\ t \geq t_0,$$

which leads us to

$$\|x\| \leq \frac{K \prod_{z=0}^{t_0-1} (1 - A(z))^{-1} + \alpha}{1 - \alpha} \|\phi\|_{t_0}. \tag{2.11}$$

Thus for every, $\varepsilon > 0$, we can find $\delta > 0$ such that $\|\phi\|_{t_0} < \delta$ implies that $\|x\| < \varepsilon$. This shows that the zero solution of (1.1) is stable and hence, it is asymptotically stable.

Necessary condition. Suppose that the zero solution of (1.1) is asymptotically stable and that the condition (2.9) fails. It follows from (2.7) that there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n \to \infty} \prod_{z=0}^{t_n-1} (1 - A(z))^{-1} \text{ exists and is finite.}$$

Hence, we can choose a positive constant L satisfying

$$0 < \prod_{z=0}^{t_n-1} (1 - A(z))^{-1} < L,\ \forall n \geq 1. \tag{2.12}$$

Then, condition (2.8) gives us

$$c_n = \sum_{s=0}^{t_n-1} \omega(s) \prod_{z=0}^{s} (1 - A(z))^{-1} \leq \alpha \prod_{z=0}^{t_n-1} (1 - A(z))^{-1} < L.$$

The sequence $\{c_n\}$ is increasing and bounded, so it has a finite limit. For any $\delta_0 > 0$, there exists $n_0 > 0$ such that

$$\sum_{s=t_{n_0}}^{t_n-1} \omega(s) \prod_{z=0}^{s} (1 - A(z))^{-1} < \frac{\delta_0}{2K},\ \forall n \geq n_0, \tag{2.13}$$

where K is as in (2.10). We choose δ_0 such that $\delta_0 < \frac{1-\alpha}{KL+1}$ and consider the solution $x(t) = x(t, t_n, \phi)$ of (1.1) with the initial data $\phi(t_{n_0}) = \delta_0$ and $|\phi(s)| \le \delta_0$, $s \le t_{n_0}$. It follows from (2.11) that

$$|x(t)| \le 1 - \delta_0, \ \forall t \ge t_{n_0}. \tag{2.14}$$

Applying the fundamental inequality $|a - b - c| \ge |a| - |b| - |c|$ and then using (2.14), (2.13) and (2.12), we get

$$\begin{aligned}
|x(t_n)| &\ge \left|\phi(t_{n_0}) \prod_{z=t_{n_0}}^{t_n-1} (1 - A(z))\right| - \left|\sum_{s=t_{n_0}}^{t_n-1} L_x(s) \prod_{z=s+1}^{t_n-1} (1 - A(z))\right| \\
&\quad - \left|\sum_{s=t_{n_0}}^{t_n-1} N_x(s) \prod_{z=s+1}^{t_n-1} (1 - A(z))\right| \\
&\ge \delta_0 \prod_{z=t_{n_0}}^{t_n-1} (1 - A(z)) - \sum_{s=t_{n_0}}^{t_n-1} \omega(s) \prod_{z=s+1}^{t_n-1} (1 - A(z)) \\
&\ge \prod_{z=t_{n_0}}^{t_n-1} (1 - A(z)) \left(\delta_0 - \prod_{z=0}^{t_{n_0}-1} (1 - A(z)) \sum_{s=t_{n_0}}^{t_n-1} \omega(s) \prod_{z=0}^{s} (1 - A(z))^{-1}\right) \\
&\ge \prod_{z=t_{n_0}}^{t_n-1} (1 - A(z)) \left(\delta_0 - K \sum_{s=t_{n_0}}^{t_n-1} \omega(s) \prod_{z=0}^{s} (1 - A(z))^{-1}\right) \\
&\ge \frac{1}{2}\delta_0 \prod_{z=t_{n_0}}^{t_n-1} (1 - A(z)) \ge \frac{\delta_0}{2LK} > 0,
\end{aligned}$$

which is a contradiction because $x(t_n) \to 0$ as $t_n \to \infty$. The proof is complete.

Let $b(t) = 0$ we get the following corollary.

Corollary 2.1 *Suppose that the following two conditions hold*

$$|A_0(z)| < 1, \ \forall z \in [t_0, \infty) \cap \mathbb{Z}, \ \lim_{t \to \infty} \prod_{z=0}^{t-1} (1 - A_0(z)) \textit{ exists}, \tag{2.15}$$

$$\sup_{t \ge 0} \sum_{s=0}^{t-1} \omega_0(s) \prod_{z=s+1}^{t-1} (1 - A_0(z)) = \alpha < 1, \tag{2.16}$$

where

$$A_0(z) = \sum_{s=z-r(z)}^{z-1} a(z, s),$$

and

$$\omega_0(s) = \sum_{w=s-r(s)}^{s-1} |a(s, w)| \sum_{u=w}^{s-1} \sum_{v=u-r(u)}^{u-1} |a(u, v)|.$$

Then the zero solution of

$$\varDelta x(t)+\sum_{s=t-r(t)}^{t-1}a(t,s)x(s)=0,$$

is asymptotically stable if and only if

$$\prod_{z=0}^{t-1}(1-A_0(z))\to 0 \text{ as } t\to\infty. \tag{2.17}$$

Let $a(t,s)=0$, we get the following.

Corollary 2.2 *Suppose that the following two conditions hold*

$$|b(z)|<1,\ \forall z\in[t_0,\infty)\cap\mathbb{Z},\ \lim_{t\to\infty}\prod_{z=0}^{t-1}(1-b(z)) \text{ exists}, \tag{2.18}$$

$$\sup_{t\geq 0}\sum_{s=0}^{t-1}|b(s)|\sum_{u=s-h(s)}^{s-1}|b(u)|\prod_{z=s+1}^{t-1}(1-b(z))=\alpha<1. \tag{2.19}$$

Then the zero solution of

$$\varDelta x(t)+b(t)x(t-h(t))=0,$$

is asymptotically stable if and only if

$$\prod_{z=0}^{t-1}(1-b(z))\to 0 \text{ as } t\to\infty. \tag{2.20}$$

Next we turn our attention to the following delay Levin-Nohel difference equation with several delays

$$\varDelta x(t)+\sum_{k=1}^{M}\sum_{s=t-r_k(t)}^{t-1}a_k(t,s)x(s)+\sum_{k=1}^{M}b_k(t)x(t-h_k(t))=0,\ t\geq t_0, \tag{2.21}$$

with the initial condition

$$x(t)=\phi(t),\ t\in[m(t_0),t_0]\cap\mathbb{Z},$$

where $\phi:[m(t_0),t_0]\cap\mathbb{Z}\to\mathbb{R}$ is a bounded sequence and for $t_0\geq 0$

$$m_k(t_0)=\inf\{s-r_k(s):s\geq t_0\},\ n_k(t_0)=\inf\{s-h(s):s\geq t_0\},$$
$$m(t_0)=\min_{1\leq k\leq M}(m_k(t_0),n_k(t_0)).$$

We assume that $b_k:[0,\infty)\cap\mathbb{Z}\to\mathbb{R}$, $a_k:[0,\infty)\cap\mathbb{Z}\times[m(0),\infty)\cap\mathbb{Z}\to\mathbb{R}$ and $\tau_k,h_k:[0,\infty)\cap\mathbb{Z}\to\mathbb{Z}^+$ with $t-\tau_k(t)\to\infty$ and $t-h_k(t)\to\infty$ as $t\to\infty$, $1\leq k\leq M$.

Lemma 2.2 *Suppose that*

$$\overline{A}(z) \neq 1,\ \forall z \in [t_0, \infty) \cap \mathbb{Z}, \tag{2.22}$$

where

$$\overline{A}(z) = \sum_{k=1}^{M} \left(\sum_{s=z-r_k(z)}^{z-1} a_k(z,s) + b_k(z) \right).$$

Then x is a solution of equation (2.21) if and only if

$$\begin{aligned} x(t) = \phi(t_0) \prod_{z=t_0}^{t-1} (1 - \overline{A}(z)) - \sum_{s=t_0}^{t-1} \overline{L}_x(s) \prod_{z=s+1}^{t-1} (1 - \overline{A}(z)) \\ - \sum_{s=t_0}^{t-1} \overline{N}_x(s) \prod_{z=s+1}^{t-1} (1 - \overline{A}(z)),\ t \geq t_0, \end{aligned}$$

where

$$\overline{L}_x(s) = \sum_{k=1}^{M} \sum_{w=s-r_k(s)}^{s-1} a_k(s,w) \sum_{u=w}^{s-1} \sum_{i=1}^{M} \left(\sum_{v=u-r_i(u)}^{u-1} a_i(u,v)x(v) + b_i(u)x(u-h_i(u)) \right),$$

$$\overline{N}_x(s) = \sum_{k=1}^{M} b_k(s) \sum_{u=s-h_k(s)}^{s-1} \sum_{i=1}^{M} \left(\sum_{v=u-r_i(u)}^{u-1} a_i(u,v)x(v) + b_i(u)x(u-h_i(u)) \right).$$

The proof follows along the lines of Lemma 2.1, and hence we omit it.

Theorem 2.2 *Suppose that the following two conditions hold*

$$\left|\overline{A}(z)\right| < 1,\ \forall z \in [t_0, \infty) \cap \mathbb{Z},\ \lim_{t\to\infty} \prod_{z=0}^{t-1} (1 - \overline{A}(z)) \textit{ exists},$$

$$\sup_{t\geq 0} \sum_{s=0}^{t-1} \overline{\omega}(s) \prod_{z=s+1}^{t-1} (1 - \overline{A}(z)) = \alpha < 1,$$

where

$$\begin{aligned} \overline{\omega}(s) = \sum_{k=1}^{M} \sum_{w=s-r_k(s)}^{s-1} |a_k(s,w)| \sum_{u=w}^{s-1} \sum_{i=1}^{M} \left(\sum_{v=u-r_i(u)}^{u-1} |a_i(u,v)| + |b_i(u)| \right) \\ + \sum_{k=1}^{M} |b_k(s)| \sum_{u=s-h_k(s)}^{s-1} \sum_{i=1}^{M} \left(\sum_{v=u-r_i(u)}^{u-1} |a_i(u,v)| + |b_i(u)| \right). \end{aligned}$$

Then the zero solution of (2.21) is asymptotically stable if and only if

$$\prod_{z=0}^{t-1} (1 - \overline{A}(z)) \to 0 \textit{ as } t \to \infty.$$

The proof is similar to that of Theorem 2.1, and hence, we omit it.

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