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Inverse boundary value problem for one partial differential equation of third order

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Abstract. *In this paper, we consider the boundary value problem for one partial differential equation of third order. The existence and uniqueness of the inverse boundary value problem for this equation are proved.*

Keywords. boundary value problem, partial differential operator of third order, spectral parameter, eigenfunction, Riesz basis

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1 Introduction

Let $D_T = \{(x,t) : 0 < x < 1,\, 0 < t < T\}$, and let $f(x,t)$, $\varphi_i(x), i = \overline{0,2}$ and $h(t)$ are the given functions defined for $x \in [0,1]$ and $t \in [0,T]$.

In this paper we shall discuss the following inverse boundary value problem: find the functions $u(x,t)$ and $a(t)$ connected in the area D_T by equation

$$u_{ttt}(x,t) + u_{xx}(x,t) = a(t)u(x,t) + f(x,t) \tag{1.1}$$

under the fulfillment of the following initial and boundary conditions, and the condition of redefinition for the function $u(x,t)$:

$$u(x,0) = \varphi_0(x),\ u_t(x,0) = \varphi_1(x),\ u_{tt}(x,T) = \varphi_2(x)\ 0 \le x \le 1, \tag{1.2}$$

$$u(0,t) = u_x(1,t) = 0,\ 0 \le t \le T, \tag{1.3}$$

$$u(1,t) = h(t),\ 0 \le t \le T. \tag{1.4}$$

Recently, inverse boundary value problems have found very wide application in various fields of science: geophysics, aerodynamics, hydrodynamics, filtration theory, explosion theory, mineral exploration, biology, medicine, computer tomography, etc. Various inverse problems for individual types of partial differential equations studied in many works. We

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note here, first of all, Tikhonov [13], Lavrent'ev [4], Lavrent'ev, Romanov and Shishatsky [5], Ivanov, Vasin and Tanina [2] and others. For more information, see, for example, Denisov [1].

Different boundary value problems for a partial differential equation of third order were investigated in [6, 7]. In [12] the inverse problem is considered to determine the solution of a third-order equation and the unknown right-hand side of this equation.

In this paper, the existence and uniqueness of the solution of the inverse boundary value problem (1.1)-(1.4) are proved.

2 Statement of the problem and reduction it to an equivalent problem

To study problem (1.1)-(1.4), we first consider the following problem

$$y'''(t) = a(t)y(t),\ 0 < t < T, \tag{2.1}$$

$$y(0) = 0,\ y'(0) = 0,\ y''(T) = 0, \tag{2.2}$$

where $a(t) \in C[0,T]$ is given function and $y(t)$ is desired function (by the solution of problem (2.1)-(2.2) we mean the function $y(t)$ belonging to $C^3(0,T)$ and satisfying conditions (2.1)-(2.2) in the usual sense).

Lemma 2.1 *Let* $a(t) \in C[0,T]$ *such that*

$$\|a(t)\|_\infty \le R = const,$$

where $\|\cdot\|_\infty$ *is the standard sup-norm in* $C[0,T]$ *and*

$$\frac{2T^3R}{3} < 1. \tag{2.3}$$

Then problem (2.1)-(2.2) *has only trivial solution.*

Proof. It is easy see that the problem

$$\begin{aligned} &y'''(t) = 0,\ 0 < t < T,\\ &y(0) = 0,\ y'(0) = 0,\ y''(T) = 0 \end{aligned} \tag{2.4}$$

has only trivial solution. Then problem (2.4) has one Green function and boundary value problem (2.1)-(2.2) is equivalent the following integral equation

$$y(t) = \int_0^T G(t,\tau)y(\tau)\,d\tau,\ \ 0 \le t \le T, \tag{2.5}$$

where

$$G(t,\tau) = \begin{cases} -\frac{t^2}{2}, & t \in [0,\tau],\\ -t\tau + \frac{\tau^2}{2}, & t \in [\tau,T]. \end{cases} \tag{2.6}$$

Setting

$$A(y(t)) = \int_0^T G(t,\tau)a(\tau)y(\tau)\,d\tau, \tag{2.7}$$

we can write Eq. (2.5) in the form

$$y(t) = Ay(t). \tag{2.8}$$

We will study equation Eq. (2.8) in the space $C[0,T]$. It is easy to see that the operator A is continuous in $C[0,T]$. Now we show that this operator is contraction in space $C[0,T]$. Indeed, for any $y(t), \overline{y}(t) \in C[0,T]$ we have

$$\|Ay(t) - A\overline{y}(t)\|_\infty \le \|a(t)\|_\infty \|y(t) - \overline{y}(t)\|_\infty \int_0^T |G(t,\tau)| \, d\tau. \tag{2.9}$$

From (2.6) we obtain

$$\int_0^T |G(t,\tau)| \, d\tau = \frac{1}{6}t^3 + \frac{1}{2}t^2 T.$$

Obviously, the function $g(t) = \frac{1}{6}t^3 + \frac{1}{2}t^2 T, \ 0 \le t \le T$ takes its maximum value in $C[0,T]$ at $t = T$ and $g(T) = \frac{2}{3}T^3$. Therefore,

$$\int_0^T |G(t,\tau)| \, d\tau \le \frac{2}{3}T^3, \ 0 \le t \le T. \tag{2.10}$$

Now from (2.9) with the use of (2.10), we get

$$\|Ay(t) - A\overline{y}(t)\|_\infty \le \frac{2}{3}T^3 \|a(t)\|_\infty \|y(t) - \overline{y}(t)\|_\infty . \tag{2.11}$$

Then by (2.3) it follows from (2.11) that the operator A is contraction in $C[0,T]$. Hence the operator A in the space $C[0,T]$ has unique fixed point which is a solution of Eq. (2.8). Thus, the integral equation (2.5) in $C[0,T]$ has unique solution. Since $y(t) = 0$ is a solution of the boundary value problem (2.1)-(2.2) this problem has only a trivial solution. The proof of this lemma is complete.

Along with problem (1.1)-(1.4) we consider the following auxiliary inverse problem: it is required to determine a pair $\{u(x,t), a(t)\}$ of functions $u(x,t)$ and $a(t)$ having properties 1 and 2 from the definition of classical solutions of problem (1.1)-(1.4), relations (1.1)-(1.3) and

$$h'''(t) + u_{xx}(1,t) = a(t)h(t) + f(1,t), \ 0 < t < T. \tag{2.12}$$

Lemma 2.2 *Let* $\varphi_i(x) \in C[0,1]$, $i = \overline{0,2}$, $f(x,t) \in C(\overline{D}_T)$, $h(t) \in C^3[0,T]$, $h(t) \ne 0$ *for* $t \in [0,T]$ *and the following matching conditions hold*

$$\varphi_0(1) = h(0), \quad \varphi_1(1) = h'(0), \quad \varphi_2(1) = h''(T). \tag{2.13}$$

Then the following statements are true:

(i) each classical solution $\{u(x,t), a(t)\}$ *of problem* (1.1)-(1.4) *is also solution of problem* (1.1)-(1.3), (2.12) *;*

(ii) each solution $\{u(x,t), a(t)\}$ *of problem* (1.1)-(1.3), (2.12) *such that*

$$\frac{2\,T^3 \|a(t)\|_\infty}{3} < 1 \tag{2.14}$$

is a classical solution of problem (1.1)-(1.4).

Proof. Let $\{u(x,t), a(t)\}$ be a solution of problem (1.1)-(1.4). Considering the condition $h(t) \in C^3[0,T]$ and differentiating three times (1.4), we obtain

$$u_{ttt}(1,t) = h'''(t),\ 0 \le t \le T. \tag{2.15}$$

Putting $x = 1$ in Eq. (2.1) we have

$$u_{ttt}(1,t) + u_{xx}(1,t) = a(t)u(1,t) + f(1,t),\ 0 \le t \le T. \tag{2.16}$$

By virtue of (1.4) and (2.15) it follows from (2.16) that (2.12) holds.

Now suppose that $\{u(x,t), a(t)\}$ is a solution of problem (1.1)-(1.3), (2.12) and the condition (2.14) hold. Then it follows from (2.12) and (2.16) that

$$\frac{d^3}{dt^3}\left(u(1,t) - h(t)\right) = a(t)\left(u(1,t) - h(t)\right),\ 0 \le t \le T. \tag{2.17}$$

Next, by virtue of condition (1.2) and matching conditions (2.13), we get

$$u\left(1,0\right) - h\left(0\right) = \varphi_0\left(1\right) - h\left(0\right) = 0,$$

$$u_t\left(1,0\right) - h'\left(0\right) = \varphi_1\left(1\right) - h'\left(0\right) = 0,$$

$$u_{tt}\left(1,T\right) - h''\left(T\right) = \varphi_2\left(1\right) - h''\left(T\right) = 0. \tag{2.18}$$

By Lemma 2.1 it follows from (2.17) and (2.18) that the condition (1.4) holds. The proof of this lemma is complete.

3 The existence and uniqueness of a classical solution of the inverse boundary value problem (1.1)-(1.4)

The first component $u(x,t)$ of solution $\{u(x,t), a(t)\}$ of problem (1.1)-(1.3), (2.12) we will seek in the following form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin \lambda_k x,\ \lambda_k = \frac{\pi}{2}(2k-1), \tag{3.1}$$

where $u_k(t) = 2\int\limits_0^1 u(x,t) \sin \lambda_k x\, dx,\ k = 1,\, 2,\, \ldots$. Then, applying the formal Fourier scheme, from (1.1) and (1.2) we obtain

$$u_k'''(t) - \lambda_k^2 u_k(t) = F_k(t; u, a),\ k = 1,\, 2,\, \ldots;\ 0 \le t \le T, \tag{3.2}$$

$$u_k(0) = \varphi_{0k}, \quad u_k'(0) = \varphi_{1k},\ u_k''(T) = \varphi_{2k}, \tag{3.3}$$

where

$$F_k(t; u, a) = f_k(t) + a(t)u_k(t), \quad f_k(t) = 2\int\limits_0^1 f(x,t) \sin \lambda_k x\, dx,$$

$$\varphi_{ik} = 2\int\limits_0^1 \varphi_i(x) \sin \lambda_k x\, dx,\ i = \overline{0,2};\ k = 1,\, 2,\, \ldots\ .$$

In order to convert problem (3.2)-(3.3) to problem with homogenous boundary conditions we introduce the desired function

$$u_k\left(t\right)=v_k\left(t\right)+\frac{\varphi_{2k}}{2}t^2+t\varphi_{1k}+\varphi_{0k}, \tag{3.4}$$

where $v_k\left(t\right)$ is a solution of the following problem

$$v_k'''(t)-\lambda_k^2 v_k(t)=g_k(t;v_k,a),\ k=1,\,2,\,\ldots;\,0\le t\le T, \tag{3.5}$$

$$v_k(0)=v_k'(0)=v_k''(T)=0,\ k=1,\,2,\,\ldots, \tag{3.6}$$

where

$$g_k(t;v_k,a)=f_k(t)+a(t)v_k\left(t\right)+\left(a(t)+\lambda_k^2\right)\left(\frac{\varphi_{2k}}{2}t^2+t\varphi_{1k}+\varphi_{0k}\right). \tag{3.7}$$

We first consider the homogeneous equation corresponding to (3.5)

$$v_k'''(t)-\lambda_k^2 v_k(t)=0,\ k=1,\,2,\,\ldots;\,0\le t\le T. \tag{3.8}$$

It is obvious that the general solution of Eq. (3.5) has the form

$$v_k\left(t\right)=c_{1k}e^{\lambda_k^{\frac{2}{3}}t}+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\left(c_{2k}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+c_{3k}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\right), \tag{3.9}$$

where c_{1k},c_{2k},c_{3k} are arbitrary constants.

We can show that the problem

$$v_k'''(t)-\lambda_k^2 v_k(t)=0,\quad v_k(0)=v_k'(0)=v_k''(T)=0 \tag{3.10}$$

has only a trivial solution. It is known (see [12]) that problem (3.10) has only one Green function, where the Green function of this problem is the function $G_k(t,\tau)$ that satisfies the following conditions:

1) the function $G_k(t,\tau)$ is continuous and has continuous derivative with respect to t for any $t,\ \tau\ \in[0,T]$;

2) for any fixed $\tau\in[0,T]$ the function $G_k(t,\tau)$ has continuous derivatives of second and third orders with respect to t in each of intervals $[0,\tau)$ and $(\tau,T]$ and the derivative of third order for $t=\tau$ has a leap equal to 1:

$$G_{ktt}(\tau+0,\tau)-G_{ktt}(\tau-0,\tau)=1. \tag{3.11}$$

3) in each of intervals $[0,\tau)$ and $(\tau,T]$ the function $G_k(t,\tau)$ that regarded as a function of t satisfies Eq. (3.8) and boundary conditions (3.6).

Since the general solution of homogeneous equation (3.8) has the form (3.9) the Green function for boundary value problem (3.10) has the form

$$G_k(t,\tau)=\begin{cases}c_{1k}e^{\lambda_k^{\frac{2}{3}}t}+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\left(c_{2k}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+c_{3k}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\right),\ t\in[0,\tau],\\ \overline{c}_{1k}e^{\lambda_k^{\frac{2}{3}}t}+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\left(\overline{c}_{2k}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+\overline{c}_{3k}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\right),\ t\in[\tau,T],\end{cases} \tag{3.12}$$

where $c_{1k},c_{2k},c_{3k},\overline{c}_{1k},\overline{c}_{2k},\overline{c}_{3k}$ are some functions that depend on τ.

Differentiating (3.10) two time on t we obtain

$$G_{kt}(t,\tau)=\begin{cases} c_{1k}\lambda_k^{\frac{2}{3}}e^{\lambda_k^{\frac{2}{3}}t}-\lambda_k^{\frac{2}{3}}e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\left(c_{2k}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{3}\right)\right.\\ \left.+c_{3k}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{3}\right)\right),\ t\in[0,\tau],\\ \overline{c}_{1k}\lambda_k^{\frac{2}{3}}e^{\lambda_k^{\frac{2}{3}}t}-\lambda_k^{\frac{2}{3}}e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\left(\overline{c}_{2k}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{3}\right)\right.\\ \left.+\overline{c}_{3k}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{3}\right)\right),\ t\in[\tau,T], \end{cases} \tag{3.13}$$

$$G_{ktt}(t,\tau)=\begin{cases} c_{1k}\lambda_k^{\frac{4}{3}}e^{\lambda_k^{\frac{2}{3}}t}+\lambda_k^{\frac{4}{3}}e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\left(c_{2k}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{2\pi}{3}\right)\right.\\ \left.+c_{3k}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{2\pi}{3}\right)\right),\ t\in[0,\tau],\\ \overline{c}_{1k}\lambda_k^{\frac{4}{3}}e^{\lambda_k^{\frac{2}{3}}t}+\lambda_k^{\frac{4}{3}}e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\left(\overline{c}_{2k}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{2\pi}{3}\right)\right.\\ \left.+\overline{c}_{3k}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{2\pi}{3}\right)\right),\ t\in[\tau,T], \end{cases} \tag{3.14}$$

Then it can be seen from (3.12) and (3.13) that the continuity of the function $G_k(t,\tau)$ and its derivatives for $t=\tau$ gives us

$$\overline{c}_{1k}e^{\lambda_k^{\frac{2}{3}}\tau}+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\left(\overline{c}_{2k}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau+\overline{c}_{3k}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau\right)$$

$$=c_{1k}e^{\lambda_k^{\frac{2}{3}}\tau}+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\left(c_{2k}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau+c_{3k}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau\right),$$

$$\overline{c}_{1k}e^{\lambda_k^{\frac{2}{3}}\tau}-e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\left(\overline{c}_{2k}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{\pi}{3}\right)+\overline{c}_{3k}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{\pi}{3}\right)\right)$$

$$=c_{1k}e^{\lambda_k^{\frac{2}{3}}\tau}-e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\left(c_{2k}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{\pi}{3}\right)+c_{3k}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{\pi}{3}\right)\right),$$

and from (3.14) we see that the condition (3.11) can be written as

$$\overline{c}_{1k}\lambda_k^{\frac{4}{3}}e^{\lambda_k^{\frac{2}{3}}\tau}+\lambda_k^{\frac{4}{3}}e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\left(\overline{c}_{2k}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{2\pi}{3}\right)\right.$$

$$\left.+\overline{c}_{3k}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{2\pi}{3}\right)\right)-\left[c_{1k}\lambda_k^{\frac{4}{3}}e^{\lambda_k^{\frac{2}{3}}\tau}\right.$$

$$\left.+\lambda_k^{\frac{4}{3}}e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\left(c_{2k}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{2\pi}{3}\right)+c_{3k}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{2\pi}{3}\right)\right)\right]=1$$

Denote by

$$\gamma_{1k}=\overline{c}_{1k}-c_{1k},\quad \gamma_{2k}=\overline{c}_{2k}-c_{2k},\quad \gamma_{3k}=\overline{c}_{3k}-c_{3k},$$

we obtain a record of the last relations in the form of a system of equations:

$$\begin{cases} \gamma_{1k}e^{\lambda_k^{\frac{2}{3}}\tau}+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\left(\gamma_{2k}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau+\gamma_{3k}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau\right)=0, \\ \overline{\gamma}_{1k}e^{\lambda_k^{\frac{2}{3}}\tau}-e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\left(\gamma_{2k}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{\pi}{3}\right)+\gamma_{3k}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{\pi}{3}\right)\right)=0, \\ \gamma_{1k}e^{\lambda_k^{\frac{2}{3}}\tau}+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\left(\gamma_{2k}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{2\pi}{3}\right)+\gamma_{3k}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau-\frac{2\pi}{3}\right)\right)= \\ \frac{1}{\lambda_k^{\frac{4}{3}}}, \end{cases} \tag{3.15}$$

whose determinant is the Wronskian of the fundamental system

$$e^{\lambda_k^{\frac{2}{3}}t},\quad e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t,\quad e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t$$

for $t=\tau$, and consequently, is different from zero. Therefore, the system (3.15) uniquely defines a function $\gamma_{ik},\ i=\overline{1,3}$:

$$\begin{cases} \gamma_{1k}=\frac{1}{3\lambda_k^{\frac{4}{3}}}e^{-\lambda_k^{\frac{2}{3}}\tau}, \\ \gamma_{2k}=\frac{2}{3\sqrt{3}\lambda_k^{\frac{4}{3}}}e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\left(\sin\frac{\sqrt{3}}{2}\left(\lambda_k^{\frac{2}{3}}\tau-\frac{\pi}{3}\right)+\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau\right), \\ \gamma_{3k}=-\frac{2}{3\sqrt{3}\lambda_k^{\frac{4}{3}}}e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\left(\cos\frac{\sqrt{3}}{2}\left(\lambda_k^{\frac{2}{3}}\tau-\frac{\pi}{3}\right)+\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\tau\right). \end{cases} \tag{3.16}$$

It is obvious that

$$\overline{c}_{1k}=c_{1k}+\gamma_{1k},\quad \overline{c}_{2k}=c_{2k}+\gamma_{2k},\quad \overline{c}_{3k}=c_{3k}+\gamma_{3k},$$

where $\gamma_{ik},\ \imath=\overline{1,3}$ determined by (3.16).

Substituting the last relation in (3.12) after some calculations, we obtain

$$G_k(t,\tau)=\begin{cases} c_{1k}e^{\lambda_k^{\frac{2}{3}}t}+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\left(c_{2k}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+c_{3k}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\right),\ t\in[0,\tau], \\ c_{1k}e^{\lambda_k^{\frac{2}{3}}t}+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\left(c_{2k}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t+c_{3k}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\right)+ \\ +\frac{1}{3\lambda_k^{\frac{4}{3}}}e^{\lambda_k^{\frac{2}{3}}(t-\tau)}-\frac{2}{3\lambda_k^{\frac{4}{3}}}e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}(t-\tau)}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}(t-\tau)+\frac{\pi}{6}\right),\ t\in[\tau,T], \end{cases} \tag{3.17}$$

where $c_{ik},\ i=\overline{1,3}$, are some arbitrary constants.

To determine $c_{ik},\ i=\overline{1,3}$, we use the boundary conditions (3.6). Then we have

$$\begin{cases} c_{1k}+c_{2k}=0, \\ c_{1k}-\frac{1}{2}c_{2k}+\frac{\sqrt{3}}{2}c_{3k}=0, \\ c_{1k}\lambda_k^{\frac{4}{3}}e^{\lambda_k^{\frac{2}{3}}T}+\lambda_k^{\frac{4}{3}}e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}T}\left(c_{2k}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{2\pi}{3}\right)+c_{3k}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{2\pi}{3}\right)\right) \\ +\frac{1}{3}e^{\lambda_k^{\frac{2}{3}}(T-\tau)}+\frac{2}{3}e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}(T-\tau)}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}(T-\tau)=0. \end{cases}$$

Solving the last system, we find

$$c_{1k}=-\frac{1}{3\lambda_k^{\frac{4}{3}}}\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)^{-1}\left(e^{\lambda_k^{\frac{2}{3}}\left(\frac{3}{2}T-\tau\right)}+2e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}(T-\tau)\right),$$

$$c_{2k}=\frac{1}{3\lambda_k^{\frac{4}{3}}}\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)^{-1}\left(e^{\lambda_k^{\frac{2}{3}}\left(\frac{3}{2}T-\tau\right)}+2e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\left(T-\tau\right)\right),$$

$$c_{3k}=\frac{\sqrt{3}}{3\lambda_k^{\frac{4}{3}}}\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)^{-1}\left(e^{\lambda_k^{\frac{2}{3}}\left(\frac{3}{2}T-\tau\right)}+2e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}\tau}\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\left(T-\tau\right)\right).$$

Substituting the resulting expression for $c_{ik},\ i=\overline{1,3}$, in (3.17) using some transformations we obtain the Green function $G_k(t,\tau)$ for the boundary value problem (3.10):

$$G_k(t,\tau)=\begin{cases}\alpha_k(T,t,\tau), t\in[0,\tau],\\ \beta_k(T,t,\tau), t\in[\tau,T],\end{cases}\tag{3.18}$$

where

$$\alpha_k(T,t,\tau)=-\frac{1}{3\lambda_k^{\frac{4}{3}}}\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)^{-1}\left\{e^{\lambda_k^{\frac{2}{3}}\left(\frac{3}{2}T+t-\tau\right)}\right.$$

$$-2e^{\lambda_k^{\frac{2}{3}}\left(\frac{3}{2}T-\frac{t}{2}-\tau\right)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{3}\right)$$

$$\left.+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\left(T-\tau\right)\left(e^{\lambda_k^{\frac{2}{3}}\left(t+\frac{1}{2}\tau\right)}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}(t-\tau)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{3}\right)\right)\right\},$$

$$\beta_k(T,t,\tau)=$$

$$=-\frac{1}{3\lambda_k^{\frac{4}{3}}}\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)^{-1}\left\{-2e^{\lambda_k^{\frac{2}{3}}\left(\frac{3}{2}T-\frac{t}{2}-\tau\right)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{3}\right)\right.$$

$$+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\left(T-\tau\right)\left(e^{\lambda_k^{\frac{2}{3}}\left(t+\frac{1}{2}\tau\right)}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}(t-\tau)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{3}\right)\right)$$

$$+2e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(3T-(t-\tau))}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\left(t-\tau\right)+\frac{\pi}{6}\right)$$

$$\left.-2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\left(e^{\lambda_k^{\frac{2}{3}}(t-\tau)}-2e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}(t-\tau)}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}\left(t-\tau\right)+\frac{\pi}{6}\right)\right)\right\}.$$

Note that the boundary value problem (3.5)-(3.6) is equivalent to the following integral equation

$$v_k(t)=\int_0^T G_k(t,\tau)g_k(\tau;v_k,a)\,d\tau,\ \ 0\le t\le T.\tag{3.19}$$

Substituting (3.19) in (3.4) and using (3.7), we get

$$u_k\left(t\right)=\frac{\varphi_{2k}}{2}t^2+t\varphi_{1k}+\varphi_{0k}+$$

$$+\int_0^T G_k(t,\tau)\left[F_k(\tau;u_k,a)+\lambda_k^2\left(\frac{\varphi_{2k}}{2}\tau^2+\tau\varphi_{1k}+\varphi_{0k}\right)\right]d\tau.\tag{3.20}$$

From here, after some calculations, we obtain

$$u_k(t)=$$

$$=\left(\sqrt{3}\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)\right)^{-1}\left\{2\varphi_{0k}\left[-e^{\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{\pi}{3}\right)\right.\right.$$

$$\left.+e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(3T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{6}\right)-e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}(T-t)-\frac{2\pi}{3}\right)\right]$$

$$+\frac{2}{\lambda_k^{\frac{2}{3}}}\varphi_{1k}\left[-e^{\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{2\pi}{3}\right)\right.$$

$$\left.+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}(T-t)-\frac{2\pi}{3}\right)+e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(3T-t)}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\right]$$

$$\left.+\frac{\sqrt{3}}{\lambda_k^{3}}\varphi_{2k}\left[e^{\lambda_k^{\frac{2}{3}}\left(\frac{T}{2}+t\right)}-2e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{3}\right)\right]\right\}$$

$$+\int_0^T G_k(t,\tau)F_k(\tau;u,a)\,d\tau. \tag{3.21}$$

After substituting (3.21) into (3.1) for to determine the component $u(x,t)$ of solution $(u(x,t),a(t))$ of problem (1.1)-(1.3), (2.12), we obtain

$$u(x,t)=$$

$$=\sum_{k=1}^{\infty}\left\{\left(\sqrt{3}\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)\right)^{-1}\left\{2\varphi_{0k}\left[-e^{\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{\pi}{3}\right)\right.\right.\right.$$

$$\left.+e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(3T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{6}\right)-e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}(T-t)-\frac{2\pi}{3}\right)\right]$$

$$+\frac{2}{\lambda_k^{\frac{2}{3}}}\varphi_{1k}\left[-e^{\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{2\pi}{3}\right)\right.$$

$$\left.+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}(T-t)-\frac{2\pi}{3}\right)+e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(3T-t)}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\right]$$

$$\left.+\frac{\sqrt{3}}{\lambda_k^{3}}\varphi_{2k}\left[e^{\lambda_k^{\frac{2}{3}}\left(\frac{T}{2}+t\right)}-2e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{3}\right)\right]\right\}$$

$$\left.+\int_0^T G_k(t,\tau)F_k(\tau;u,a)\,d\tau\right\}\sin\lambda_k x. \tag{3.22}$$

Now by (3.1) from (2.12) we find

$$a(t)=[h(t)]^{-1}\left\{h'''(t)-f(1,t)-\sum_{k=1}^{\infty}(-1)^{k+1}\lambda_k^2u_k(t)\right\}. \tag{3.23}$$

In order to obtain the equation for the second component $a(t)$ of solution $\{u(x,t),a(t)\}$ of problem (1.1)-(1.3), (2.12), we substitute (3.21) in (3.23):

$$a(t)=[h(t)]^{-1}\left\{h'''(t)-f(1,t)\right.$$

$$-\sum_{k=1}^{\infty}\left\{\left(\sqrt{3}\left(e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}+2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T\right)\right)^{-1}\left\{2\varphi_{0k}\left[-e^{\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{\pi}{3}\right)\right.\right.\right.$$

$$\left.+e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(3T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{6}\right)-e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}(T-t)-\frac{2\pi}{3}\right)\right]$$

$$+\frac{2}{\lambda_k^{\frac{2}{3}}}\varphi_{1k}\left[-e^{\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T-\frac{2\pi}{3}\right)\right.$$

$$\left.+e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}t}\sin\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}(T-t)-\frac{2\pi}{3}\right)+e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(3T-t)}\sin\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t\right]$$

$$\left.+\frac{\sqrt{3}}{\lambda_k^{3}}\varphi_{2k}\left[e^{\lambda_k^{\frac{2}{3}}\left(\frac{T}{2}+t\right)}-2e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(T-t)}\cos\left(\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}t-\frac{\pi}{3}\right)\right]\right\}$$

$$\left.\left.+\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau\right\}(-1)^{k+1}\lambda_k^2\right\}. \tag{3.24}$$

Therefore, the solution of problem (1.1)-(1.3), (2.12) is reduces to the system (3.22)-(3.23) with respect to the unknown functions $a(t)$ and $u(x,t)$.

The following lemma plays an important role in studying the question of the uniqueness of a solution to problem (1.1)-(1.3), (2.12).

Lemma 3.1 *If* $\{u(x,t),a(t)\}$ *is an any solution of problem* (1.1)-(1.3), (2.12)*, then the functions*

$$u_k(t)=2\int_0^1 u(x,t)\sin\lambda_k x\,dx,\ \ k=1,\,2,\,\dots\ ,$$

satisfy on $[0,T]$ *system* (3.21).

Proof. Let $\{u(x,t),a(t)\}$ be the any solution of problem (1.1)-(1.3), (2.12). Then multiplying both sides of Eq. (1.1) on the function $2\sin\lambda_k x,\ k=1,\,2,\,\dots$, integrating resulting equality over x from 0 to 1 and using the relations

$$2\int_0^1 u_{ttt}(x,t)\sin\lambda_k x\,dx=\frac{d^3}{dt^3}(2\int_0^1 u(x,t)\sin\lambda_k x\,dx)=u_k'''(t),\ \ k=1,\,2,\,\dots\ ,$$

$$2\int_0^1 u_{xx}(x,t)\sin\lambda_k x\,dx = -\lambda_k^2(2\int_0^1 u(x,t)\sin\lambda_k x\,dx) = -\lambda_k^2 u_k(t),\ k=1,\,2,\,\ldots$$

we obtain that Eq. (3.2) holds.

In a similar way from (1.2) we obtain that condition (3.3) holds.

Thus, $u_k(t),\ k=1,\,2,\,\ldots$, is a solution of problem (3.2)-(3.3). It follows immediately that the functions $u_k(t),\ k=1,\,2,\,\ldots$, satisfy on $[0,T]$ system (3.21). The proof of this theorem is complete.

It is obvious that if $u_k(t) = 2\int_0^1 u(x,t)\sin\lambda_k x\,dx\ k=1,\,2,\,\ldots$, is a solution of system (3.21), then the pair $\{u(x,t),a(t)\}$ of functions $u(x,t)=\sum_{k=1}^{\infty} u_k(t)\sin\lambda_k(x)$ and $a(t)$ is a solution of system (3.22), (3.24).

It follows from Lemma 3.1 that the following result is true.

Corollary 3.1 *Suppose that the system* (3.22), (3.24) *has an unique solution. If the problem* (1.1)-(1.3), (2.12) *has a solution, then this solution is unique.*

Now, in order to study problem (1.1)-(1.3), (2.12), we consider the following spaces:

1. Denote by $B_{2,T}^3$ (see [3]) set of all functions $u(x,t)$ of the form

$$u(x,t)=\sum_{k=1}^{\infty} u_k(t)\sin\lambda_k x,\ \lambda_k=\frac{\pi}{2}(2k-1),$$

considered D_T, where each of functions $u_k(t),\ k=1,\,2,\,\ldots$, is continuous on $[0,T]$ and

$$J_T(u)\equiv\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right\}^{1/2}<+\infty.$$

The norm in this space is defined as follows:

$$\|u(x,t)\|_{B_{2,T}^3}=J_T(u).$$

2. By E_T^3 we denote the space which consists of a topological product

$$B_{2,T}^3\times C[0,T],$$

and the norm of element $z=\{u,a\}$ is defined by the formula

$$\|z\|_{E_T^3}=\|u(x,t)\|_{B_{2,T}^3}+\|a(t)\|_{C[0,T]}\,.$$

Note that $B_{2,T}^3$ and E_T^3 are Banach spaces.

Now we consider in the space E_T^3 the operator

$$\Phi(u,a)=\{\Phi_1(u,a),\Phi_2(u,a)\}\,,$$

where

$$\Phi_1(u,a)=\tilde{u}(x,t)\equiv\sum_{k=1}^{\infty}\tilde{u}_k(t)\sin\lambda_k x,$$

$$\Phi_2(u,a)=\tilde{a}(t),$$

and $\tilde{u}_k(t),\ k=1,\,2,\ldots$ and $\tilde{a}(t)$ are equal to the right-hand sides of (3.21) and (3.24), respectively.

Lemma 3.2 *The inequality holds*

$$\cos x \geq -\frac{1}{4}e^{x}, \quad 0 \leq x < +\infty. \tag{3.25}$$

The proof of this lemma is similar to that of [10, Lemma 4].

It follows from (3.25) that

$$e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T \geq \frac{1}{2}e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T}.$$

By virtue of this relation we have

$$\frac{e^{\lambda_k^{\frac{2}{3}}t}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2, \quad \frac{e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(3T-t)}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2,$$

$$\frac{e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}t}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2, \quad \frac{e^{\lambda_k^{\frac{2}{3}}\left(\frac{T}{2}+t\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2, \; 0 \leq t \leq T,$$

$$\frac{e^{\lambda_k^{\frac{2}{3}}\left(t+\frac{1}{2}\tau\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2, \quad \frac{e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}(t-\tau)}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2,$$

$$\frac{e^{\lambda_k^{\frac{2}{3}}\left(t+\frac{1}{2}\tau\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2, \quad \frac{e^{-\frac{1}{2}\lambda_k^{\frac{2}{3}}(t-\tau)}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2,$$

$$\frac{e^{\lambda_k^{\frac{2}{3}}\left(\frac{3}{2}T-\frac{t}{2}-\tau\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2, \quad \frac{e^{\lambda_k^{\frac{2}{3}}\left(t+\frac{1}{2}\tau\right)}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2,$$

$$\frac{e^{\frac{1}{2}\lambda_k^{\frac{2}{3}}(3T-(t-\tau))}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2, \quad \frac{e^{\lambda_k^{\frac{2}{3}}(t-\tau)}}{e^{\frac{3}{2}\lambda_k^{\frac{2}{3}}T} + 2\cos\frac{\sqrt{3}}{2}\lambda_k^{\frac{2}{3}}T} \leq 2, \; 0 \leq \tau \leq t \leq T.$$

Using this relation we get

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|\tilde{u}_k(t)\right\|_{\infty}\right)^2\right)^{\frac{1}{2}}$$

$$\leq 12\sqrt{2}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{0k}\right|\right)^2\right)^{\frac{1}{2}} + 12\sqrt{2}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{1k}\right|\right)^2\right)$$

$$+6\sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\varphi_{2k}\right|\right)^2\right)^{\frac{1}{2}} + 16\sqrt{5T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_k(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$+16\sqrt{5}T\left\|a(t)\right\|_{\infty}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{\infty}\right)^2\right)^{\frac{1}{2}}, \tag{3.26}$$

$$\|\tilde{a}(t)\|_{\infty} \leq \left\|[h(t)]^{-1}\right\|_{\infty} \left\{\left\|h'''(t) - f(1,t)\right\|_{\infty}\right.$$

$$+\left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \left[4\sqrt{3}\left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{0k}|)^2\right)^{\frac{1}{2}} + 4\sqrt{3}\left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{1k}|)^2\right)^{\frac{1}{2}}\right.$$

$$+6\left(\sum_{k=1}^{\infty} (\lambda_k^2 |\varphi_{2k}|)^2\right)^{\frac{1}{2}} + 16\sqrt{T}\left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}}$$

$$\left.\left.+16T\,\|a(t)\|_{\infty}\left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{\infty})^2\right)^{\frac{1}{2}}\right]\right\}. \tag{3.27}$$

Suppose that data of problem (2.1)-(2.3), (2.16) satisfied the following conditions:

1. $\varphi_0(x) \in C^2[0,1], \varphi_0'''(x) \in L_2(0,1),\ \ \varphi_0(0) = \varphi_0'(1) = \varphi_0''(0) = 0;$
2. $\varphi_1(x) \in C^2[0,1], \varphi_1'''(x) \in L_2(0,1), \varphi_1(0) = \varphi_1'(1) = \varphi_1''(0) = 0;$
3. $\varphi_2(x) \in C^1[0,1], \varphi_2''(x) \in L_2(0,1), \varphi_2(0) = \varphi_2'(1) = 0;$
4. $f(x,t), f_x(x,t) \in C(\overline{D}_T), f_{xx}(x,t) \in L_2(D_T),\ f(0,t) = f_x(1,t) = 0,\ 0 \leq t \leq T;$
5. $h(t) \in C^3[0,T],\ h(t) \neq 0,\ 0, \leq t \leq T.$

Then from (3.26) and (3.27) we obtain

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \leq A_1(T) + B_1(T)\,\|a(t)\|_{\infty}\,\|u(x,t)\|_{B^3_{2,T}}, \tag{3.28}$$

$$\|\tilde{a}(t)\|_{\infty} \leq A_2(T) + B_2(T)\,\|a(t)\|_{\infty}\,\|u(x,t)\|_{B^3_{2,T}}, \tag{3.29}$$

respectively, where

$$A_1(T) = 12\sqrt{2}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)} + 12\sqrt{2}\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}$$

$$+6\sqrt{5}\left\|\varphi_2''(x)\right\|_{L_2(0,1)} + 16\sqrt{5T}\,\|f_{xx}(x,t)\|_{L_2(D_T)},$$

$$B_1(T) = 16\sqrt{5}T,$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{\infty} \left\{\left\|h'''(t) - f(1,t)\right\|_{\infty}\right.$$

$$+\left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \left[4\sqrt{3}\left\|\varphi_0'''(x)\right\|_{L_2(0,1)} + 4\sqrt{3}\left\|\varphi_1'''(x)\right\|_{L_2(0,1)}\right.$$

$$\left.\left.+6\left\|\varphi_2''(x)\right\|_{L_2(0,1)} + 16\sqrt{T}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right]\right\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{\infty}\left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} 16T.$$

By virtue of (3.28) and (3.29) we have

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{\infty} \leq A(T) + B(T)\,\|a(t)\|_{\infty}\,\|u(x,t)\|_{B^3_{2,T}}, \tag{3.30}$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T).$$

Therefore, we can prove the following theorem.

Theorem 3.1 *Let conditions 1-5 and condition*

$$(A(T)+2)^2B(T)<1. \tag{3.31}$$

hold. Then the problem (1.1)-(1.3), (2.12) *in the ball* $K=K_R(\|z\|_{E_T^3}\le R=A(T)+2)$ *of the space* E_T^3 *has a unique solution.*

Proof. In the space E_T^3 we consider the equation

$$z=\Phi z, \tag{3.32}$$

where $z=\{u,a\}$, the components $\Phi_i(u,a),\ i=1,\,2,$ of operator $\Phi(u,a)$ are determined by the right-hand sides of equations (3.22) and (3.24).

Consider the operator $\Phi(u,a)$ in the ball $K=K_R$ of E_T^3. Similarly to (3.30) we obtain that for any $z,z_1,z_2\in K_R$ the following estimates are true:

$$\|\Phi z\|_{E_T^3}\le A(T)+B(T)\,\|a(t)\|_\infty\,\|u(x,t)\|_{B_{2,T}^5}\,, \tag{3.33}$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^3}\le B(T)R(\|a_1(t)-a_2(t)\|_\infty+\|u(x,t)-\overline{u}(x,t)\|_{B_{2,T}^3}). \tag{3.34}$$

Then by (3.31) it follows from estimates (3.33) and (3.34) that the operator $\Phi(u,a)$ acts in the ball $K=K_R$ and is constricting. Hence in $K=K_R$ the operator $\Phi(u,a)$ has an unique fixed point $\{u,a\}$ that is unique in the ball $K=K_R$ solution of problem (3.32), i.e. it is an unique solution in $K=K_R$ of system (3.22), (3.24).

The function $u(x,t)$ as an element of the space $B_{2,T}^3$ is continuous and has continuous derivatives $u_x(x,t),\ u_{xx}(x,t)$ in $\overline{D}_T$. It is easy to see from (3.22) that

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_k'''(t)\right\|_\infty\right)^2\right)^{\frac{1}{2}}\le\sqrt{3}\left(\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_\infty\right)^2\right)^{\frac{1}{2}}\right.$$

$$\left.+\left\|\|f_x(x,t)\|_\infty\right\|_{L_2(0,1)}+\|a(t)\|_\infty\,\|u(x,t)\|_{B_{2,T}^3}\right).$$

Hence It follows that $u_{ttt}(x,t)$ is continuous in $\overline{D}_T$. It is easy to verify that the equation (1.1) and conditions (1.2), (1.3) and (2.12) are satisfied in the usual sense. Consequently, $\{u(x,t),a(t)\}$ is a solution of problem (1.1)-(1.3), (2.12). By virtue of corollary 3.1 it is unique in the ball $K=K_R$. The proof of this theorem is complete.

Using Lemma 2.2 is proved the following lemma.

Theorem 3.2 *Let all conditions of Theorem 3.1 be satisfied,*

$$\frac{2}{3}(A(T)+2)T^3<1$$

and the following matching conditions

$$\varphi_0(1)=h(0),\quad \varphi_1(1)\,dx=h'(0),\quad \varphi_2(1)=h''(T).$$

Then problem (1.1)-(1.4) *has in the ball* $K=K_R(\|z\|_{E_T^3}\le R=A(T)+2)$ *of the space* E_T^3 *an unique classical solution.*

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