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Spanne-type characterization of parabolic fractional integral and its commutators in parabolic generalized Orlicz-Morrey spaces

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Abstract. *In this paper, we give necessary and sufficient condition for the Spanne-type boundedness of the parabolic fractional integral operator and its commutators with some BMO functions on the parabolic generalized Orlicz-Morrey spaces.*

Keywords. Parabolic generalized Orlicz-Morrey space; parabolic fractional integral; commutator; BMO

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1 Introduction

The theory of boundedness of classical operators of the real analysis, such as the fractional maximal operator, Riesz potential and the singular integral operators etc, from one Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Lebesgue spaces, Orlicz spaces also play an important role.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have positive real part. Let $A_t = t^P \ (t > 0)$, and set $\gamma = trP$. Then, there exists a quasi-distance ρ associated with P such that

$$\begin{aligned} &(a) \;\; \rho(A_t x) = t\rho(x), \;\; t>0, \;\; \text{for every} \;\; x \in \mathbb{R}^n; \\ &(b) \;\; \rho(0) = 0, \;\; \rho(x-y) = \rho(y-x) \geq 0 \\ &\qquad \text{and} \;\; \rho(x-y) \leq k(\rho(x-z) + \rho(y-z)); \\ &(c) \;\; dx = \rho^{\gamma-1} d\sigma(w) d\rho, \;\; \text{where} \;\; \rho = \rho(x), w = A_{\rho^{-1}} x \\ &\qquad \text{and } d\sigma(w) \text{is a } C^\infty \text{measure on the ellipsoid } \{w : \rho(w) = 1\}. \end{aligned}$$

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([4,5]). The balls

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with respect to ρ, centered at x of radius r, are just the ellipsoids $\mathcal{E}(x,r) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$, with the Lebesgue measure $|\mathcal{E}(x,r)| = v_\rho r^\gamma$, where v_ρ is the volume of the unit ellipsoid in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}(x,r) = \mathbb{R}^n \setminus \mathcal{E}(x,r)$ be the complement of $\mathcal{E}(x,r)$. If $P = I$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_I(x,r) = B(x,r)$. Let $S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$.

Let $S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$. The parabolic maximal function $M^P f$ and the parabolic fractional integral $I^P_\alpha f$, $0 < \alpha < \gamma$, of a function $f \in L^{\rm loc}_1(\mathbb{R}^n)$ are defined by

$$M^P f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y)| dy,$$

$$I^P_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\rho(x-y)^{\gamma-\alpha}} dy.$$

If $P = I$, then $M \equiv M^I_0$ is the Hardy-Littlewood maximal operator and $I_\alpha \equiv I^I_\alpha$ is the fractional integral operator It is well known that the parabolic fractional integral operators play an important role in harmonic analysis (see [19]).

In this work we present the boundedness for parabolic fractional integral operator I^P_α (Theorem 4.1) and its commutators $[b, I^P_\alpha]$ (Theorem 4.3) in the parabolic generalized Orlicz-Morrey spaces $M^P_{\Phi,\varphi}(\mathbb{R}^n)$. Moreover, we give necessary and sufficient condition for the Spanne-type boundedness of the parabolic fractional integral operator (Theorem 4.2) and its commutators with some BMO functions (Theorem 4.4) on the parabolic generalized Orlicz-Morrey spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

2.1 On Young Functions and Orlicz Spaces

First, we recall the definition of Young functions.

Definition 2.1 *A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous,* $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ *and* $\lim\limits_{r\to\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. The set of Young functions such that $0 < \Phi(r) < \infty$ for $0 < r < \infty$ will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let $\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}$. If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \qquad \text{for } r \geq 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty) \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if $\Phi(2r) \leq C\Phi(r)$, $r > 0$ for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if $\Phi(r) \leq \frac{1}{2C}\Phi(Cr)$, $r \geq 0$ for some $C > 1$.

The Orlicz space and weak Orlicz space are defined as follows.

Definition 2.2 *(Orlicz Space). For a Young function Φ, the set*

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{\rm loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \, \text{for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L_\Phi(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \leq r \leq 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L_\Phi(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. The space $L_\Phi^{\rm loc}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_{\mathcal{E}} \in L_\Phi(\mathbb{R}^n)$ for all parabolic balls $\mathcal{E} \subset \mathbb{R}^n$.

$L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \leq 1 \right\}.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let $m(\Omega,\ f,\ t) = |\{x \in \Omega : |f(x)| > t\}|$. In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,\ t)$. The weak Orlicz space $WL_\Phi(\mathbb{R}^n) = \{f \in L_1^{\rm loc}(\mathbb{R}^n) : \|f\|_{WL_\Phi} < \infty\}$ is defined by the norm $\|f\|_{WL_\Phi} = \inf \Big\{ \lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big) \ \leq 1 \Big\}$.

The following analogue of the Hölder's inequality is well known (see, for example, [15]).

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and functions f, g measurable on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid $\int_\Omega |f(x)g(x)|dx \leq 2\|f\|_{L_\Phi(\Omega)}\|g\|_{L_{\widetilde{\Phi}}(\Omega)}$.*

By elementary calculations we have the following property.

Lemma 2.1 *Let Φ be a Young function and $\mathcal{E}$ be a parabolic balls in $\mathbb{R}^n$. Then*

$$\|\chi_{\mathcal{E}}\|_{L_\Phi} = \|\chi_{\mathcal{E}}\|_{WL_\Phi} = \frac{1}{\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)}.$$

By Theorem 2.1, Lemma 2.1 and (2.1) we get the following estimate.

Lemma 2.2 *For a Young function Φ and for the parabolic balls $\mathcal{E} = \mathcal{E}(x, r)$ the following inequality is valid:*

$$\int_{\mathcal{E}} |f(y)|dy \leq 2|\mathcal{E}|\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)\|f\|_{L_\Phi(\mathcal{E})}.$$

The following theorem is an analogue of Lebesgue differentiation theorem in Orlicz spaces.

Theorem 2.2 *[12] Suppose that Φ is a Young function and let $f \in L_\Phi(\mathbb{R}^n)$ be nonnegative. Then*

$$\liminf_{r\to 0+} \frac{\|f\chi_{\mathcal{E}(x,r)}\|_{L_\Phi}}{\|\chi_{\mathcal{E}(x,r)}\|_{L_\Phi}} \geq f(x), \quad \textit{for almost every } x \in \mathbb{R}^n.$$

3 Parabolic fractional integral and its commutators in Orlicz spaces

In [1] the boundedness of the parabolic maximal operator M^P in Orlicz spaces $L_\Phi(\mathbb{R}^n)$ was obtained, see also [2].

Theorem 3.1 *[1] Let Φ any Young function. Then the parabolic maximal operator M^P is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$ bounded in $L_\Phi(\mathbb{R}^n)$.*

In [3] the boundedness of the parabolic fractional integral operator I^P_α in Orlicz spaces $L_\Phi(\mathbb{R}^n)$ was obtained, see also [10, 16].

Theorem 3.2 *[3] Let $0 < \alpha < \gamma$, Φ, Ψ be Young functions, $\Phi, \Psi \in \mathcal{Y}$ and*

$$\int_r^\infty t^{\alpha-1}\Phi^{-1}\big(t^{-\gamma}\big)dt \lesssim r^\alpha \Phi^{-1}\big(t^{-\gamma}\big),\ 0<r<\infty, \tag{3.1}$$

holds. Then the condition

$$r^{-\frac{\alpha}{\gamma}}\Phi^{-1}(r) \le C\Psi^{-1}(r) \tag{3.2}$$

for all $r > 0$, where $C > 0$ does not depend on r, is necessary and sufficient for the boundedness of I^P_α from $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, the condition (3.2) is necessary and sufficient for the boundedness of I^P_α from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$.

The commutators $[b, I^P_\alpha]$, $|b, I^P_\alpha|$ generated by $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and the operator I^P_α are defined by

$$[b, I^P_\alpha]f(x) = \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{\rho(x-y)^{\gamma-\alpha}} f(y)dy,$$

$$|b, I^P_\alpha|f(x) = \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|}{\rho(x-y)^{\gamma-\alpha}} f(y)dy, \qquad 0<\alpha<\gamma,$$

respectively.

We recall that the space $BMO(\mathbb{R}^n) = \{b \in L_1^{\rm loc}(\mathbb{R}^n)\ :\ \|b\|_* < \infty\}$ is defined by the seminorm

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|\mathcal{E}(x,r)|}\int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}|dy < \infty,$$

where $b_{\mathcal{E}(x,r)} = |\mathcal{E}(x,r)|^{-1}\int_{\mathcal{E}(x,r)} b(y)dy$. We will need the following property of BMO-functions:

$$\big|b_{\mathcal{E}(x,r)} - b_{\mathcal{E}(x,t)}\big| \le C\|b\|_* \ln\frac{t}{r} \quad \text{for} \quad 0<2r<t, \tag{3.3}$$

where C does not depend on b, x, r and t. We refer for instance to [11] and [13] for details on this space and properties.

Lemma 3.1 *[3] Let $b \in BMO(\mathbb{R}^n)$ and Φ be a Young function with $\Phi \in \Delta_2$. Then*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(r^{-\gamma}\big)\, \big\|b(\cdot) - b_{\mathcal{E}(x,r)}\big\|_{L_\Phi(\mathcal{E}(x,r))}.$$

Lemma 3.2 *If $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$, then*

$$r_0^\alpha |b(x) - b_{\mathcal{E}_0}| \le C|b, I^P_\alpha|\chi_{\mathcal{E}_0}(x)$$

for every $x \in \mathcal{E}_0$.

The known boundedness statements for the commutator operator $[b, I^P_\alpha]$ on Orlicz spaces run as follows, see [3] and [10].

Theorem 3.3 *[3] Let* $0 < \alpha < \gamma$*,* $b \in BMO(\mathbb{R}^n)$ *and* $\Phi, \Psi \in \mathcal{Y}$*.*

1. *If* $\Phi \in \nabla_2$ *and* $\Psi \in \Delta_2$*, then the condition*

$$r^{\alpha}\Phi^{-1}\big(r^{-\gamma}\big) + \int_r^{\infty} \Big(1 + \ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1} dt \le C\Psi^{-1}\big(r^{-\gamma}\big) \tag{3.4}$$

for all $r > 0$*, where* $C > 0$ *does not depend on* r*, is sufficient for the boundedness of* $[b, I^P_\alpha]$ *from* $L_\Phi(\mathbb{R}^n)$ *to* $L_\Psi(\mathbb{R}^n)$*.*

2. *If* $\Psi \in \Delta_2$*, then the condition* (3.2) *is necessary for the boundedness of* $|b, I^P_\alpha|$ *from* $L_\Phi(\mathbb{R}^n)$ *to* $L_\Psi(\mathbb{R}^n)$*.*

3. *Let* $\Phi \in \nabla_2$ *and* $\Psi \in \Delta_2$*. If the condition*

$$\int_r^{\infty} \Big(1 + \ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1} dt \le C r^{\alpha}\Phi^{-1}\big(r^{-\gamma}\big) \tag{3.5}$$

holds for all $r > 0$*, where* $C > 0$ *does not depend on* r*, then the condition* (3.2) *is necessary and sufficient for the boundedness of* $|b, I^P_\alpha|$ *from* $L_\Phi(\mathbb{R}^n)$ *to* $L_\Psi(\mathbb{R}^n)$*.*

4 Parabolic fractional integral and its commutators in parabolic generalized Orlicz-Morrey spaces

Various versions of generalized Orlicz-Morrey spaces were introduced in [14], [18] and [7]. We used the definition of [7] which runs as follows.

Definition 4.1 *Let* $\varphi(x, r)$ *be a positive measurable function on* $\mathbb{R}^n \times (0, \infty)$ *and* Φ *any Young function. We denote by* $M^P_{\Phi,\varphi}(\mathbb{R}^n)$ *the parabolic generalized parabolic Orlicz-Morrey space, the space of all functions* $f \in L^{\mathrm{loc}}_{\Phi}(\mathbb{R}^n)$ *for which*

$$\|f\|_{M^P_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1}\Phi^{-1}(|\mathcal{E}(x, r)|^{-1})\|f\|_{L_\Phi(\mathcal{E}(x,r))} < \infty.$$

Lemma 4.1 *Let* Φ *be a Young function and* φ *be a positive measurable function on* $\mathbb{R}^n \times (0, \infty)$*.*

(i) If

$$\sup_{t<r<\infty} \frac{\Phi^{-1}(|\mathcal{E}(x, r)|^{-1})}{\varphi(x, r)} = \infty \quad \textit{for some } t > 0 \textit{ and for all } x \in \mathbb{R}^n, \tag{4.1}$$

then $M^P_{\Phi,\varphi}(\mathbb{R}^n) = \Theta$*.*

(ii) If

$$\sup_{0<r<\tau} \varphi(x, r)^{-1} = \infty \quad \textit{for some } \tau > 0 \textit{ and for all } x \in \mathbb{R}^n, \tag{4.2}$$

then $M^P_{\Phi,\varphi}(\mathbb{R}^n) = \Theta$*.*

Proof. (i) Let (4.1) be satisfied and f be not equivalent to zero. Then $\sup_{x\in\mathbb{R}^n} \|f\|_{L_\Phi(\mathcal{E}(x,t))} > 0$, hence

$$\begin{aligned}\|f\|_{M^P_{\Phi,\varphi}} &\ge \sup_{x\in\mathbb{R}^n} \sup_{t<r<\infty} \varphi(x, r)^{-1}\Phi^{-1}(|\mathcal{E}(x, r)|^{-1})\|f\|_{L_\Phi(\mathcal{E}(x,r))} \\ &\ge \sup_{x\in\mathbb{R}^n} \|f\|_{L_\Phi(\mathcal{E}(x,t))} \sup_{t<r<\infty} \varphi(x, r)^{-1}\Phi^{-1}(|\mathcal{E}(x, r)|^{-1}).\end{aligned}$$

Therefore $\|f\|_{M^P_{\Phi,\varphi}} = \infty$.

(ii) Let $f \in M^P_{\Phi,\varphi}(\mathbb{R}^n)$ and (4.2) be satisfied. Then there are two possibilities:

Case 1: $\sup_{0<r<t} \varphi(x,r)^{-1} = \infty$ for all $t > 0$.
Case 2: $\sup_{0<r<t} \varphi(x,r)^{-1} < \infty$ for some $s \in (0,\tau)$.

For Case 1, by Theorem 2.2, for almost all $x \in \mathbb{R}^n$,

$$\lim_{r\to 0+} \frac{\|f\chi_{\mathcal{E}(x,r)}\|_{L_\Phi}}{\|\chi_{\mathcal{E}(x,r)}\|_{L_\Phi}} \geq |f(x)|. \tag{4.3}$$

We claim that $f(x) = 0$ for all those x. Indeed, fix x and assume $|f(x)| > 0$. Then by Lemma 2.1 and (4.3) there exists $t_0 > 0$ such that

$$\Phi^{-1}\big(|\mathcal{E}(x,r)|^{-1}\big)\|f\|_{L_\Phi(\mathcal{E}(x,r))} \geq \frac{|f(x)|}{2}$$

for all $0 < r \leq t_0$. Consequently,

$$\begin{aligned}\|f\|_{M^P_{\Phi,\varphi}} &\geq \sup_{0<r<t_0} \varphi(x,r)^{-1}\Phi^{-1}\big(|\mathcal{E}(x,r)|^{-1}\big)\|f\|_{L_\Phi(\mathcal{E}(x,r))}\\ &\geq \frac{|f(x)|}{2} \sup_{0<r<t_0} \varphi(x,r)^{-1}.\end{aligned}$$

Hence $\|f\|_{M^P_{\Phi,\varphi}} = \infty$, so $f \notin M^P_{\Phi,\varphi}(\mathbb{R}^n)$ and we have arrived at a contradiction.

Note that Case 2 implies that $\sup_{s<r<\tau} \varphi(x,r)^{-1} = \infty$, hence

$$\begin{aligned}\sup_{s<r<\infty} \varphi(x,r)^{-1}\Phi^{-1}(|\mathcal{E}(x,r)|^{-1}) &\geq \sup_{s<r<\tau} \varphi(x,r)^{-1}\Phi^{-1}(|\mathcal{E}(x,r)|^{-1})\\ &\geq \Phi^{-1}(|\mathcal{E}(x,\tau)|^{-1}) \sup_{s<r<\tau} \varphi(x,r)^{-1} = \infty,\end{aligned}$$

which is the case in (i).

Remark 4.1 Let Φ be a Young function. We denote by $\Omega_{\Phi,P}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n} \Big\| \frac{\Phi^{-1}(|\mathcal{E}(x,r)|^{-1})}{\varphi(x,r)} \Big\|_{L_\infty(t,\infty)} < \infty,$$

and

$$\sup_{x\in\mathbb{R}^n} \Big\|\varphi(x,r)^{-1}\Big\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 4.1, we always assume that $\varphi \in \Omega_{\Phi,P}$.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \leq C\varphi(s) \qquad (\text{resp. } \varphi(r) \geq C\varphi(s)) \quad \text{for } r \leq s.$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto \frac{1}{\Phi^{-1}(t^{-\gamma})}\varphi(t)$ is almost increasing.

Lemma 4.2 *[2] Let $\mathcal{E}_0 := \mathcal{E}(x_0,r_0)$. If $\varphi \in \mathcal{G}_\Phi$, then there exist $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \leq \|\chi_{\mathcal{E}_0}\|_{M^P_{\Phi,\varphi}} \leq \frac{C}{\varphi(r_0)}.$$

The following Guliyev-type local estimate for the parabolic fractional integral operator I^P_α in Orlicz space is valid.

Lemma 4.3 *Let* $0 < \alpha < \gamma$, Φ, Ψ *Young functions and* (Φ, Ψ) *satisfy the conditions* (3.1) *and* (3.2). *Then*

$$\|I^P_\alpha f\|_{WL_\Psi(\mathcal{E})} \lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \Psi^{-1}\big(t^{-\gamma}\big) \|f\|_{L_\Phi(\mathcal{E}(x,t))} \frac{dt}{t} \tag{4.4}$$

holds for any ball $\mathcal{E} = \mathcal{E}(x, r)$ *and for all* $f \in L^{\rm loc}_\Phi(\mathbb{R}^n)$.

If $\Phi \in \nabla_2$, *then*

$$\|I^P_\alpha f\|_{L_\Psi(\mathcal{E})} \lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \Psi^{-1}\big(t^{-\gamma}\big) \|f\|_{L_\Phi(\mathcal{E}(x,t))} \frac{dt}{t} \tag{4.5}$$

holds for any ball $\mathcal{E} = \mathcal{E}(x, r)$ *and for all* $f \in L^{\rm loc}_\Phi(\mathbb{R}^n)$.

Proof. Let $0 < \alpha < \gamma$, $\Phi \in \nabla_2$ and (Φ, Ψ) satisfy the conditions (3.1) and (3.2). We put $f = f_1 + f_2$, where $f_1 = f\chi_{\mathcal{E}(x,2kr)}$ and $f_2 = f\chi_{{}^{\complement}\mathcal{E}(x,2kr)}$, where k is the constant from the triangle inequality.

Estimation of $I^P_\alpha f_1$: By Theorem 3.2 we have

$$\|I^P_\alpha f_1\|_{L_\Psi(\mathcal{E})} \le \|I^P_\alpha f_1\|_{L_\Psi(\mathbb{R}^n)} \lesssim \|f_1\|_{L_\Phi(\mathbb{R}^n)} = \|f\|_{L_\Phi(\mathcal{E}(x,2kr))}.$$

By using the monotonicity of the functions $\|f\|_{L_\Phi(\mathcal{E}(x,t))}$, $\Phi^{-1}\big(t\big)$ with respect to t we get,

$$\begin{aligned} &\frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \Psi^{-1}\big(t^{-\gamma}\big) \|f\|_{L_\Phi(\mathcal{E}(x,t))} \frac{dt}{t} \\ &\qquad \ge \frac{\|f\|_{L_\Phi(\mathcal{E}(x,2kr))}}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \Psi^{-1}\big(t^{-\gamma}\big) \frac{dt}{t} \gtrsim \|f\|_{L_\Phi(\mathcal{E}(x,2kr))}. \end{aligned} \tag{4.6}$$

Consequently we have

$$\|I^P_\alpha f_1\|_{L_\Psi(\mathcal{E})} \lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \Psi^{-1}\big(t^{-\gamma}\big) \|f\|_{L_\Phi(\mathcal{E}(x,t))} \frac{dt}{t}. \tag{4.7}$$

Estimation of $I^P_\alpha f_2$: Let y be an arbitrary point from $\mathcal{E}$.

A geometric observation shows that $y \in \mathcal{E}$, $z \in {}^{\complement}\mathcal{E}(x, 2kr)$ implies $\frac{1}{2k}\rho(x - z) \le \rho(y - z) \le \frac{2k+1}{2}\rho(x - z)$. Therefore, by Lemma 2.2

$$\begin{aligned} \big|I^P_\alpha f_2(y)\big| &\lesssim \int_{{}^{\complement}\mathcal{E}(x,2kr)} \frac{|f(z)|}{\rho(x - z)^{\gamma-\alpha}} dz \approx \int_{{}^{\complement}\mathcal{E}(x,2kr)} |f(z)| dz \int_{\rho(x-z)}^{\infty} \frac{dt}{t^{\gamma+1-\alpha}} \\ &\approx \int_{2kr}^{\infty} \int_{2kr \le \rho(x-z) < t} |f(z)| dz \frac{dt}{t^{\gamma+1-\alpha}} \lesssim \int_{2kr}^{\infty} \int_{\mathcal{E}(x,t)} |f(z)| dz \frac{dt}{t^{\gamma+1-\alpha}} \\ &\lesssim \int_{2kr}^{\infty} \|f\|_{L_\Phi(\mathcal{E}(x,t))}\, t^\alpha\, \Phi^{-1}\big(t^{-\gamma}\big) \frac{dt}{t} \lesssim \int_{2kr}^{\infty} \|f\|_{L_\Phi(\mathcal{E}(x,t))}\, \Psi^{-1}\big(t^{-\gamma}\big) \frac{dt}{t}. \end{aligned} \tag{4.8}$$

Thus the function $I^P_\alpha f_2(y)$, with fixed x and r, is dominated by the expression not depending on y. Then we integrate the obtained estimate for $I^P_\alpha f_2(y)$ in y over $\mathcal{E}$, we get

$$\|I^P_\alpha f_2\|_{L_\Psi(\mathcal{E})} \lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \|f\|_{L_\Phi(\mathcal{E}(x,t))}\, \Psi^{-1}\big(t^{-\gamma}\big) \frac{dt}{t}. \tag{4.9}$$

Gathering the estimates (4.7) and (4.9) we arrive at (4.5).

Let now Φ be an arbitrary Young function. It is obvious that

$$\|I_\alpha^P f\|_{WL_\Phi(\mathcal{E})} \le \|I_\alpha^P f_1\|_{WL_\Phi(\mathcal{E})} + \|I_\alpha^P f_2\|_{WL_\Phi(\mathcal{E})}.$$

By the boundedness of the operator I_α^P from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$, provided by Theorem 3.2, we have

$$\|I_\alpha^P f_1\|_{WL_\Phi(\mathcal{E})} \lesssim \|f\|_{L_\Phi(\mathcal{E}(x,2kr))}.$$

By using (4.6), (4.9) and Lemma 2.1 we arrive at (4.4).

Theorem 4.1 *Let $0 < \alpha < \gamma$, Φ, Ψ Young functions and (Φ,Ψ) satisfy the conditions* (3.1) *and* (3.2)*. Assume that the functions (φ_1,φ_2) and (Φ,Ψ) satisfy the conditions* (3.1), (3.2) *and*

$$\int_r^\infty \Psi^{-1}\big(t^{-\gamma}\big) \operatorname*{ess\,inf}_{t<s<1} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \frac{dt}{t} \le C\,\varphi_2(x,r), \tag{4.10}$$

where C does not depend on r. Then the operator I_α^P is bounded from $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM^P_{\Psi,\varphi_2}(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$, the operator I_α^P is bounded from $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^P_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof. Note that $\Big(\operatorname*{ess\,inf}_{x\in A} f(x)\Big)^{-1} = \operatorname*{ess\,sup}_{x\in A} \frac{1}{f(x)}$ is true for any real-valued nonnegative function f and measurable on A and the fact that $\|f\|_{L_\Phi(\mathcal{E}(x,t))}$ is a nondecreasing function of t

$$\begin{aligned}
\frac{\|f\|_{L_\Phi(\mathcal{E}(x,t))}}{\operatorname*{ess\,inf}_{0<t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-\gamma}\big)}} &= \operatorname*{ess\,sup}_{0<t<s<\infty} \frac{\Phi^{-1}\big(s^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))}}{\varphi_1(x,s)} \\
&\le \sup_{x\in\mathbb{R}^n, r>0} \frac{\Phi^{-1}\big(s^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,s))}}{\varphi_1(x,s)} = \|f\|_{M^P_{\Phi,\varphi_1}}.
\end{aligned}$$

Since (φ_1,φ_2) and (Φ,Ψ) satisfy the condition (4.10),

$$\begin{aligned}
&\int_r^\infty \|f\|_{L_\Phi(\mathcal{E}(x,t))} \Psi^{-1}\big(t^{-\gamma}\big)\frac{dt}{t} \\
&\le \int_r^\infty \frac{\|f\|_{L_\Phi(\mathcal{E}(x,t))}}{\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-\gamma}\big)}} \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \Psi^{-1}\big(t^{-\gamma}\big)\frac{dt}{t} \\
&\lesssim \|f\|_{M^P_{\Phi,\varphi_1}} \int_r^\infty \Big(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-\gamma}\big)}\Big) \Psi^{-1}\big(t^{-\gamma}\big)\frac{dt}{t} \\
&\lesssim \varphi_2(x,r)\|f\|_{M^P_{\Phi,\varphi_1}}.
\end{aligned} \tag{4.11}$$

Then by (4.10) and (4.11) we get

$$\|I_\alpha^P f\|_{M^P_{\Psi,\varphi_2}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)} \int_r^\infty \Psi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))}\frac{dt}{t} \lesssim \|f\|_{M^P_{\Phi,\varphi_1}}.$$

The estimate $\|I_\alpha^P f\|_{WM^P_{\Psi,\varphi_2}} \lesssim \|f\|_{M^P_{\Phi,\varphi_1}}$ can be proved similarly by the help of local estimate (4.4).

Remark 4.2 Note that Theorem 4.1 in the isotropic case $P = I$ were proved in [8], see also [6, 17].

For proving our main results, we need the following estimate.

Lemma 4.4 *If $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$, then $r_0^\alpha \lesssim I_\alpha^P \chi_{\mathcal{E}_0}(x)$ for every $x \in \mathcal{E}_0$.*

Proof. It is well-known that

$$\mathrm{M}_\alpha^P f(x) \leq 2^{\gamma-\alpha} M_\alpha^P f(x), \tag{4.12}$$

where $\mathrm{M}_\alpha^P(f)(x) = \sup\limits_{\mathcal{E} \ni x} |\mathcal{E}|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}} |f(y)|dy$.

Now let $x \in \mathcal{E}_0$. By using (4.12), we get

$$\begin{aligned} I_\alpha^P \chi_{\mathcal{E}_0}(x) &\gtrsim M_\alpha^P \chi_{\mathcal{E}_0}(x) \gtrsim \mathrm{M}_\alpha^P \chi_{\mathcal{E}_0}(x) \gtrsim \sup_{\mathcal{E} \ni x} |\mathcal{E}|^{-1+\frac{\alpha}{\gamma}} |\mathcal{E} \cap \mathcal{E}_0| \\ &\gtrsim |\mathcal{E}_0|^{-1+\frac{\alpha}{\gamma}} |\mathcal{E}_0 \cap \mathcal{E}_0| = r_0^\alpha. \end{aligned}$$

The following theorem gives necessary and sufficient conditions for Spanne-type boundedness of the operator I_α^P from $M_{\Phi,\varphi_1}^P(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^P(\mathbb{R}^n)$.

Theorem 4.2 *(Spanne-type result) Let $0 < \alpha < \gamma$, (Φ, Ψ) be Young functions, and let $\varphi_1 \in \Omega_{\Phi,P}$, $\varphi_2 \in \Omega_{\Psi,P}$.*

1. *If the functions (Φ, Ψ) satisfy the conditions* (3.1) *and* (3.2)*, then the condition*

$$\int_r^\infty \Psi^{-1}\big(r^{-\gamma}\big) \operatorname*{ess\,inf}_{r<s<1} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \frac{dt}{t} \leq C\, \varphi_2(t), \tag{4.13}$$

for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for the boundedness of I_α^P from $M_{\Phi,\varphi_1}^P(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^P(\mathbb{R}^n)$.

2. *If the function $\varphi_1 \in \mathcal{G}_\Phi$, then the condition*

$$t^\alpha \varphi_1(t) \leq C \varphi_2(t), \tag{4.14}$$

for all $t > 0$, where $C > 0$ does not depend on t, is necessary for the boundedness of I_α^P from $M_{\Phi,\varphi_1}^P(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^P(\mathbb{R}^n)$.

3. *Let the functions (Φ, Ψ) satisfy the conditions* (3.1) *and* (3.2)*. If $\varphi_1 \in \mathcal{G}_\Phi$ satisfies the condition*

$$\int_r^\infty \frac{\Psi^{-1}\big(t^{-\gamma}\big)}{\Phi^{-1}\big(t^{-\gamma}\big)} \varphi_1(t) \frac{dt}{t} \leq C r^\alpha \varphi_1(t), \tag{4.15}$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition (4.14) *is necessary and sufficient for the boundedness of I_α^P from $M_{\Phi,\varphi_1}^P(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^P(\mathbb{R}^n)$.*

Proof. The first statement of the theorem follows from Theorem 4.1.

We shall now prove the second part. Let $\mathcal{E}_0 = \mathcal{E}(x_0, t_0)$ and $x \in \mathcal{E}_0$. By Lemma 4.4 we have $t_0^\alpha \lesssim I_\alpha \chi_{\mathcal{E}_0}(x)$. Therefore, by Lemma 2.1 and Lemma 4.2

$$\begin{aligned} t_0^\alpha &\lesssim \Psi^{-1}(|\mathcal{E}_0|^{-1}) \|I_\alpha^P \chi_{\mathcal{E}_0}\|_{L_\Psi(\mathcal{E}_0)} \lesssim \varphi_2(t_0) \|I_\alpha^P \chi_{\mathcal{E}_0}\|_{M_{\Psi,\varphi_2}^P} \\ &\lesssim \varphi_2(t_0) \|\chi_{\mathcal{E}_0}\|_{M_{\Phi,\varphi_1}^P} \leq C \frac{\varphi_2(t_0)}{\varphi_1(t_0)} \end{aligned}$$

Since this is true for every $t_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem.

The following Guliyev-type local estimate for the commutator of the parabolic fractional integral operator $[b, I_\alpha^P]$ in Orlicz space is valid.

Lemma 4.5 *Let $0 < \alpha < \gamma$, $b \in BMO(\mathbb{R}^n)$, Φ, Ψ Young functions and (Φ, Ψ) satisfy the conditions* (3.1) *and* (3.2). *Let $\Psi^{-1}(r) \approx \Phi^{-1}(r)\, r^{-\frac{\alpha}{\gamma}}$ and $\Phi \in \Delta_2 \cap \nabla_2$, then the inequality*

$$\|[b, I_\alpha^P]f\|_{L_\Psi(\mathcal{E}(x_0,r))} \lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \Big(1 + \ln\frac{t}{r}\Big) \Psi^{-1}\big(t^{-\gamma}\big) \|f\|_{L_\Phi(\mathcal{E}(x_0,t))} \frac{dt}{t}$$

holds for any ball $\mathcal{E}(x_0, r)$ and for all $f \in L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. For $\mathcal{E} = \mathcal{E}(x_0, r)$, write $f = f_1 + f_2$ with $f_1 = f\chi_{2kB}$ and $f_2 = f\chi_{\complement_{(2k\mathcal{E})}}$, where k is the constant from the triangle inequality, so that

$$\big\|[b, I_\alpha^P]f\big\|_{L_\Psi(\mathcal{E})} \le \big\|[b, I_\alpha^P]f_1\big\|_{L_\Psi(\mathcal{E})} + \big\|[b, I_\alpha^P]f_2\big\|_{L_\Psi(\mathcal{E})}.$$

By the boundedness of the operator $[b, I_\alpha^P]$ from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ provided by Theorem 3.3, we obtain

$$\|[b, I_\alpha^P]f_1\|_{L_\Psi(\mathcal{E})} \le \|[b, I_\alpha^P]f_1\|_{L_\Psi(\mathbb{R}^n)} \lesssim \|b\|_* \|f_1\|_{L_\Phi(\mathbb{R}^n)} = \|b\|_* \|f\|_{L_\Phi(2k\mathcal{E})}. \tag{4.16}$$

As we proceed in the proof of Lemma 4.3, we have for $x \in \mathcal{E}$

$$\big|[b, I_\alpha^P](f_2)(x)\big| \lesssim \int_{\complement_{\mathcal{E}(x,2kr)}} \frac{|b(y) - b(x)|\,|f(y)|}{\rho(x_0 - y)^{\gamma-\alpha}} dy.$$

Then

$$\begin{aligned}
\|[b, I_\alpha^P]f_2\|_{L_\Psi(\mathcal{E})} &\lesssim \Big\| \int_{\complement_{\mathcal{E}(x,2kr)}} \frac{|b(y) - b(x)|\,|f(z)|}{\rho(x_0 - y)^{\gamma-\alpha}} dy \Big\|_{L_\Psi(\mathcal{E})} \\
&\lesssim J_1 + J_2 = \Big\| \int_{\complement_{\mathcal{E}(x,2kr)}} \frac{|b(y) - b_{\mathcal{E}}|\,|f(y)|}{\rho(x_0 - y)^{\gamma-\alpha}} dy \Big\|_{L_\Psi(\mathcal{E})} \\
&\quad + \Big\| \int_{\complement_{\mathcal{E}(x,2kr)}} \frac{|b(\cdot) - b_{\mathcal{E}}|\,|f(y)|}{\rho(x_0 - y)^{\gamma-\alpha}} dy \Big\|_{L_\Psi(\mathcal{E})}.
\end{aligned}$$

For the term J_1 by Lemma 2.1 we obtain

$$J_1 \approx \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{\complement_{\mathcal{E}(x,2kr)}} \frac{|b(y) - b_{\mathcal{E}}|\,|f(y)|}{\rho(x_0 - y)^{\gamma-\alpha}} dy$$

and split it as follows:

$$\begin{aligned}
J_1 &\lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{\complement_{\mathcal{E}(x,2kr)}} |b(y) - b_{\mathcal{E}}|\,|f(y)| dy \int_{\rho(x_0-y)}^{\infty} \frac{dt}{t^{\gamma+1-\alpha}} \\
&\approx \int_{2kr}^{\infty} \int_{2kr \le \rho(x_0-y) < t} |b(y) - b_{\mathcal{E}}|\,|f(y)| dy \frac{dt}{t^{\gamma+1-\alpha}} \\
&\lesssim \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |b(y) - b_{\mathcal{E}}|\,|f(y)| dy \frac{dt}{t^{\gamma+1-\alpha}}.
\end{aligned}$$

Applying Hölder's inequality, by Lemmas 2.2 and 3.1 and (3.3) we get

$$\begin{aligned}
J_1 &\lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |b(y) - b_{\mathcal{E}(x_0,t)}||f(y)| dy \frac{dt}{t^{\gamma+1-\alpha}} \\
&\quad + \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} |b_{\mathcal{E}(x_0,r)} - b_{\mathcal{E}(x_0,t)}| \int_{\mathcal{E}(x_0,t)} |f(y)| dy \frac{dt}{t^{\gamma+1-\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \big\|b(\cdot)-b_{\mathcal{E}(x_0,t)}\big\|_{L_{\widetilde{\Phi}}(\mathcal{E})} \|f\|_{L_\Phi(\mathcal{E}(x_0,t))} \frac{dt}{t^{\gamma+1-\alpha}} \\
&\quad + \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} |b_{\mathcal{E}(x_0,r)}-b_{\mathcal{E}(x_0,t)}| \|f\|_{L_\Phi(\mathcal{E}(x_0,t))} \Phi^{-1}\big(t^{-\gamma}\big) \frac{dt}{t^{1-\alpha}} \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_\Phi(\mathcal{E}(x_0,t))} \Psi^{-1}\big(t^{-\gamma}\big) \frac{dt}{t}.
\end{aligned}
$$

For J_2 we obtain

$$
\begin{aligned}
J_2 &\approx \|b(\cdot)-b_B\|_{L_\Psi(\mathcal{E})} \int_{\complement_{\mathcal{E}(x,2kr)}} \frac{|f(y)|}{\rho(x_0-y)^{\gamma-\alpha}} dy \\
&\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{\complement_{\mathcal{E}(x,2kr)}} \frac{|f(y)|}{\rho(x_0-y)^{\gamma-\alpha}} dy \\
&\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \|f\|_{L_\Phi(\mathcal{E}(x,t))} \Psi^{-1}\big(t^{-\gamma}\big) \frac{dt}{t}.
\end{aligned}
$$

gathering the estimates for J_1 and J_2, we get

$$
\|[b,I_\alpha^P]f_2\|_{L_\Psi(\mathcal{E})} \lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \Psi^{-1}\big(t^{-\gamma}\big) \|f\|_{L_\Phi(\mathcal{E}(x_0,t))} \frac{dt}{t}. \tag{4.17}
$$

By using (4.6) we unite (4.17) with (4.16), which completes the proof.

Theorem 4.3 *Let $0<\alpha<\gamma$, $b\in BMO(\mathbb{R}^n)$, Φ, Ψ Young functions and (Φ,Ψ) satisfy the conditions* (3.1) *and* (3.2). *Let $\Psi^{-1}(r)\approx\Phi^{-1}(r)\,r^{-\frac{\alpha}{\gamma}}$, $\Phi\in\Delta_2\cap\nabla_2$, and the functions (φ_1,φ_2) and (Φ,Ψ) satisfy the condition*

$$
\int_r^{\infty} \Big(1+\ln\frac{t}{r}\Big)\Psi^{-1}\big(t^{-\gamma}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \frac{dt}{t} \le C\,\varphi_2(x,r), \tag{4.18}
$$

where C does not depend on x, r. Then the operator $[b,I_\alpha^P]$ is bounded from $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^P_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof. The proof is similar to the proof of Theorem 4.1 thanks to Lemma 4.5.

Remark 4.3 Note that Theorem 4.3 in the isotropic case $P=I$ were proved in [9].

For proving our main results, we need the following estimate.

Lemma 4.6 *If $b\in L^1_{\rm loc}(\mathbb{R}^n)$ and $\mathcal{E}_0:=\mathcal{E}(x_0,r_0)$, then*

$$
r_0^\alpha|b(x)-b_{\mathcal{E}_0}| \lesssim |b,I_\alpha^P|\chi_{\mathcal{E}_0}(x) \ \text{ for every } \ x\in\mathcal{E}_0.
$$

Proof. It is well-known that

$$
\mathrm{M}^P_{b,\alpha}f(x) \le 2^{\gamma-\alpha} M^P_{b,\alpha}f(x), \tag{4.19}
$$

where $\mathrm{M}^P_{b,\alpha}(f)(x)=\sup\limits_{\mathcal{E}\ni x}|\mathcal{E}|^{-1+\frac{\alpha}{\gamma}}\int_{\mathcal{E}}|b(x)-b(y)||f(y)|dy$.

Now let $x \in \mathcal{E}_0$. By using (4.19), we get

$$
\begin{aligned}
|b, I_\alpha^P| &\gtrsim M_{b,\alpha}^P \chi_{\mathcal{E}_0}(x) \gtrsim \mathrm{M}_{b,\alpha}^P f(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}} |b(x) - b(y)| \chi_{\mathcal{E}_0} dy \\
&= \sup_{\mathcal{E} \ni x} |\mathcal{E}|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E} \cap \mathcal{E}_0} |b(x) - b(y)| dy \gtrsim |\mathcal{E}_0|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}_0 \cap \mathcal{E}_0} |b(x) - b(y)| dy \\
&\gtrsim |\mathcal{E}_0|^{-1+\frac{\alpha}{\gamma}} \Big| \int_{\mathcal{E}_0} (b(x) - b(y)) dy \Big| = r_0^\alpha |b(x) - b_{\mathcal{E}_0}|.
\end{aligned}
$$

The following theorem gives necessary and sufficient conditions for Spanne-type boundedness of the operator $[b, I_\alpha^P]$ from $M_{\Phi,\varphi_1}^P(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^P(\mathbb{R}^n)$.

Theorem 4.4 *Let $0 < \alpha < \gamma$, $b \in BMO(\mathbb{R}^n)$, (Φ, Ψ) be Young functions, and let $\varphi_1 \in \Omega_{\Phi,P}$, $\varphi_2 \in \Omega_{\Psi,P}$.*

1. *Let $\Psi^{-1}(t) \approx t^{-\alpha/\gamma} \Phi^{-1}(t)$ and $\Phi, \Psi \in \Delta_2 \cap \nabla_2$, then the condition*

$$
\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \Psi^{-1}\big(t^{-\gamma}\big) \operatorname*{ess\,inf}_{t<s<1} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \frac{dt}{t} \le C\, \varphi_2(r), \tag{4.20}
$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $[b, I_\alpha^P]$ from $M_{\Phi,\varphi_1}^P(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^P(\mathbb{R}^n)$.

2. *If $\Psi \in \Delta_2$ and $\varphi_1 \in \mathcal{G}_\Phi$, then the condition* (4.14) *is necessary for the boundedness of $|b, I_\alpha^P|$ from $M_{\Phi,\varphi_1}^P(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^P(\mathbb{R}^n)$.*

3. *Let $\Psi^{-1}(t) \approx t^{-\alpha/\gamma} \Phi^{-1}(t)$ and $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. If $\varphi_1 \in \mathcal{G}_\Phi$ satisfies the condition*

$$
\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) t^\alpha\, \varphi_1(t) \frac{dt}{t} \le C r^\alpha \varphi_1(r), \tag{4.21}
$$

for all $r > 0$, where $C > 0$ does not depend on r, then the condition (4.14) *is necessary and sufficient for the boundedness of $|b, I_\alpha^P|$ from $M_{\Phi,\varphi_1}^P(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^P(\mathbb{R}^n)$.*

Proof. The first statement of the theorem follows from Theorem 4.3.

We shall now prove the second part. Let $\mathcal{E}_0 = \mathcal{E}(x_0, r_0)$ and $x \in \mathcal{E}_0$. By Lemma 4.6 we have $r_0^\alpha |b(x) - b_{\mathcal{E}_0}| \lesssim |b, I_\alpha^P| \chi_{\mathcal{E}_0}(x)$. Therefore, by Lemma 3.1 and Lemma 4.2

$$
\begin{aligned}
r_0^\alpha &\lesssim \frac{\| |b, I_\alpha^P| \chi_{\mathcal{E}_0} \|_{L_\Psi(\mathcal{E}_0)}}{\| b(\cdot) - b_{\mathcal{E}_0} \|_{L_\Psi(\mathcal{E}_0)}} \lesssim \frac{1}{\|b\|_*} \| |b, I_\alpha^P| \chi_{\mathcal{E}_0} \|_{L_\Psi(\mathcal{E}_0)} \Psi^{-1}(|\mathcal{E}_0|^{-1}) \\
&\lesssim \frac{1}{\|b\|_*} \varphi_2(r_0) \| |b, I_\alpha^P| \chi_{\mathcal{E}_0} \|_{M_{\Psi,\varphi_2}^P} \lesssim \varphi_2(r_0) \| \chi_{\mathcal{E}_0} \|_{M_{\Phi,\varphi_1}^P} \lesssim \frac{\varphi_2(r_0)}{\varphi_1(r_0)}.
\end{aligned}
$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem.

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