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Existence of solutions for a resonant problem under Landesman-Lazer type conditions involving more general elliptic operators in divergence form

Rabil Ayazoglu (Mashiyev) * · **Sidika Şule Şener** · **Tuba Ağırman Aydın**

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Abstract. *The present paper is concerned with the resonant problem*

$$-div\,(a\,(x,\nabla u)) = \lambda_1\,|u|^{p-2}\,u + f\,(x,u) - g(x) \quad in \quad \Omega,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 2)$, $p \in (1,\infty)$ and div$(a\,(x,\nabla u))$ is a general elliptic operators in divergence form. By assuming a Landesman-Lazer type condition and using a variational method based on the Minimum Principle, we show the existence of a weak solution in the Sobolev space $W_0^{1,p}\,(\Omega)$.

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1 Introduction and preliminaries

In this work, we obtain existence and multiplicity results for equations involving more general elliptic operators in divergence form

$$\begin{cases} -\text{div}\,(a\,(x,\nabla u)) = \lambda_1\,|u|^{p-2}\,u + f\,(x,u) - g(x) \quad \text{in} \quad \Omega, \\ u \in W_0^{1,p}\,(\Omega)\,, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 2)$, $p \in (1,\infty)$ and div$(a\,(x,\nabla u))$ is a more general elliptic operators in divergence form, $g \in L^{p'}\,(\Omega)$, where p' is the conjugate exponent of p with $1/p + 1/p' = 1$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded

* Corresponding author

R. Ayazoglu (Mashiyev)
Faculty of Education, Bayburt University, 69000 Bayburt, Turkey
E-mail: rabilmashiyev@gmail.com

S.Ş. Şener
Faculty of Science, Ataturk University, 25000 Erzurum, Turkey
E-mail: senersule@atauni.edu.tr

T.A. Aydın
Faculty of Education, Bayburt University, 69000 Bayburt, Turkey
E-mail: tubaaydin@bayburt.edu.tr

Carathéodory function. Such operators arise, for example, from the expression of the p-Laplacian in curvilinear coordinates. In the case of the p-Laplacian, this is usually achieved by using the uniform convexity of the Sobolev space $E := W_0^{1,p}(\Omega)$ with the norm

$$\|u\|_{W_0^{1,p}(\Omega)} := \|u\|_E = \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}.$$

In order to extend this idea to more general equations, we introduce a notion of uniformly convex functional.

Let X is a Banach space.

Definition 1.1 *We shall say that the convex functional $A : X \to \mathbb{R}$ is uniformly convex on the (convex) set $\Omega \subset X$ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$A\left(\frac{x+y}{2}\right) \leq \frac{1}{2}A(x) + \frac{1}{2}A(y) - \delta(\varepsilon)$$

for $x, y \in \Omega$ and $\|x - y\|_X > \varepsilon$.

If functional A is uniformly convex on every ball of X, we shall say that functional A is locally uniformly convex.

Example 1 $x \to x^p$ *is strongly p-monotone if* $p \geq 2$.

Moreover, although the well-known Poincaré inequality, i.e.,

$$\|u\|_p \leq C \|\nabla u\|_p, \tag{1.2}$$

holds true in $L^p(\Omega)$, where $C > 0$ is a consatat and $\|u\|_{L^p(\Omega)} := \|u\|_p$.

Moreover, let λ_1 denote the first eigenvalue for $-\Delta_p$ on Ω with zero Dirichlet boundary condition which has the variational characterization

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in W_0^{1,p}(\Omega) \setminus \{0\} \text{ with } \int_\Omega |u|^p \, dx = 1 \right\}.$$

Recall that λ_1 is simple, positive and there exists a unique positive eigenfunction ϕ_1 whose norm in $W_0^{1,p}(\Omega)$ equals one (see [1]).

Resonance problems of quasilinear elliptic partial differential equations have been studied extensively in the usual Sobolev spaces. Since the celebrated paper by Landesman and Lazer (see [11]), many existence results were obtained under various nonlinearity growth conditions and the Landesman–Lazer conditions (see [2], [3], [6], [8], [9], [12], [14] and references therein).

As we know, the geometry of the problem to (1.1) depends strongly on the values of r in the estimate below

$$|F(x,u)| \leq C(h(x) + |u|^q),$$

where C is a positive constant, and $F(x,s) = \int_0^s f(x,t)\, dt$ and $h \in L^{p'}(\Omega)$ with $h(x) \geq 0$ for any $x \in \Omega$. We can discuss three distinct cases:

(i) $q < p$ (sublinear-like),

(ii) $q > p$ (superlinear-like),

(iii) $q = p$ (of resonance type).

For the cases (i), (ii) and some other mixed cases there are many papers so we refer the reader to [10], [18] for (i), to [5], [7] for (ii), and to [13], [16], [17] for mixed cases. For the case (iii), which is the main subject of the present paper, the solution of (1.1) depend in an essential manner on the asymptotic behavior of f. Let assume, for example, that f is asymptotic linear, that is $\frac{f(x,u)}{|u|^{p-2}u}$ has a finite limit as $|u| \to \infty$. If the term $\lambda_1 + \frac{f(x,u)}{|u|^{p-2}u}$ meets the eigenvalue λ_1, then problem (1.1) is said to be with resonance at infinity. For the treatment of resonance and the existence of a solution, it is sufficient that $g \in L^{p'}(\Omega)$ satisfy the Landesman-Lazer's condition.

Assume that $a : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ is continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x,\xi)$, i.e. $a(x,\xi) = \nabla_\xi A(x,\xi)$, and that there are positive real number C_0 and nonnegative measurable function h on Ω such that $h \in L^{p'}(\Omega)$ for a.e. $x \in \Omega$. We can give the following examples for the operators A and a:

(i) Set $A(x,\xi) = \frac{1}{p}|\xi|^p$, $a(x,\xi) = |\xi|^{p-2}\xi$, where $p \geq 2$. Then, we get the p-Laplace operator $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$.

(ii) Set $A(x,\xi) = \frac{1}{p}\left[\left(1+|\xi|^2\right)^{\frac{p}{2}} - 1\right]$, $a(x,\xi) = \left(1+|\xi|^2\right)^{\frac{p-2}{2}}\xi$, where $p \geq 2$. Then, we obtain the generalized mean curvature operator $\operatorname{div}\left(\left(1+|\nabla u|^2\right)^{\frac{p-2}{2}}\nabla u\right)$.

Suppose that a and A satisfy the following hypotheses:

(A_1) The following inequality holds

$$|a(x,\xi)| \leq C_0\left(h(x) + |\xi|^{p-1}\right)\ \forall x \in \Omega, \xi \in \mathbb{R}^N, h \in L^{p'}(\Omega)$$

for some constant $C_0 > 0$;

(A_2) A is p-uniformly convex: There exists a constant $k > 0$ such that

$$A(x,\frac{\xi+\psi}{2}) \leq \frac{1}{2}A(x,\xi) + \frac{1}{2}A(x,\psi) - k|\xi-\psi|^p$$

for all $x \in \Omega$ and $\xi, \psi \in \mathbb{R}^N$;

(A_3) The following inequality holds

$$|\xi|^p \leq a(x,\xi)\cdot\xi \leq pA(x,\xi)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$;

(A_4) $A(x,0) = 0$.

Moreover, to construct our basic results, we also suppose the following assumptions exist.

(g_1) $g \in L^{p'}(\Omega)$;

(f_1) $|f(x,s)| \leq \gamma(x)$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, where $\gamma \in L^{p'}(\Omega)$;

(f_2) $\limsup\limits_{s\to+\infty} f(x,s) = f^{+\infty}(x) \in L^\infty(\Omega)$, $\liminf\limits_{s\to-\infty} f(x,s) = f_{-\infty}(x) \in L^\infty(\Omega)$;

(f_3) $\int_\Omega f^{+\infty}(x)\phi_1(x)\,dx < \int_\Omega g(x)\phi_1(x)\,dx < \int_\Omega f_{-\infty}(x)\phi_1(x)\,dx$.

As we know, under (f_2), problem (1.1) may not have a solution. However, in [11] Landesman and Lazer have showed that the condition (f_3) (so called Landesman-Lazer's condition) is a sufficient condition for the existence of solution of (1.1).

In this paper by introducing a of Landesman–Lazer type condition (see (f_3)) we shall prove an existence result for a p-Laplacian type operator on resonance in bounded domain with the nonlinearities f and g to be functions. We also point out that in that papers, the property $a(x,\xi)\cdot\xi = pA(x,\xi)$, which may not hold under our assumptions by (A_3), play an important role in the arguments.

Define the energy functional $I_{\lambda_1} : E \to \mathbb{R}$ associated to (1.1) by

$$I_{\lambda_1}(u) = \int_\Omega A(x,\nabla u)\,dx - \frac{\lambda_1}{p}\int_\Omega |u|^p\,dx - \int_\Omega F(x,u)\,dx + \int_\Omega gu dx,$$

where $F(x,u) = \int_0^u f(x,t)dt$.

Letting

$$\Lambda(u) = \int_\Omega A(x, \nabla u)\,dx,$$

and

$$J_{\lambda_1}(u) = \frac{\lambda_1}{p}\int_\Omega |u|^p\,dx + \int_\Omega F(x,u)\,dx - \int_\Omega gudx.$$

As we know, standard arguments imply that $J_{\lambda_1} \in C^1(E,\mathbb{R})$ and its derivative given by

$$\langle J'_{\lambda_1}(u), \upsilon\rangle = \lambda_1 \int_\Omega |u|^{p-2}\,u\upsilon dx + \int_\Omega f(x,u)\,\upsilon dx - \int_\Omega g\upsilon dx$$

for all $u, \upsilon \in E$.

We say that $u \in E$ is a *weak solution* of problem (1.1) if

$$\int_\Omega a(x,\nabla u)\,\nabla \upsilon dx - \lambda_1 \int_\Omega |u|^{p-2}\,u\upsilon dx - \int_\Omega f(x,u)\,\upsilon dx + \int_\Omega g\upsilon dx = 0$$

for all $\upsilon \in E$.

2 Auxiliary results

Lemma 2.1

(i) *A verifies the growth condition*

$$|A(x,\xi)| \le C_0(h(x)\,|\xi| + |\xi|^p),$$

for all $\xi \in \mathbb{R}^N$ *and a.e.* $x \in \Omega$.

(ii) *A is* p*-homogeneous*

$$A(x, z\xi) \le A(x,\xi)\,z^p,$$

for all $z \ge 1, \xi \in \mathbb{R}^N$ *and a.e.* $x \in \Omega$.

Proof.

(i) For any $\xi \in \mathbb{R}^N$, we have

$$A(x,\xi) = \int_0^1 \frac{d}{dt}A(x,t\xi)\,dt = \int_0^1 a(x,t\xi)\cdot\xi dt$$

By hypothesis $(A_\mathbf{1})$, we have

$$\begin{aligned}|A(x,\xi)| &\le \int_0^1 |a(x,t\xi)|\cdot|\xi|\,dt \le C_0\int_0^1 (h(x) + |\xi|^{p-1}\,|t|^{p-1})\,|\xi|\,dt \\ &\le C_0\int_0^1 (h(x)\,|\xi| + |\xi|^p\,|t|^{p-1})\,dt \\ &\le C_0(h(x)\,|\xi| + |\xi|^p).\end{aligned}$$

(ii) To see that, let us define $g(t) = A(t\xi)$. Then, by (A_3)

$$g'(t) = a(x,t\xi)\cdot\xi = \frac{1}{t}a(x,t\xi)\cdot t\xi \le \frac{p}{t}A(x,t\xi) = \frac{p}{t}g(t),$$

then

$$\frac{g'(t)}{g(t)} \leq \frac{p}{t},$$

and integrating both side over $(1, z)$, we have

$$\log g(z) - \log g(1) \leq p \log z.$$

Then,

$$\frac{g(z)}{g(1)} \leq z^p,$$

so we conclude that

$$A(x, z\xi) \leq A(x, \xi) z^p.$$

The proof is complete.

Lemma 2.2

(i) *The functional Λ is well-defined on E.*

(ii) *The functional Λ is of class $C^1(E, \mathbb{R})$ and*

$$\langle \Lambda'(u), \varphi \rangle = \int_\Omega a(x, \nabla u) \cdot \nabla \varphi dx,$$

for all $u, \varphi \in E$.

(iii) *The functional Λ is weakly lower semi-continuos on E.*

(iv) *For all $u, v \in E$*

$$\Lambda(\frac{u+v}{2}) \leq \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(v) - k\|u - v\|_E^p.$$

(v) *For all $u, v \in E$*

$$\Lambda(u) - \Lambda(v) \geq \langle \Lambda'(v), u - v \rangle.$$

Proof. (i) By (i) in Lemma 2.1 and (1.2), we have

$$\begin{aligned}\Lambda(u) &= \int_\Omega A(x, \nabla u) dx \leq C_0 \int_\Omega h(x) |\nabla u| dx + C_0 \int_\Omega |\nabla u|^p dx \\ &\leq C_0 C \|h\|_{p'} \|u\|_E + C_0 C \|u\|_E^p < \infty.\end{aligned}$$

Hence, Λ is well defined on E.

(ii) Let $u, \varphi \in E$, $x \in \Omega$, and $0 < |r| < 1$. Then, by the mean value theorem, there exists $v \in [0, 1]$ such that

$$\begin{aligned}&\left|\frac{A(x, \nabla u(x) + r\nabla\varphi(x)) - A(x, \nabla u(x))}{r}\right| \\ &= \left|\int_0^1 a(x, \nabla u(x) + vr\nabla\varphi(x)) \nabla\varphi(x) dv\right| \\ &\leq C_0 \int_0^1 (h(x) + |\nabla u(x) + vr\nabla\varphi(x)|^{p-1}) |\nabla\varphi(x)| dv \\ &\leq C_0(h(x) + (|\nabla u(x)| + |\nabla\varphi(x)|)^{p-1}) |\nabla\varphi(x)| \\ &\leq C_0 h(x) |\nabla\varphi(x)| + C_0 |\nabla\varphi(x)| (|\nabla u(x)| + |\nabla\varphi(x)|)^{p-1} \\ &\leq C_0 h(x) |\nabla\varphi(x)| + C_0 2^{p-1} |\nabla\varphi(x)| (|\nabla u(x)|^{p-1} + |\nabla\varphi(x)|^{p-1}) \\ &\leq C_0 h(x) |\nabla\varphi(x)| + C_0 2^{p-1} |\nabla\varphi(x)| |\nabla u(x)|^{p-1} + C_0 2^{p-1} |\nabla\varphi(x)|^p.\end{aligned}$$

By help of the (1.2), we can see $h(x)|\nabla\varphi(x)|$, $|\nabla\varphi(x)||\nabla u(x)|^{p-1}$, and $|\nabla\varphi(x)|^p$ are integrable on Ω, so the right-hand side is integrable on Ω. Applying the Lebesgue Dominated convergence theorem, we have

$$\begin{aligned}\left\langle \Lambda'(u),\varphi\right\rangle &= \lim_{r\to 0}\int\limits_{\Omega}\frac{A(x,\nabla u+r\nabla\varphi)-A(x,\nabla u)}{r}dx\\ &= \int\limits_{\Omega} a(x,\nabla u)\cdot\nabla\varphi dx\end{aligned}$$

Next, let show the continuity of Λ' on E. Suppose $u_n\to u$ in E and let define $\theta(x,u)=a(x,\nabla u)$. Using the hypothesis (A_1), we conclude that $\theta(x,u_n)\to\theta(x,u)$ in $(L^{p'}(\Omega))^N$ a.e. $x\in\Omega$. Then, we have

$$\left|\left\langle \Lambda'(u_n)-\Lambda'(u),\varphi\right\rangle\right|\le\|\theta(x,u_n)-\theta(x,u)\|_{p'}\|\nabla\varphi\|_p,$$

and so

$$\left\|\Lambda'(u_n)-\Lambda'(u)\right\|\le\|\theta(x,u_n)-\theta(x,u)\|_{p'}\to 0,$$

as $n\to\infty$.

(iii) By corollary III.8 in Brezis [4], it is enough to show that Λ is lower semi-continuous. Since Λ is convex (by condition (A_2)), we deduce that for any $\upsilon\in E$, the following inequality holds

$$\int_\Omega A(x,\nabla\upsilon)\,dx\ge\int_\Omega A(x,\nabla u)\,dx+\int_\Omega a(x,\nabla u)\cdot(\nabla\upsilon-\nabla u)\,dx.$$

Using condition (A_1), we have

$$\begin{aligned}&\int_\Omega A(x,\nabla\upsilon)\,dx\\ &\ge\int_\Omega A(x,\nabla u)\,dx-\int_\Omega|a(x,\nabla u)||\nabla\upsilon-\nabla u|\,dx\\ &\ge\int_\Omega A(x,\nabla u)\,dx-C_0\int_\Omega h(x)|\nabla(\upsilon-u)|\,dx-C_0\int_\Omega|\nabla(\upsilon-u)||\nabla u|^{p-1}\,dx\\ &\ge\int_\Omega A(x,\nabla u)\,dx-C_1\|h\|_{p'}\|\nabla(\upsilon-u)\|_p-C_2\|\nabla(\upsilon-u)\|_p\left\||\nabla u|^{p-1}\right\|_{p'}\\ &\ge\int_\Omega A(x,\nabla u)\,dx-C_3\,||\upsilon-u||_E-C_4\,||\upsilon-u||_E\,\|u\|_E^{p-1}\\ &\ge\int_\Omega A(x,\nabla u)\,dx-C_3\,||\upsilon-u||_E-C_5\,||\upsilon-u||_E\\ &\ge\int_\Omega A(x,\nabla u)\,dx-\varepsilon\end{aligned}$$

for all $\upsilon\in E$ with $||\upsilon-u||_E<\delta=\dfrac{\varepsilon}{C_3+C_5}$. So, we deduce that Λ is weakly lower semi-continuous.

(iv) Using condition (A_2), we have

$$\begin{aligned}\Lambda(\frac{u+v}{2}) &= \int_\Omega A(x,\frac{\nabla u+\nabla v}{2})dx\\ &\le \frac{1}{2}\int_\Omega A\left(x,\nabla u\right)dx+\frac{1}{2}\int_\Omega A\left(x,\nabla v\right)dx-k\int_\Omega \left|\nabla u-\nabla v\right|^p dx\\ &\le \frac{1}{2}\Lambda\left(u\right)+\frac{1}{2}\Lambda\left(v\right)-k\left\|u-v\right\|_E^p.\end{aligned}$$

(v) Since Λ is convex (by condition (A_2)), we can find $t\in(0,1)$ such that

$$\begin{aligned}&\frac{\Lambda\left(v+t\left(u-v\right)\right)-\Lambda\left(v\right)}{t}\\ =\ &\frac{\Lambda\left(\left(1-t\right)v+tu\right)-\Lambda\left(v\right)}{t}\\ \le\ &\frac{\left(1-t\right)\Lambda\left(v\right)+t\Lambda\left(u\right)-\Lambda\left(v\right)}{t}=\Lambda\left(u\right)-\Lambda\left(v\right).\end{aligned}$$

Letting $t\longrightarrow 0$, we have

$$\lim_{t\to 0}\frac{\Lambda\left(v+t\left(u-v\right)\right)-\Lambda\left(v\right)}{t}=\left\langle \Lambda'\left(v\right),u-v\right\rangle.$$

Thus, we obtain

$$\left\langle \Lambda'\left(v\right),u-v\right\rangle \le \Lambda\left(u\right)-\Lambda\left(v\right).$$

The proof of Lemma 2.2 is complete.

Lemma 2.3 *(Palais-Smale condition $(PS)_c$, see [15]).* *Let X be a real Banach space. A functional $I\in C^1(X,\mathbb{R})$ satisfies the Palais-Smale condition, $(PS)_c$ condition for short, if any sequence $\{u_n\}$ in X such that*

$$|I(u_n)|\le c \text{ and } I'(u_n)\to 0, \tag{2.1}$$

has a convergent subsequence, where $c\in\mathbb{R}$.

Lemma 2.4 *(The Minimum Principle, see [15]).* *Let X be a real Banach space and $I\in C^1(X,\mathbb{R})$. Assume that*
(i) I is bounded from below, $c=\inf I$,
(ii) I satisfies (PS) condition.
Then, there exists $u_0\in X$ such that $I(u_0)=c$.

3 The main results and proofs

The main theorem which we deal with in the present paper is

Theorem 3.1 *Suppose the conditions (A_1)-(A_4) and (f_1)-(f_3) hold. Then, the problem (1.1) has at least one nontrivial weak solution in E.*

Lemma 3.1 *I_{λ_1} is well-defined on E and of class $C^1(E,\mathbb{R})$, and its derivative given by*

$$\langle I'_{\lambda_1}(u),v\rangle=\int_\Omega a\left(x,\nabla u\right)\nabla v dx-\lambda_1\int_\Omega |u|^{p-2}uvdx-\int_\Omega f\left(x,u\right)vdx+\int_\Omega gvdx$$

for all $u,v\in E$.

Proof. This comes from (i) and (ii) in Lemma 2.2 and properties of J_{λ_1}.

Lemma 3.2 *I_{λ_1} satisfies $(PS)_c$ condition on E provided that the condition (f_3) hold.*

Proof. First, we prove that $\{u_n\}$ is bounded in E. We assume by contradiction that $\|u_n\|_E \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_E}$ for all n. Thus $\{v_n\}$ is bounded in E. Since E is reflexive, we can assume that there exists a subsequence which we still denote by $\{v_n\}$ converging weakly to a certain v in E. Since we have the embedding $E \hookrightarrow\hookrightarrow L^p(\Omega)$ (compact), then $\{v_n\}$ converges strongly to a certain v in $L^p(\Omega)$.

$$I_{\lambda_1}(u_n) = \int_\Omega A(x, \nabla u_n)\, dx - \frac{\lambda_1}{p}\int_\Omega |u_n|^p\, dx - \int_\Omega F(x, u_n)\, dx + \int_\Omega g u_n dx. \quad (3.1)$$

Taking into account (A_3) together with (2.1), and dividing (3.1) by $\|u_n\|_E^p$, we have

$$\frac{1}{p}\int_\Omega \frac{|\nabla u_n|^p}{\|u_n\|_E^p} dx - \frac{\lambda_1}{p}\int_\Omega \frac{|u_n|^p}{\|u_n\|_E^p} dx - \int_\Omega \frac{F(x,u_n)}{\|u_n\|_E^p} dx + \int_\Omega \frac{g u_n}{\|u_n\|_E^p} dx \leq \frac{c}{\|u_n\|_E^p}. \quad (3.2)$$

Now we take the limit of both sides, we have

$$\limsup_{n\to+\infty}\left(\frac{1}{p}\int_\Omega |\nabla v_n|^p\, dx - \frac{\lambda_1}{p}\int_\Omega |v_n|^p\, dx - \int_\Omega \frac{F(x,u_n)}{\|u_n\|_E^p} dx + \int_\Omega \frac{g u_n}{\|u_n\|_E^p} dx\right) \leq 0. \quad (3.3)$$

Moreover, using (f_1) and L'Hôpital's rule, we have

$$\int_\Omega \frac{F(x,u_n)}{\|u_n\|_E^p} dx = \frac{1}{\|u_n\|_E^{p-1}}\int_\Omega \frac{F(x,u_n)}{\|u_n\|_E} dx \to 0 \text{ as } n \to \infty.$$

Similarly, considering that $g \in L^{p'}(\Omega)$, we also get

$$\int_\Omega \frac{g u_n}{\|u_n\|_E^p} dx = \frac{1}{\|u_n\|_E^{p-1}}\int_\Omega g v_n dx \to 0 \text{ as } n \to \infty.$$

Hence, it follows

$$\limsup_{n\to+\infty}\left(\int_\Omega \frac{F(x,u_n)}{\|u_n\|_E^p} dx + \int_\Omega \frac{g u_n}{\|u_n\|_E^p} dx\right) = 0. \quad (3.4)$$

Then, considering (3.3) together with (3.4), we get

$$\limsup_{n\to+\infty}\int_\Omega |\nabla v_n|^p\, dx \leq \lambda_1 \int_\Omega |v_n|^p\, dx.$$

Since $\{v_n\}$ converges strongly to v in $L^p(\Omega)$, we have

$$\limsup_{n\to+\infty}\int_\Omega |v_n|^p\, dx = \int_\Omega |v|^p\, dx.$$

Thus, we can write

$$\limsup_{n\to+\infty}\int_\Omega |\nabla v_n|^p\, dx \leq \lambda_1 \int_\Omega |v|^p\, dx.$$

Using the weak lower semi-continuity of norm, we have

$$\lambda_1 \int_\Omega |v|^p \, dx \leq \int\limits_\Omega |\nabla v|^p \, dx \leq \liminf_{n \to +\infty} \int_\Omega |\nabla v_n|^p \, dx$$
$$\leq \limsup_{n \to +\infty} \int_\Omega |\nabla v_n|^p \, dx \leq \lambda_1 \int_\Omega |v|^p \, dx.$$

Hence, we get $\{v_n\}$ converges strongly to a certain v in E, and

$$\int_\Omega |\nabla v|^p \, dx = \lambda_1 \int_\Omega |v|^p \, dx.$$

This implies, by the definition of ϕ_1, that $v = \pm\phi_1$.

Moreover, using (2.1), we can obtain the following two inequalities

$$-cp \leq pI_{\lambda_1}(u_n) \leq cp \tag{3.5}$$

and

$$-\varepsilon_n \|u_n\|_E \leq \langle I'_{\lambda_1}(u_n), u_n \rangle_{E^*} \leq \varepsilon_n \|u_n\|_E, \tag{3.6}$$

where $\varepsilon_n \to 0$ and E^* is dual space of E.

By considering the following two cases, we shall conclude that $\{u_n\}$ is bounded in E.

We then consider the following two cases.

Case 1. Suppose that $v_n \to -\phi_1$. Since $u_n \to -\infty$ as $n \to +\infty$, by (f_2), we have

$$f(x, u_n) \to f_{-\infty}(x) \text{ a.e. } x \in \Omega,$$

and

$$\frac{F(x, u_n)}{u_n} \to f_{-\infty}(x) \text{ a.e. } x \in \Omega. \tag{3.7}$$

Hence, letting n tend to infinity, and considering the Lebesgue Dominated Theorem, we get

$$\lim_{n \to +\infty} \int_\Omega \left(f(x, u_n) \frac{u_n}{\|u_n\|_E} - \frac{pF(x, u_n)}{u_n} \frac{u_n}{\|u_n\|_E} \right) dx$$
$$= \lim_{n \to +\infty} \int_\Omega \left(f(x, u_n) v_n - \frac{pF(x, u_n)}{u_n} v_n \right) dx = (p-1) \int_\Omega f_{-\infty}(x) \phi_1 dx. \tag{3.8}$$

On the other hand, by summing up (3.5) and (3.6),

$$-cp - \varepsilon_n \|u_n\|_E$$
$$\leq \int_\Omega \left(a(x, \nabla u_n) \nabla u_n - pA(x, \nabla u_n) \right) dx + \lambda_1 p \int_\Omega \frac{1}{p} |u_n|^p \, dx$$
$$-\lambda_1 \int_\Omega |u_n|^p \, dx + \int_\Omega \left(pF(x, u_n) \, dx - f(x, u_n) u_n \right) dx + (1-p) \int_\Omega g u_n dx$$
$$\leq \int_\Omega \left(pF(x, u_n) \, dx - f(x, u_n) u_n \right) dx + (1-p) \int_\Omega g u_n dx.$$

Dividing by $\|u_n\|_E$, we have

$$\frac{-cp}{\|u_n\|_E} - \varepsilon_n \leq \int_\Omega \left(p \frac{F(x, u_n)}{u_n} v_n - f(x, u_n) v_n \right) dx + (1-p) \int_\Omega g v_n dx. \tag{3.9}$$

Since $g \in L^{p'}(\Omega)$ and $\|v_n - (-\phi_1)\|_E \to 0$, we obtain

$$\lim_{n\to+\infty} \int_\Omega g v_n dx = -\int_\Omega g\phi_1 dx. \tag{3.10}$$

In (3.9), taking limit to both sides and using (3.8) and (3.10), we get

$$(1-p)\int_\Omega f_{-\infty}(x)\,\phi_1 dx - (1-p)\int_\Omega g\phi_1 dx \geq 0$$

that is, since $p>1$,

$$\int_\Omega g\phi_1 dx \geq \int_\Omega f_{-\infty}(x)\,\phi_1 dx,$$

which contradicts (f_3).

Case 2. Suppose that $v_n \to \phi_1$. Since $u_n \to +\infty$ as $n \to +\infty$, by (f_2), we have

$$f(x,u_n) \to f^{+\infty}(x) \text{ a.e. } x \in \Omega,$$

and

$$\frac{F(x,u_n)}{u_n} \to f^{+\infty}(x) \text{ a.e. } x \in \Omega. \tag{3.11}$$

Hence, letting n tend to infinity, and considering the Lebesgue Dominated Theorem, we get

$$\begin{aligned}
&\lim_{n\to+\infty} \int_\Omega \left(f(x,u_n)\frac{u_n}{\|u_n\|_E} - \frac{pF(x,u_n)}{u_n}\frac{u_n}{\|u_n\|_E}\right) dx \\
&= \lim_{n\to+\infty} \int_\Omega \left(f(x,u_n)\,v_n - \frac{pF(x,u_n)}{u_n} v_n\right) dx \\
&= (1-p)\int_\Omega f^{+\infty}(x)\,\phi_1 dx.
\end{aligned} \tag{3.12}$$

By summing up (3.5) and (3.6) again, we have

$$\begin{aligned}
& cp + \varepsilon_n \|u_n\|_E \\
&\geq \int_\Omega \left(pA(x,\nabla u_n) - a(x,\nabla u_n)\nabla u_n\right) dx - \lambda_1 p \int_\Omega \frac{1}{p}|u_n|^p dx \\
&\quad + \lambda_1 \int_\Omega |u_n|^p dx + \int_\Omega \left(f(x,u_n)\,u_n - pF(x,u_n)\right) dx + (p-1)\int_\Omega g u_n dx \\
&\geq \int_\Omega \left(f(x,u_n)\,u_n - pF(x,u_n)\right) dx + (p-1)\int_\Omega g u_n dx.
\end{aligned}$$

Dividing by $\|u_n\|_E$, we get

$$\int_\Omega \left(f(x,u_n)\,v_n - p\frac{F(x,u_n)}{u_n} v_n\right) dx + (p-1)\int_\Omega g v_n dx \leq \frac{cp}{\|u_n\|_E} + \varepsilon_n \tag{3.13}$$

Since $g \in L^{p'}(\Omega)$ and $\|v_n - \phi_1\|_E \to 0$, we have

$$\lim_{n\to+\infty} \int_\Omega g v_n dx = \int_\Omega g\phi_1 dx \tag{3.14}$$

In (3.13), taking limit to both sides and using (3.12) and (3.14), we conclude

$$(1-p)\int_\Omega f^{+\infty}(x)\,\phi_1 dx + (p-1)\int_\Omega g\phi_1 dx \leq 0,$$

that is, since $p > 1$,

$$\int_{\Omega} g\phi_1 dx \leq \int_{\Omega} f^{+\infty}(x)\, \phi_1 dx,$$

which contradicts (f_3).

From the two cases above, we conclude $\{u_n\}$ is bounded in E. Hence, $u_n \rightharpoonup u$ in E. Now, we shall show $u_n \to u$ in E. By the embedding $E \hookrightarrow\hookrightarrow L^p(\Omega)$ and (2.1), it follows

$$\left\langle I'_{\lambda_1}(u_n), u_n - u \right\rangle \to 0 \text{ as } n \to \infty.$$

On the other hand, we have

$$\begin{aligned} &\left\langle I'_{\lambda_1}(u_n), u_n - u \right\rangle \\ &= \int_{\Omega} a(x, \nabla u_n)(\nabla u_n - \nabla u)\, dx - \lambda_1 \int_{\Omega} |u_n|^{p-2} u_n (u_n - u)\, dx \\ &\quad - \int_{\Omega} f(x, u_n)(u_n - u)\, dx + \int_{\Omega} g(u_n - u)\, dx, \end{aligned}$$

thus

$$\begin{aligned} \int_{\Omega} a(x, \nabla u_n)(\nabla u_n - \nabla u)\, dx &= \left\langle I'_{\lambda_1}(u_n), u_n - u \right\rangle + \lambda_1 \int_{\Omega} |u_n|^{p-2} u_n (u_n - u)\, dx \\ &+ \int_{\Omega} f(x, u_n)(u_n - u)\, dx - \int_{\Omega} g(u_n - u)\, dx. \end{aligned}$$

Using the fact that $\{u_n\}$ converges strongly to a certain u in $L^p(\Omega)$ then $\|u_n\|_p \leq C$,$(C > 0)$ we have

$$\begin{aligned} \int_{\Omega} |u_n|^{p-2} u_n (u_n - u)\, dx &\leq \left| \int_{\Omega} |u_n|^{p-2} u_n (u_n - u)\, dx \right| \leq \left\| |u_n|^{p-1} \right\|_{p'} \|u_n - u\|_p \\ &\leq \|u_n\|_p^{p-1} \|u_n - u\|_p \leq C^{p-1} \|u_n - u\|_p . \end{aligned}$$

Moreover, by the embedding $E \hookrightarrow\hookrightarrow L^p(\Omega)$, we have $\|u_n - u\|_p \to 0$ as $n \to +\infty$. Hence, we deduce

$$\lim_{n \to +\infty} \int_{\Omega} |u_n|^{p-2} u_n (u_n - u)\, dx = 0.$$

Using similar arguments and considering the hypotheses on f (see (f_1)) and g (see (g_1)), we also deduce that

$$\lim_{n \to +\infty} \int_{\Omega} f(x, u_n)(u_n - u)\, dx = 0,$$

and

$$\lim_{n \to +\infty} \int_{\Omega} g(u_n - u)\, dx = 0.$$

All the above pieces of information imply

$$\int_{\Omega} a(x, \nabla u_n)(\nabla u_n - \nabla u)\, dx = 0,$$

that is,

$$\lim_{n \to +\infty} \left\langle \Lambda'(u_n), u_n - u \right\rangle = 0.$$

By using (v) in Lemma 2.2, we have

$$0=\lim_{n\to\infty}\left\langle \Lambda'(u_n),u-u_n\right\rangle \leq \lim_{n\to\infty}\left(\Lambda(u)-\Lambda(u_n)\right)=\Lambda(u)-\lim_{n\to\infty}\Lambda(u_n)$$

or

$$\lim_{n\to\infty}\Lambda(u_n)\leq\Lambda(u).$$

This fact and relation (iii) in Lemma 2.2 imply

$$\lim_{n\to\infty}\Lambda(u_n)=\Lambda(u).$$

We assume by contradiction that $\{u_n\}$ does not converge strongly to u in E. Then, there exists $\varepsilon>0$ and a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that $\|u_{n_m}-u\|_E\geq\varepsilon$. On the other hand, by (iv) in Lemma 2.2, we have

$$\frac{1}{2}\Lambda(u_{n_m})+\frac{1}{2}\Lambda(u)-\Lambda\left(\frac{u_{n_m}+u}{2}\right)\geq k\|u_{n_m}-u\|_E\geq k\varepsilon.$$

Letting $m\to\infty$ in the above inequality, we obtain

$$\limsup_{n\to\infty}\Lambda\left(\frac{u_{n_m}+u}{2}\right)\leq\Lambda(u)-k\varepsilon^p.$$

Moreover, we have $\left\{\frac{u_{n_m}+u}{2}\right\}$ converges weakly to u in E. Using (iii) in Lemma 2.2, we obtain

$$\Lambda(u)\leq\liminf_{n\to\infty}\Lambda\left(\frac{u_{n_m}+u}{2}\right),$$

and that is a contradiction. It follows that $\{u_n\}$ converges strongly to u in E. The proof of Lemma 3.2 is complete.

Lemma 3.3 *The functional I_{λ_1} is coercive on E provided (f_3) holds.*

Proof. We firstly note that, in the proof of the Lemma 3.2, we showed that if $I_{\lambda_1}(u_n)$ is a sequence bounded from above with $\|u_n\|_E\to\infty$ then (up to a subsequence), $v_n=\frac{u_n}{\|u_n\|_E}\to\pm\phi_1$ in E. Using this fact, we shall prove that I_{λ_1} is coercive on E provided (f_3) holds. Indeed, if I_{λ_1} is not coercive, it is possible to choose a sequence $\{u_n\}\subset E$ such that $\|u_n\|_E\to\infty$, $I_{\lambda_1}(u_n)\leq M$, where M is a constant, and $v_n=\frac{u_n}{\|u_n\|}\to\pm\phi_1$ in E.

By the assumption (A_3), we have

$$\begin{aligned}
&I_{\lambda_1}(u_n)\\
&\geq\frac{1}{p}\int_\Omega|\nabla u_n|^p\,dx-\frac{\lambda_1}{p}\int_\Omega|u_n|^p\,dx-\int_\Omega F(x,u_n)\,dx+\int_\Omega gu_n dx\\
&\geq-\int_\Omega F(x,u_n)\,dx+\int_\Omega gu_n dx.
\end{aligned}\tag{3.15}$$

Now, we shall investigate two cases:

Case 1. Assume that $v_n\to\phi_1$. Dividing (3.15) by $\|u_n\|_E$ and using (3.11), we have

$$\begin{aligned}
&-\int_\Omega f^{+\infty}(x)\,\phi_1dx+\int_\Omega g\phi_1dx\\
&=\lim_{n\to+\infty}\left(-\int_\Omega\frac{F(x,u_n)}{\|u_n\|_E}dx+\int_\Omega g\frac{u_n}{\|u_n\|_E}dx\right)\\
&\leq\limsup_{n\to+\infty}\frac{I_{\lambda_1}(u_n)}{\|u_n\|_E}\leq\limsup_{n\to+\infty}\frac{M}{\|u_n\|_E}=0,
\end{aligned}$$

that is,

$$\int_{\Omega} g\phi_1 dx \le \int_{\Omega} f^{+\infty}(x)\,\phi_1 dx,$$

which contradicts (f_3).

Case 2. Assume that $v_n \to -\phi_1$. Dividing (3.15) by $\|u_n\|_E$ and using (3.7), we have

$$\begin{aligned}
&\int_{\Omega} f_{-\infty}(x)\,\phi_1 dx - \int_{\Omega} g\phi_1 dx \\
&= \lim_{n\to+\infty}\left(-\int_{\Omega}\frac{F(x,u_n)}{\|u_n\|_E}dx + \int_{\Omega} g\frac{u_n}{\|u_n\|_E}dx\right) \\
&\le \limsup_{n\to+\infty}\frac{I_{\lambda_1}(u_n)}{\|u_n\|_E} \le \limsup_{n\to+\infty}\frac{M}{\|u_n\|_E} = 0,
\end{aligned}$$

that is,

$$\int_{\Omega} f_{-\infty}(x)\,\phi_1 dx \le \int_{\Omega} g\phi_1 dx,$$

which contradicts (f_3). Hence, the functional I_{λ_1} is coercive on E.

Proof of Theorem 3.1 is completed. The coerciveness and $(PS)_c$ condition are enough to prove that I_{λ_1} attains its infimum in E (see Lemma 2.4). Hence, the problem (1.1) has at least a weak solution in E. The proof is complete.

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