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# Maximal operator with rough kernel and its commutators in generalized weighted Morrey spaces

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**Abstract.** *Let $\Omega \in L_q(S^{n-1})$ be a homogeneous function of degree zero with $q > 1$. In this paper, we study the boundedness of the maximal operator with rough kernels $M_\Omega$ and its commutators $[b, M_\Omega]$ on generalized weighted Morrey spaces $M_{p,\varphi}(w)$. We find the sufficient conditions on the pair $(\varphi_1, \varphi_2)$ with $q' \le p < \infty$, $p \ne 1$ and $w \in A_{p/q'}$ or $1 < p \le q$ and $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the operators $M_\Omega$ from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ for $1 < p < \infty$. We find the sufficient conditions on the pair $(\varphi_1, \varphi_2)$ with $b \in BMO(\mathbb{R}^n)$ and $q' \le p < \infty$, $p \ne 1$, $w \in A_{p/q'}$ or $1 < p \le q$, $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the operators $[b, M_\Omega]$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1 < p < \infty$. In all cases the conditions for the boundedness of the operators $T_\Omega$, $[b, T_\Omega]$ are given in terms of supremal-type inequalities on $(\varphi_1, \varphi_2)$ and $w$, which do not assume any assumption on monotonicity of $\varphi_1(x, r)$, $\varphi_2(x, r)$ in $r$.*

**Keywords.** Maximal operator; rough kernel; generalized weighted Morrey spaces; commutator; $A_p$ weights

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## 1 Introduction

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [6,7] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let $K$ be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [8] states that the commutator operator $[b, K]f = K(bf) - b\,Kf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [9–11, 15, 24, 26]).

The classical Morrey spaces were originally introduced by Morrey in [34] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [9, 10, 12, 15, 19]. Guliyev, Mizuhara and Nakai [17, 33, 38] introduced generalized Morrey spaces

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$M^{p,\varphi}(\mathbb{R}^n)$ (see, also [18,19,21,39]). Recently, Komori and Shirai [31] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [20] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L^{p,\kappa}(w)$. In [20] Guliyev also studied the boundedness of the classical operators and its commutators in these spaces $M_{p,\varphi}(w)$, see also Guliyev et al. [3,11,13,22,25,26,28,29].

Watson [40] and independently by Duoandikoetxea [14] established weighted $L_p$ boundedness for the singular integral operators with rough kernels and their commutators.

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$.

Suppose that $\Omega$ satisfies the following conditions.

$(i)$ $\Omega$ is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{1.1}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The maximal operator with rough kernel $M_\Omega$ is defined by

$$M_\Omega f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |\Omega(x-y)| \, |f(y)| dy.$$

The commutators generated by a suitable function $b$ and the operator $M_\Omega$ is formally defined by

$$[b, M_\Omega]f = M_\Omega(bf) - bM_\Omega(f).$$

It is obvious that when $\Omega \equiv 1$, $M_\Omega$ is the Hardy-Littlewood maximal operator $M$. For $b \in L_1^{\rm loc}(\mathbb{R}^n)$ the commutator of the maximal operator $M_{\Omega,b}$ is defined by

$$M_{\Omega,b} f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |\Omega(x-y)| \, |b(x)-b(y)| \, |f(y)| dy. \tag{1.2}$$

Therefore, it will be an interesting thing to study the property of $M_\Omega$. The main purpose of this paper is to show that maximal operator with rough kernels $M_\Omega$ is bounded from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$, $1 < p < \infty$. We find the sufficient conditions on the pair $(\varphi_1, \varphi_2)$ with $b \in BMO(\mathbb{R}^n)$ and $q' \leq p < \infty$, $p \neq 1$, $w \in A_{p/q'}$ or $1 < p \leq q$, $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the commutator operators $[b, M_\Omega]$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1 < p < \infty$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant $C$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.

## 2 Preliminaries

Next we will give the weighted boundedness of maximal operator $M_\Omega$ with rough kernel and its commutator. In their proof, the weighted boundedness of the maximal operator $M_\Omega$ with rough kernel (for its definition, see (1.2)) is needed, while the latter itself is of great significance.

**Theorem 2.1** *[14] Suppose that $\Omega$ satisfies the condition* (1.1) *and $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$. Then for every $q' \leq p < \infty$, $p \neq 1$ and $w \in A_{p/q'}$ or $1 < p \leq q$, $p \neq \infty$ and $w^{1-p'} \in A_{p'/q'}$, there is a constant $C$ independent of $f$ such that*

$$\|M_\Omega(f)\|_{L_{p,w}} \leq C \|f\|_{L_{p,w}}.$$

**Theorem 2.2** *[4] Suppose that $\Omega$ satisfies the condition* (1.1) *and $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$. Let also $b \in BMO(\mathbb{R}^n)$. Then for every $q' \le p < \infty$, $p \ne 1$ and $w \in A_{p/q'}$ or $1 < p \le q$, $p \ne \infty$ and $w^{1-p'} \in A_{p'/q'}$, there is a constant $C$ independent of $f$ such that*

$$\|M_{\Omega,b}(f)\|_{L_{p,w}} \le C\|f\|_{L_{p,w}}.$$

For a function $b$ defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M]$ and $M_b$ are valid :

Let $b$ be any non-negative locally integrable function. Then

$$|[b, M]f(x)| \le M_b(f)(x), \qquad x \in \mathbb{R}^n$$

holds for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

If $b$ is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M]f(x)| \le M_b(f)(x) + 2b^-(x)Mf(x), \qquad x \in \mathbb{R}^n \tag{2.1}$$

holds for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (see, for example, [1]).

In the sequel $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$ stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$ , and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively. We also denote

$$\mathbb{A} = \{\varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0^+} \varphi = 0\}.$$

Let $u$ be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(t) := \|ug\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty),$$

The following theorem was proved in [5].

**Theorem 2.3** *[5] Suppose that $v_1$ and $v_2$ are nonnegative measurable functions such that $0 < \|v_1\|_{L_\infty(0,\cdot)} < \infty$ for every $t > 0$. Let $u$ be a continuous nonnegative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(\mathbb{R}_+)$ to $L_{\infty,v_2}(\mathbb{R}_+)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \big(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)}\big) \right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

## 3 Generalized weighted Morrey spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [34] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [16,32].

We recall that a weight function $w$ is in the Muckenhoupt class $A_p$ [35], $1 < p < \infty$, if

$$[w]_{A_p} := \sup_B [w]_{A_p(B)} = \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1} \tag{3.1}$$

where the sup is taken with respect to all the balls $B$ and $\frac{1}{p}+\frac{1}{p'}=1$. Note that, for all balls $B$ using Hölder's inequality, we have that

$$[w]_{A_p(B)}^{1/p} = |B|^{-1}\|w\|_{L_1(B)}^{1/p}\,\|w^{-1/p}\|_{L_{p'}(B)} \geq 1. \tag{3.2}$$

For $p = 1$, the class $A_1$ is defined by the condition $Mw(x) \leq Cw(x)$ with $[w]_{A_1} = \sup\limits_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p=\infty$ $A_\infty = \bigcup_{1\leq p<\infty} A_p$ and $[w]_{A_\infty} = \inf\limits_{1\leq p<\infty}[w]_{A_p}$.

**Remark 3.1** It is known that

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}(B)}^{q'/p'} = |B|^{-1}\|w^{1-p'}\|_{L_1(B)}^{q'/p'}\,\|w^{q'/p}\|_{L_{(p'/q')'}(B)}.$$

Moreover, we can write $w^{1-p'} \in A_{p'/q'} \Rightarrow w^{1-p'} \in A_{p'}$ because of $w^{1-p'} \in A_{p'/q'} \subset A_{p'}$. Therefore, we get

$$w^{1-p'} \in A_{p'/q'} \Rightarrow w^{1-p'} \in A_{p'} \Rightarrow [w^{1-p'}]_{A_{p'}(B)}^{1/p'} = |B|^{-1}\|w^{1-p'}\|_{L_1(B)}^{1/p'}\,\|w^{1/p}\|_{L_p(B)}. \tag{3.3}$$

But the opposite is not true.

**Remark 3.2** Let's write $w^{1-p'} \in A_{p'/q'}$ and used the definitions $A_p$ classes we get the following

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{\frac{q(p-1)}{p(q-1)}} = |B|^{-1}\|w^{1-p'}\|_{L_1(B)}^{\frac{q(p-1)}{p(q-1)}}\,\|w^{q'/p}\|_{L_{(p'/q')'}(B)}$$
$$\Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{1/p'} = |B|^{-\frac{q-1}{q}}\|w^{1-p'}\|_{L_1(B)}^{1/p'}\,\|w\|_{L_{\frac{q}{q-p}}(B)}^{1/p}, \tag{3.4}$$

where the following equalities are provided.

$$1-p' = -\frac{p'}{p},\quad \frac{q'}{p} = \frac{q}{p(q-1)},\quad \frac{q'}{p'} = \frac{q(p-1)}{p(q-1)},\quad \left(\frac{q}{p}\right)' = \frac{q}{q-p},\quad \left(\frac{p'}{q'}\right)' = \frac{p(q-1)}{q-p}.$$

Then from eq.(3.3) and eq.(3.4) we have

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{1/p'} = |B|^{\frac{1}{q}}[w^{1-p'}]_{A_{p'}(B)}^{1/p'}\,\|w^{1/p}\|_{L_p(B)}^{-1}\,\|w\|_{L_{\frac{q}{q-p}}(B)}^{1/p}. \tag{3.5}$$

We define the generalized weighed Morrey spaces as follows.

**Definition 3.1** *[20] Let $1 \le p < \infty$, $\varphi$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $w$ be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))},$$

*where $L_{p,w}(B(x,r))$ denotes the weighted $L_p$-space of measurable functions $f$ for which*

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

*Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL^{\rm loc}_{p,w}(\mathbb{R}^n)$ for which*

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

*where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions $f$ for which*

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\, |f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

**Remark 3.3** (1) If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(v,w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(5) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

The following statement, was proved in [30].

**Theorem 3.1** *Let $1 \le p < \infty$, $w \in A_p$ and $(\varphi_1, \varphi_2)$ satisfy the condition*

$$\sup_{t>r} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \le C\, \varphi_2(x,r), \tag{3.6}$$

*where $C$ does not depend on $x$ and $r$. Then the operator $M$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.*

The following statement, was proved in [30], see also [20].

**Theorem 3.2** *Let $1 < p < \infty$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$ and $(\varphi_1, \varphi_2)$ satisfy the condition*

$$\sup_{t>r} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \le C\, \varphi_2(x,r), \tag{3.7}$$

*where $C$ does not depend on $x$ and $r$. Then the operator $M_b$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.*

Note that, in the case $w = 1$ Theorem 3.1 was proved in [23, 37], see also [2].

## 4 Maximal operator with rough kernels $M_\Omega$ in the spaces $M_{p,\varphi}(w)$

In the following lemma we get Guliyev weighted local estimate (see, for example, [17, 19] in the case $w = 1$ and [20] in the case $w \in A_p$) for the operator $T_\Omega$.

**Lemma 4.1** *Suppose that $\Omega$ be satisfies the condition* (1.1) *and $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$.*

*If $q' \leq p < \infty$, $p \neq 1$ and $w \in A_{p/q'}$, then the inequality*

$$\|M_\Omega(f)\|_{L_{p,w}(B(x,r))} \lesssim w(B(x,r))^{\frac{1}{p}} \sup_{t>2r} \|f\|_{L_{p,w}(B(x,t))}\, w(B(x,t))^{-\frac{1}{p}}$$

*holds for any ball $B(x,r)$, and for all $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$.*

*If $1 < p \leq q$, $p \neq \infty$ and $w^{1-p'} \in A_{p'/q'}$, then the inequality*

$$\|M_\Omega(f)\|_{L_{p,w}(B(x,r))} \lesssim \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,r))} \sup_{t>2r} \|f\|_{L_{p,w}(B(x,t))}\, \|w\|^{-1/p}_{L_{\frac{q}{q-p}}(B(x,t))}$$

*holds for any ball $B(x,r)$, and for all $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$.*

**Proof.** Let $\Omega$ be satisfies the condition (1.1) and $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$.

Note that

$$\|\Omega(x-\cdot)\|_{L_q(B(x,t))} \leq c_0\, \|\Omega\|_{L_q(S^{n-1})}\, |B(0, t+|x-x_0|)|^{\frac{1}{q}}, \tag{4.1}$$

where $c_0 = (nv_n)^{-1/q}$ and $v_n = |B(0,1)|$ (see, [23]).

For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x,r)$ for the ball centered at $x_0$ and of radius $r$, $2B = B(x_0, 2r)$. We represent $f$ as

$$f = f_1 + f_2, \;\; f_1(y) = f(y)\chi_{2B}(y), \;\; f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \;\; r > 0 \tag{4.2}$$

and have

$$\|M_\Omega(f)\|_{L_{p,w}(B)} \leq \|M_\Omega(f_1)\|_{L_{p,w}(B)} + \|M_\Omega(f_2)\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $M_\Omega(f_1) \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of $M_\Omega$ in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_{p/q'}$ and $q' \leq p < \infty$, $p \neq 1$ (see Theorem 2.2) it follows that

$$\begin{aligned}
\|M_\Omega(f_1)\|_{L_{p,w}(B)} &\leq \|M_\Omega(f_1)\|_{L_{p,w}(\mathbb{R}^n)} \\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\, [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\, \|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\
&\approx \|\Omega\|_{L_q(S^{n-1})}\, [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\, \|f\|_{L_{p,w}(2B)} \\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\, [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\, w(B)^{\frac{1}{p}} \sup_{t>2r} \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}}.
\end{aligned}$$

Let $z$ be an arbitrary point in $B \equiv B(x,r)$. If $B(z,t) \cap {}^{\complement}B(x,2r) \neq \varnothing$, then $t > r$. Indeed, if $y \in B(z,t) \cap {}^{\complement}B(x,2r)$, then we get $t > |y-z| \geq |x-y| - |x-z| > 2r - r = r$.

On the other hand, $B(z,t)\cap {}^{\complement}B(x,2r)\subset B(x,2t)$. Indeed, if $y\in B(z,t)\cap {}^{\complement}B(x,2r)$, then we get $|x-y|\le|y-z|+|x-z|<t+r<2t$. Hence, for all $z\in B$

$$\begin{aligned}M_\Omega f_2(z) &= \sup_{t>0}|B(z,t)|^{-1}\int_{B(z,t)}|\Omega(z-y)|\,|f_2(y)|dy\\ &\le \sup_{t>r}|B(x,2t)|^{-1}\int_{B(z,t)\cap {}^{\complement}B(x,2r)}|\Omega(z-y)|\,|f(y)|dy\\ &\le \sup_{t>r}|B(x,2t)|^{-1}\int_{B(x,2t)}|\Omega(z-y)|\,|f(y)|dy\\ &= \sup_{t>2r}|B(x,t)|^{-1}\int_{B(x,t)}|\Omega(z-y)|\,|f(y)|dy.\end{aligned}$$

By applying Hölder's inequality for $q'\le p<\infty$, $p\ne 1$ and $w\in A_{p/q'}$, we get

$$\begin{aligned}&M_\Omega f_2(z)\le \sup_{t>2r}|B(x,t)|^{-1}\int_{B(x,t)}|\Omega(z-y)|\,|f(y)|dy\\ &\lesssim \sup_{t>2r}|B(x,t)|^{-1}\|\Omega(z-\cdot)\|_{L_q(B(x,t))}\,\|f\|_{L_{q'}(B(x,t))}\\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\sup_{t>2r}|B(x,t)|^{-1}\|f\|_{L_{p,w}(B(x,t))}\,\|w^{-q'/p}\|^{\frac{1}{q'}}_{L_{(p/q')'}(B(x,t))}\,|B(0,t+|x-z|)|^{\frac1q}\\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac1p}_{A_{\frac{p}{q'}}}\sup_{t>2r}|B(x,t)|^{-1}\|f\|_{L_{p,w}(B(x,t))}\,w(B(x,t))^{-\frac1p}\,|B(x,t)|^{\frac{1}{q'}}\,|B(0,t+r)|^{\frac1q}\\ &\approx \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac1p}_{A_{\frac{p}{q'}}}\sup_{t>2r}\|f\|_{L_{p,w}(B(x,t))}w(B(x,t))^{-\frac1p}. \end{aligned}\tag{4.3}$$

Moreover, for all $q'\le p<\infty$, $p\ne 1$ the inequality

$$\|M_\Omega\,(f_2)\,\|_{L_{p,w}(B)}\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac1p}_{A_{\frac{p}{q'}}}\,w(B)^{\frac1p}\sup_{t>2r}\|f\|_{L_{p,w}(B(x,t))}w(B(x,t))^{-\frac1p}$$

is valid. Thus

$$\|M_\Omega(f)\|_{L_{p,w}(B)}\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac1p}_{A_{\frac{p}{q'}}}\,w(B)^{\frac1p}\sup_{t>2r}\|f\|_{L_{p,w}(B(x,t))}w(B(x,t))^{-\frac1p}.$$

Let also $1<p\le q$, $p\ne\infty$ and $w^{1-p'}\in A_{p'/q'}$. Since $f_1\in L_{p,w}(\mathbb{R}^n)$, $M_\Omega\,(f_1)\in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of $M_\Omega$ in $L_{p,w}(\mathbb{R}^n)$ for $w^{1-p'}\in A_{p'/q'}$ and $1<p<q$ (see Theorem 2.2) it follows that

$$\begin{aligned}\|M_\Omega\,(f_1)\,\|_{L_{p,w}(B)}\le \|M_\Omega\,(f_1)\,\|_{L_{p,w}(\mathbb{R}^n)} &\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{q'}}}\,\|f_1\|_{L_{p,w}(\mathbb{R}^n)}\\ &\approx \|\Omega\|_{L_q(S^{n-1})}\,[w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{q'}}}\,\|f\|_{L_{p,w}(2B)}.\end{aligned}$$

If $1 < p \le q$, $p \ne \infty$ and $w^{1-p'} \in A_{p'/q'}$, then Minkowski theorem and Hölder inequality,

$$\|M_\Omega(f_2)\|_{L_{p,w}(B)} \le \left(\int_B \left(\sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} |\Omega(x-y)|\,|f(y)|dy\right)^p w(x)dx\right)^{\frac{1}{p}}$$

$$\le \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} \|\Omega(\cdot - y)\|_{L_{p,w}(B)} |f(y)|\,dy$$

$$\lesssim \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} \|\Omega(\cdot - y)\|_{L_q(B)}\, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(B)}\, |f(y)|\,dy$$

$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(B)} \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} |B(0, r+|x-y|)|^{\frac{1}{q}}\, |f(y)|\,dy$$

$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(B)} \sup_{t>2r} |B(x,t)|^{-1} \|f\|_{L_1(B(x,t))}\, |B(0,r+t)|^{\frac{1}{q}}$$

$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \sup_{t>2r} |B(x,t)|^{-1} \|f\|_{L_{p,w}(B(x,t))}\, \|w^{-p'/p}\|^{\frac{1}{p'}}_{L_1(B(x,t))}\, |B(x,t)|^{\frac{1}{q}}$$

$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \sup_{t>2r} |B(x,t)|^{-1} \|f\|_{L_{p,w}(B(x,t))}\, \|w^{1-p'}\|^{\frac{1}{p'}}_{L_1(B(x,t))}\, |B(x,t)|^{\frac{1}{q}}$$

is obtained. By applying (3.3) for $\|w^{1-p'}\|^{\frac{1}{p'}}_{L_1(B(x,t))}$ and (3.5) for $\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)}$ we have the following inequality

$$\|M_\Omega(f_2)\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\, [w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{q'}}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \sup_{t>2r} \|f\|_{L_{p,w}(B(x,t))}\, \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,t))}$$

is valid. Thus

$$\|M_\Omega(f)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\, [w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{q'}}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \sup_{t>2r} \|f\|_{L_{p,w}(B(x,t))}\, \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,t))}.$$

Thus we complete the proof of Lemma 4.1.

**Theorem 4.1** *Suppose that $\Omega$ be satisfies the condition* (1.1) *and $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$. Let also, for $q' \le p < \infty$, $w \in A_{p/q'}$ the pair $(\varphi_1, \varphi_2)$ satisfies the condition* (3.6) *and for $1 < p \le q$, $w^{1-p'} \in A_{p'/q'}$ the pair $(\varphi_1, \varphi_2)$ satisfies the condition*

$$\sup_{t>r} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,\tau))}}{\|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,t))}} \le C\, \varphi_2(x,r)\, \frac{w(B(x,r))^{\frac{1}{p}}}{\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}}, \tag{4.4}$$

*where $C$ does not depend on $x$ and $r$.*

*Then the operator $M_\Omega$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover*

$$\|M_\Omega(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

**Proof.** When $q' \le p < \infty$, $w \in A_{p/q'}$, by Lemma 4.1 and Theorem 2.3 with $\nu_2(r) = \varphi_2(x,r)^{-1}, \nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}, g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = w(B(x,r))^{-\frac{1}{p}}$ we have

$$\begin{aligned}\|M_\Omega(f)\|_{M_{p,\varphi_2}(w)} &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|M_\Omega(f)\|_{L_{p,w}(B(x,r))}\\ &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} \|f\|_{L_{p,w}(B(x,t))}\, w(B(x,t))^{-\frac{1}{p}}\\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))}\\ &= \|f\|_{M_{p,\varphi_1}(w)}.\end{aligned}$$

For the case of $1 < p \le q$, $w^{1-p'} \in A_{p'/q'}$, by Lemma 4.1 and Theorem 2.3 with $\nu_2(r) = \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}$, $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}$ we have

$$\begin{aligned}&\|M_\Omega(f)\|_{M_{p,\varphi_2}(w)} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|M_\Omega(f)\|_{L_{p,w}(B(x,r))}\\ &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \sup_{t>r} \|f\|_{L_{p,w}(B(x,t))}\, \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,t))}\\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))}\\ &= \|f\|_{M_{p,\varphi_1}(w)}.\end{aligned}$$

## 5 Commutator of maximal operator with rough kernels $[b, M_\Omega]$ in the spaces $M_{p,\varphi}(w)$

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

**Definition 5.1** *Suppose that* $b \in L_1^{\text{loc}}(\mathbb{R}^n)$*, and let*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

*where*

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y) dy.$$

*Define*

$$BMO(\mathbb{R}^n) = \{b \in L_1^{\text{loc}}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

**Lemma 5.1** *[36] Let* $w \in A_\infty$*. Then the norm* $\|\cdot\|_*$ *is equivalent to the norm*

$$\|b\|_{*,w} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{w(B(x,r))} \int_{B(x,r)} |b(y) - b_{B(x,r),w}|\, w(y)\, dy,$$

*where*

$$b_{B(x,r),w} = \frac{1}{w(B(x,r))} \int_{B(x,r)} b(y)\, w(y)\, dy.$$

The following lemma is proved in [20].

**Lemma 5.2** *1 Let $w \in A_\infty$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$ and $r_1, r_2 > 0$. Then,*

$$\Big(\frac{1}{w(B(x,r_1))}\int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^p w(y)dy\Big)^{\frac{1}{p}} \le C\Big(1 + \Big|\ln\frac{r_1}{r_2}\Big|\Big)\|b\|_*,$$

*where $C > 0$ is independent of $f$, $w$, $x$, $r_1$ and $r_2$.*

*2 Let $w \in A_p$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 < p < \infty$, $x \in \mathbb{R}^n$ and $r_1, r_2 > 0$. Then,*

$$\Big(\frac{1}{w^{1-p'}(B(x,r_1))}\int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{p'} w(y)^{1-p'}dy\Big)^{\frac{1}{p'}} \le C\Big(1 + \Big|\ln\frac{r_1}{r_2}\Big|\Big)\|b\|_*,$$

*where $C > 0$ is independent of $b$, $w$, $x$, $r_1$ and $r_2$.*

**Remark 5.1** ([27])

(1) The John-Nirenberg inequality : There are constants $C_1$, $C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B : |b(x) - b_B| > \beta\}| \le C_1|B|e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy\right)^{\frac{1}{p}} \tag{5.1}$$

for $1 < p < \infty$.

(3) Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_* \ln\frac{t}{r} \ \text{ for } \ 0 < 2r < t, \tag{5.2}$$

where $C$ is independent of $b$, $x$, $r$ and $t$.

In the following lemma we get Guliyev weighted local estimate (see, for example, [20] ) for the maximal commutator operator $M_{\Omega,b}$.

**Lemma 5.3** *Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Suppose that $\Omega$ be satisfies the condition* (1.1) *and $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$.*

*If $q' \le p < \infty$ and $w \in A_{p/q'}$, then the inequality*

$$\|M_{\Omega,b}(f)\|_{L_{p,w}(B(x,r))} \lesssim \|b\|_*\, w(B(x,r))^{\frac{1}{p}} \sup_{t>2r}\Big(1 + \ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}}$$

*holds for any ball $B(x,r)$, and for all $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.*

*If $1 < p < q$ and $w^{1-p'} \in A_{p'/q'}$, then the inequality*

$$\|M_{\Omega,b}(f)\|_{L_{p,w}(B(x,r))} \lesssim \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))} \sup_{t>2r}\Big(1 + \ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x,t))}\,\|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,t))}$$

*holds for any ball $B(x,r)$, and for all $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.*

**Proof.** Let $p \in (1,\infty)$ and $b \in BMO(\mathbb{R}^n)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x,r)$ for the ball centered at $x$ and of radius $r$, $2B = B(x,2r)$. We represent $f$ as (4.2) and have

$$\|M_{\Omega,b}(f)\|_{L_{p,w}(B)} \le \|M_{\Omega,b}(f_1)\|_{L_{p,w}(B)} + \|M_{\Omega,b}(f_2)\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $M_{\Omega,b}(f_1) \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of $M_{\Omega,b}$ in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_{p/q'}$ and $q' \le p < \infty$ (see Theorem 2.2) it follows that

$$\begin{aligned}\|M_{\Omega,b}(f_1)\|_{L_{p,w}(B)} &\le \|M_{\Omega,b}(f_1)\|_{L_{p,w}(\mathbb{R}^n)} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}}\,\|b\|_*\,\|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\approx \|\Omega\|_{L_q(S^{n-1})}\,[w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}}\,\|b\|_*\,\|f\|_{L_{p,w}(2B)}.\end{aligned}$$

Let $z$ be an arbitrary point in $B \equiv B(x,r)$. If $B(z,t) \cap {}^{\complement}B(x,2r) \ne \varnothing$, then $t > r$. Indeed, if $y \in B(z,t) \cap {}^{\complement}B(x,2r)$, then we get $t > |y-z| \ge |x-y| - |x-z| > 2r - r = r$.

On the other hand, $B(z,t) \cap {}^{\complement}B(x,2r) \subset B(x,2t)$. Indeed, if $y \in B(z,t) \cap {}^{\complement}B(x,2r)$, then we get $|x-y| \le |y-z| + |x-z| < t + r < 2t$. Hence, for all $z \in B$

$$\begin{aligned}M_{\Omega,b}f_2(z) &= \sup_{t>0} |B(z,t)|^{-1} \int_{B(z,t)} |b(y)-b(z)|\,|\Omega(y-z)|\,|f_2(y)|\,dy \\ &= \sup_{t>0} |B(z,t)|^{-1} \int_{B(z,t)\cap {}^{\complement}B(x,2r)} |b(y)-b(z)|\,|\Omega(y-z)|\,|f(y)|\,dy \\ &\lesssim \sup_{t>r} |B(x,2t)|^{-1} \int_{B(x,2t)} |b(y)-b(z)|\,|\Omega(y-z)|\,|f(y)|\,dy \\ &= \sup_{t>2r} |B(x,2t)|^{-1} \int_{B(x,t)} |b(y)-b(z)|\,|\Omega(y-z)|\,|f(y)|\,dy.\end{aligned}$$

Therefore, for all $z \in B$ we have

$$M_{\Omega,b}f_2(z) \lesssim \sup_{t>2r} |B(x,2t)|^{-1} \int_{B(x,t)} |b(y)-b(z)|\,|\Omega(y-z)|\,|f(y)|\,dy.$$

By applying Hölder's inequality for $q' \le p < \infty$, $p \ne 1$ and $w \in A_{p/q'}$, we get

$$\begin{aligned}&M_{\Omega,b}f_2(z) \le \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} |b(y)-b(z)|\,|\Omega(z-y)|\,|f(y)|dy \\ &\lesssim \sup_{t>2r} |B(x,t)|^{-1} \|\Omega(z-\cdot)\|_{L_q(B(x,t))}\,\|(b(y)-b(z))\,f\|_{L_{q'}(B(x,t))} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})} \sup_{t>2r} |B(x,t)|^{-1} \|f\|_{L_{p,w}(B(x,t))}\,\|(b(y)-b(z))\,w^{-q'/p}\|_{L_{(p/q')'}(B(x,t))}^{\frac{1}{q'}}\,|B(0,t+|x-z|)|^{\frac{1}{q}} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}} \sup_{t>2r} |B(x,t)|^{-1} \|f\|_{L_{p,w}(B(x,t))}\,w(B(x,t))^{-\frac{1}{p}}\,|B(x,t)|^{\frac{1}{q'}}\,|B(0,t+r)|^{\frac{1}{q}} \\ &\approx \|\Omega\|_{L_q(S^{n-1})}\,[w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}} \sup_{t>2r} \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}}. \qquad (5.3)\end{aligned}$$

Moreover, for all $q' \le p < \infty$, $p \ne 1$ the inequality

$$\|M_{\Omega}(f_2)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}}\,w(B)^{\frac{1}{p}} \sup_{t>2r} \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}}$$

is valid. Thus

$$\|M_{\Omega}(f)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\, [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\, w(B)^{\frac{1}{p}} \sup_{t>2r} \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}}.$$

If $1 < p \le q$, $p \ne \infty$ and $w^{1-p'} \in A_{p'/q'}$, then Minkowski theorem and Hölder inequality,

$$\begin{aligned}
&\|M_{\Omega,b} f_2\|_{L_{p,w}(B)}\\
&\lesssim \Big( \int_B \Big( \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} |b(y)-b(z)|\, |\Omega(y-z)|\, |f(y)| dy \Big)^p w(z)\, dz \Big)^{\frac{1}{p}}\\
&\lesssim \Big( \int_B \Big( \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} |b(y)-b_{B,w}|\, |\Omega(y-z)|\, |f(y)| dy \Big)^p w(z)\, dz \Big)^{\frac{1}{p}}\\
&+ \Big( \int_B \Big( \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} |b(z)-b_{B,w}|\, |\Omega(y-z)|\, |f(y)| dy \Big)^p w(z)\, dz \Big)^{\frac{1}{p}}\\
&= J_1 + J_2.
\end{aligned}$$

Let us estimate $J_1$. Applying Hölder's inequality and by Lemma 5.2 we get

$$\begin{aligned}
J_1 &= \Big( \int_B \Big( \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} |b(y)-b_{B,w}|\, |\Omega(y-z)|\, |f(y)| dy \Big)^p w(z)\, dz \Big)^{\frac{1}{p}}\\
&\le \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} \|\Omega(y-\cdot)\|_{L_{p,w}(B)}\, |b(y)-b_{B,w}|\, |f(y)|\, dy\\
&\lesssim \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} \|\Omega(y-\cdot)\|_{L_q(B)}\, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(B)}\, |b(y)-b_{B,w}|\, |f(y)|\, dy\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(B)} \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} |B(0, r+|x-y|)|^{\frac{1}{q}}\, |b(y)-b_{B,w}|\, |f(y)|\, dy\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(B)} \sup_{t>2r} |B(x,t)|^{-1+\frac{1}{q}} \int_{B(x,t)} |b(y)-b_{B,w}|\, |f(y)|\, dy\\
&\le \|\Omega\|_{L_q(S^{n-1})}\, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(B)} \sup_{t>2r} |B(x,t)|^{-1+\frac{1}{q}} \Big( \int_{B(x,t)} |b(y)-b_{B,w}|^{p'} w(y)^{1-p'} dy \Big)^{\frac{1}{p'}} \|f\|_{L_{p,w}(B(x,t))}\\
&\lesssim \|b\|_*\, \|\Omega\|_{L_q(S^{n-1})}\, \|w\|^{\frac{1}{p}}_{L_{(q/p)'}(B)} \sup_{t>2r} |B(x,t)|^{-1+\frac{1}{q}} \Big(1+\ln\frac{t}{r}\Big) \|w^{1-p'}\|^{\frac{1}{p'}}_{L_1(B(x,t))}\, \|f\|_{L_{p,w}(B(x,t))}\\
&\lesssim \|b\|_*\, \|\Omega\|_{L_q(S^{n-1})}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \sup_{t>2r} |B(x,t)|^{-1+\frac{1}{q}} \Big(1+\ln\frac{t}{r}\Big) \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,t))}\, |B(x,t)|^{\frac{1}{q'}}\, \|f\|_{L_{p,w}(B(x,t))}\\
&= \|b\|_*\, \|\Omega\|_{L_q(S^{n-1})}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \sup_{t>2r} |B(x,t)|^{\frac{1}{q}}\, \ln\Big(e+\frac{t}{r}\Big) \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,t))}\, \|f\|_{L_{p,w}(B(x,t))}.
\end{aligned}$$

In order to estimate $J_2$ note that

$$\begin{aligned}
J_2 &= \Big( \int_B \Big( \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} |b(z)-b_{B,w}|\, |\Omega(y-z)|\, |f(y)| dy \Big)^p w(z)\, dz \Big)^{\frac{1}{p}}\\
&\le \sup_{t>2r} |B(x,t)|^{-1} \int_{B(x,t)} \Big( \int_B \big| \big(b(z)-b_{B,w}\big)\, \Omega(y-z) \big|^p w(z) dz \Big)^{\frac{1}{p}} |f(y)| dy.
\end{aligned}$$

With similar techniques for $1 < p \le q$, $w^{1-p'} \in A_{p'/q'}$ can be achieved and the proof is finished.

**Theorem 5.1** *Suppose that $\Omega$ be satisfies the condition* (1.1) *and $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$. Let $b \in BMO(\mathbb{R}^n)$. Let also, for $q' \le p < \infty$, $w \in A_{p/q'}$ the pair $(\varphi_1, \varphi_2)$ satisfies the condition* (3.7) *and for $1 < p \le q$, $w^{1-p'} \in A_{p'/q'}$ the pair $(\varphi_1, \varphi_2)$ satisfies the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,\tau))}}{\|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,t))}} \frac{dt}{t} \le C\, \varphi_2(x,r) \frac{w(B(x,r))^{\frac{1}{p}}}{\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}}, \quad (5.4)$$

*where $C$ does not depend on $x$ and $r$.*

*Then the operator $M_{\Omega,b}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.*

$$\|M_{\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

**Proof.** When $q' \le p < \infty$, $w \in A_{p/q'}$, by Lemma 5.3 and Theorem 2.3 with $\nu_2(r) = \varphi_2(x,r)^{-1}$, $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = w(B(x,r))^{-\frac{1}{p}} r^{-1}$ we have

$$\begin{aligned}
&\|M_{\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|\mu_{\Omega,b}(f)\|_{L_{p,w}(B(x,r))} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x,t))}\, w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))} \\
&= \|b\|_* \|f\|_{M_{p,\varphi_1}(w)}.
\end{aligned}$$

For the case of $1 < p \le q$, $w^{1-p'} \in A_{p'/q'}$, by Lemma 4.1 and Theorem 2.3 with $\nu_2(r) = \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}$, $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))} r^{-1}$ we have

$$\begin{aligned}
&\|M_{\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|\mu_\Omega(f)\|_{L_{p,w}(B(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \\
&\times \int_r^\infty \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x,t))}\, \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,t))} \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))} = \|f\|_{M_{p,\varphi_1}(w)}.
\end{aligned}$$

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## References


1. Agcayazi, M., Gogatishvili, A., Koca, K., Mustafayev, R.: *A note on maximal commutators and commutators of maximal functions*, J. Math. Soc. Japan. **67** (2), 581-593 (2015).
2. Akbulut, A., Guliyev, V.S., Mustafayev R.: *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem. **137** (1), 27-43 (2012).
3. Akbulut, A., Hamzayev, V.H., Safarov, Z.V.: *Rough fractional multilinear integral operators on generalized weighted Morrey spaces*, Azerb. J. Math. **6** (2), 128-142 (2016).
4. Alvarez, J., Bagby, R.J., Kurtz, D.S., Pérez C.: *Weighted estimates for commutators of linear operators*, Studia Math. **104**, 195-209, (1993).
5. Burenkov, V., Gogatishvili, A., Guliyev, V., Mustafayev, R.: *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ. **55** (8-10), 739-758 (2010).
6. Calderon, A.P.: *Commutators of singular integral operators*, Proc. Natl. Acad. Sci. USA **53**, 1092-1099 (1965).
7. Calderon, A.P.: *Cauchy integrals on Lipschitz curves and related operators*, Proc. Natl. Acad. Sci. USA **74** (4), 1324-1327 (1977).
8. Coifman, R., Rochberg, R., Weiss G.: *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (2), 611-635 (1976).
9. Di Fazio, G., Ragusa, M.A.: *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112**, 241-256 (1993).
10. Fan, D., Lu, S., Yang, D.: *Boundedness of operators in Morrey spaces on homogeneous spaces and its applications*, Acta Math. Sinica (N. S.) **14**, 625-634 (1998).
11. Eroglu, A., Omarova, M.N., Muradova, Sh.A.: *Elliptic equations with measurable coefficients in generalized weighted Morrey spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **43** (2), 197-213 (2017).
12. Eroglu, A., Azizov, J.V.: *A note on the fractional integral operators in generalized Morrey spaces on the Heisenberg group*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **37** (1), Mathematics, 86-91 (2017).
13. Deringoz, F.: *Parametric Marcinkiewicz integral operator and its higher order commutators on generalized weighted Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **37** (4), Mathematics, 24-32 (2017).
14. Duoandikoetxea, J.: *Weighted norm inequalities for homogeneous singular integrals*, Trans. Amer. Math. Soc. **336**, 869-880 (1993).
15. Gadjiev, T., Galandarova, Sh., Guliyev, V.: *Regularity in generalized Morrey spaces of solutions to higher order nondivergence elliptic equations with VMO coefficients*, Electron. J. Qual. Theory Differ. Equ. Paper No. 55, 17 pp (2019).
16. Giaquinta, M.: *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton Univ. Press, Princeton, NJ (1983).
17. Guliyev, V.S.: *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$, Doctoral dissertation, (in Russian), Moscow, Mat. Inst. Steklov, 329 pp., (1994).
18. Guliyev, V.S.: *Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications,* Baku. 1999, 1-332. (Russian)
19. Guliyev, V.S.: *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. Art. ID 503948, 20 pp (2009).
20. Guliyev, V.S.: *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J. **3** (3), 33-61 (2012).

21. Guliyev, V.S.: *Local generalized Morrey spaces and singular integrals with rough kernel*, Azerb. J. Math. **3** (2), 79-94 (2013).
22. Guliyev, V.S., Karaman, T., Mustafayev, R.Ch., Serbetci, A.: *Commutators of sublinear operators generated by Calderón-Zygmund operator on generalized weighted Morrey spaces*, Czechoslovak Math. J. **64** (139) (2), 365-386 (2014).
23. Guliyev, V.S., Balakishiyev, A.S.: *Parabolic fractional maximal and integral operators with rough kernels in parabolic generalized Morrey spaces*, J. Math. Inequal. 9 (1), 257-276 (2015).
24. Guliyev, V., Gadjiev, T., Galandarova, Sh.: *Dirichlet boundary value problems for uniformly elliptic equations in modified local generalized Sobolev-Morrey spaces*, Electron. J. Qual. Theory Differ. Equ. Paper No. 71, 17 pp., (2017).
25. Guliyev, V.S., Ahmadli A.A., Omarova M.N., Softova L.: *Global regularity in Orlicz-Morrey spaces of solutions to nondivergence elliptic equations with VMO coefficients*, Electron. J. Differential Equations Paper No. 110, 24 pp., (2018).
26. Guliyev, V.S., Omarova, M.N., Softova, L.: *The Dirichlet problem in a class of generalized weighted Morrey spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **45** (2), 270-285 (2019).
27. Janson, S.: *On functions with conditions on the mean oscillation*, Ark. Mat. **14** (2), 189-196 (1976).
28. Hamzayev, V.H.: *Sublinear operators with rough kernel generated by Calderon-Zygmund operators and their commutators on generalized weighted Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **38** (1), Mathematics, 79-94 (2018).
29. Ismayilova, A.F.: *Commutators of the Marcinkiewicz integral on generalized weighted Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **38** (4), Mathematics, 79-92 (2018).
30. Ismayilova, A.F.: *Fractional maximal operator and its commutators on generalized weighted Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. 39 (4), Mathematics, 84-95 (2019).
31. Komori, Y., Shirai, S.: *Weighted Morrey spaces and a singular integral operator*, Math. Nachr. **282** (2), 219-231 (2009).
32. Kufner, A., John, O., Fuçik S.: *Function Spaces*, Noordhoff International Publishing: Leyden, Publishing House Czechoslovak Academy of Sciences: Prague (1977).
33. Mizuhara, T.: *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo, 183-189 (1991).
34. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126-166 (1938).
35. Muckenhoupt, B.: *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165**, 207-226 (1972).
36. Muckenhoupt, B., Wheeden, R.: *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192**, 261-274 (1974).
37. Muradova, Sh.A., Hamzayev, V.H.: *Anisotropic maximal and singular integral operators in anisotropic generalized Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. 34 (4), Mathematics and Mechanics, 73-84 (2014).
38. Nakai, E.: *Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166**, 95-103 (1994).
39. Sawano, Y.: *A thought on generalized Morrey spaces*, J. Indonesian Math. Soc. **25** (3), 210-281 (2019).
40. Watson, D.: *Weighted estimates for singular integrals via Fourier transform estimates*, Duke Math. J. 60, 389-399 (1990).