

Some notes on the gh -lifts of affine connections

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Abstract. *Let M be a differentiable manifold with dimension n and T^*M be its cotangent bundle. In this paper, we determine the gh -lift of the affine connection via the musical isomorphism on the cotangent bundle T^*M . We obtain the torsion tensor, curvature tensor and geodesic curve of the gh -lift of Levi-Civita connection.*

Keywords. Horizontal lift, connection, curvature tensor, musical isomorphism, geodesic.

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1 Introduction

The connections were studied from an infinitesimal perspective in Riemannian geometry. Several authors investigated the lifts of connections on the different type bundle. Pogoda [8] defined the horizontal lift of basic connections of order r to a transverse naturel vector bundle and studied its properties. Salimov and Fattayev [9] determined the horizontal lift and complete lifts of the linear connection from a smooth manifold to its coframe bundle. Several authors investigated the lifts of connections on the cotangent bundle T^*M . Yano and Patterson [10] applied the horizontal lifts which was defined by them to the symmetric connections on the cotangent bundle T^*M . Kures [6] determined all natural operators transforming classical torsion-free linear connections on a manifold M into classical linear connections on the cotangent bundle T^*M . Also the Riemannian manifolds and the cotangent bundles have been studied by many authors [3,7,4,5]. In this paper, calculating the coefficients of gh -lift ${}^{GH}\nabla^*$ of the affine connection via the musical isomorphism we have determined the gh -lift ${}^{GH}\nabla^*$ of the affine connection ∇ . After using the coefficients of the gh -lift ${}^{GH}\nabla^*$ we have determined the torsion tensor and the curvature tensor of the gh -lift ${}^{GH}\nabla^*$ of Levi-Civita connection. Finally we have investigated properties of the geodesic of the gh -lift ${}^{GH}\nabla^*$ to the cotangent bundle T^*M .

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2 Preliminaries

Let M be a pseudo-Riemannian manifold with n dimension. The tangent bundle on M is denoted by $TM = \cup_{x \in M} T_x M$. The local coordinates on TM are $(x^i, x^{\bar{i}}) = (x^i, y^i)$ where (x^i) are local coordinates on M and (y^i) are vector space coordinates according to the basis $\partial/\partial x^i$, i.e. $y_x = y^i \frac{\partial}{\partial x^i} \in T_x M$. The cotangent bundle on M is denoted by $T^*M = \cup_{x \in M} T_x^* M$. The local coordinates on T^*M are $(x^i, \tilde{x}^{\bar{i}}) = (x^i, p_i)$ where (x^i) are local coordinates on M and p_i are vector space coordinates according to the basis dx^i , i.e. $p_x = p_i dx^i \in T_x^* M$. We denote by $\mathfrak{S}_s^r(M)$ the set of all tensor fields of type (r, s) on M . Throughout this paper we assume the manifolds, tensor fields and connections to be differentiable of class C^∞ . We use the ranges of the index i being $\{1, \dots, n\}$ and the index \bar{i} being $\{n+1, \dots, 2n\}$.

Let g be a pseudo Riemannian metric. $g^\sharp : T^*M \rightarrow TM$ is the musical isomorphism associated with g pseudo Riemannian metric with inverse given by $g^\flat : TM \rightarrow T^*M$.

The musical isomorphism g^\sharp described by

$$g^\sharp : \tilde{x}^M = (x^m, \tilde{x}^{\bar{m}}) = (x^m, p_m) \rightarrow x^J = (x^j, x^{\bar{j}}) = (\delta_m^j x^m, y^j = g^{jm} p_m). \quad (2.1)$$

In music notation, the sharp symbol \sharp increase a note by a half step. And similar to this musical notation, the musical isomorphism g^\sharp increase the indice of the vector space coordinate.

The musical isomorphism g^\flat is described by

$$g^\flat : x^I = (x^i, x^{\bar{i}}) = (x^i, y^i) \rightarrow \tilde{x}^K = (x^k, \tilde{x}^{\bar{k}}) = (\delta_i^k x^i, p_k = g_{ki} y^i). \quad (2.2)$$

In another music notation, the flat symbol \flat lowers a note by a half step. Similar to this musical notation, the musical isomorphism g^\flat low the indice of the vector space coordinate.

The Jacobian matrix of g^\flat is obtained by

$$(g_*^\flat) = \left(\tilde{A}_J^M \right) = \left(\frac{\partial \tilde{x}^M}{\partial x^J} \right) = \begin{pmatrix} \delta_j^m & 0 \\ y^s \partial_j g_{ms} & g_{mj} \end{pmatrix} \quad (2.3)$$

and the Jacobian matrix of g^\sharp is obtained by

$$(g_*^\sharp) = \left(A_M^J \right) = \left(\frac{\partial x^J}{\partial \tilde{x}^M} \right) = \begin{pmatrix} \delta_m^j & 0 \\ p_s \partial_m g^{js} & g^{jm} \end{pmatrix}, \quad (2.4)$$

where δ is the Kronecker delta [1].

Yano and Patterson [10] defined the horizontal lift and applied this notion to connection in M . The gh -lift is newly defined in [2]. We obtain gh -lift ${}^{GH} \nabla^*$ of the affine connection ∇ by transferring the horizontal lift of affine connection ∇ from tangent bundle to cotangent bundle via the musical isomorphism. The problems of transferring the lifts to the cotangent bundle were studied in [1,2].

3 The gh -lift of affine connection

Let ∇ be an affine connection on M manifold. The horizontal lift ${}^H \nabla$ of the affine connection ∇ to the tangent bundle TM is formed with equation

$${}^H \nabla = {}^C \nabla - {}^V R \quad (3.1)$$

where ${}^V R$ is vertical lift of curvature tensor of ∇ and ${}^C \nabla$ is complete lift of affine connection to the tangent bundle TM . And also the horizontal lift ${}^H \nabla$ of the affine connection ∇ provides the conditions

$$\begin{aligned} {}^H \nabla_{VX} {}^V Y &= 0, & {}^H \nabla_{VX} {}^H Y &= 0, \\ {}^H \nabla_{HX} {}^V Y &= {}^V (\nabla_X Y), & {}^H \nabla_{HX} {}^H Y &= {}^H (\nabla_X Y) \end{aligned} \quad (3.2)$$

for any $X, Y \in \mathfrak{S}_0^1(M)$.

Let Γ_{ab}^c be coefficient of ∇ according to the local coordinates (x^h) on M . Let ${}^H \Gamma_{AB}^C$ be coefficient of ${}^H \nabla$ according to the induced coordinates (x^h, y^h) to the tangent bundle TM . The non-zero coefficients ${}^H \Gamma_{AB}^C$ of the horizontal lift ${}^H \nabla$ to the tangent bundle TM are given by

$${}^H \Gamma_{ab}^c = \Gamma_{ab}^c, \quad {}^H \Gamma_{\bar{a}\bar{b}}^{\bar{c}} = \Gamma_{ab}^c, \quad {}^H \Gamma_{\bar{a}\bar{b}}^c = \Gamma_{ab}^c, \quad {}^H \Gamma_{\bar{a}\bar{b}}^{\bar{c}} = y^s \partial_s \Gamma_{ab}^c - y^s R_{sab}^c \quad (3.3)$$

where R_{sab}^c is the curvature tensor of ∇ .

The horizontal lift ${}^H \nabla^*$ of the affine connection ∇ to the cotangent bundle T^*M is formed with equation

$${}^H \nabla^* = {}^C \nabla^* - {}^V R^* \quad (3.4)$$

where ∇ is symmetric. ${}^V R^*$ is vertical lift of curvature tensor of ∇ and ${}^C \nabla^*$ is complete lift of ∇ to the cotangent bundle T^*M . Let ${}^H \Gamma_{AB}^C$ be coefficient of ${}^H \nabla^*$ according to the induced coordinates (x^h, p_h) to the cotangent bundle T^*M . The non-zero coefficients ${}^H \Gamma_{AB}^C$ of the horizontal lift ${}^H \nabla^*$ are given by

$${}^H \Gamma_{ab}^c = \Gamma_{ab}^c, \quad {}^H \Gamma_{\bar{a}\bar{b}}^{\bar{c}} = -\Gamma_{ac}^b, \quad {}^H \Gamma_{\bar{a}\bar{b}}^c = -\Gamma_{cb}^a, \quad {}^H \Gamma_{\bar{a}\bar{b}}^{\bar{c}} = p_s \left(-\partial_a \Gamma_{bc}^s + \Gamma_{kc}^s \Gamma_{ab}^k + \Gamma_{bt}^s \Gamma_{ac}^t \right) \quad (3.5)$$

according to the induced coordinates (x^h, p_h) to the cotangent bundle T^*M [11].

We obtain the $\frac{\partial^2 \tilde{x}^K}{\partial x^A \partial x^B}$ that have components

$$\begin{aligned} \frac{\partial^2 \tilde{x}^k}{\partial x^a \partial x^b} &= 0, & \frac{\partial^2 \tilde{x}^k}{\partial x^a \partial x^{\bar{b}}} &= 0, & \frac{\partial^2 \tilde{x}^k}{\partial x^{\bar{a}} \partial x^b} &= 0, & \frac{\partial^2 \tilde{x}^k}{\partial x^{\bar{a}} \partial x^{\bar{b}}} &= 0 \\ \frac{\partial^2 \tilde{x}^{\bar{k}}}{\partial x^a \partial x^b} &= \frac{p_s \partial^2 g^{ks}}{\partial x^a \partial x^b}, & \frac{\partial^2 \tilde{x}^{\bar{k}}}{\partial x^a \partial x^{\bar{b}}} &= \partial_a g^{kb}, & \frac{\partial^2 \tilde{x}^{\bar{k}}}{\partial x^{\bar{a}} \partial x^b} &= \partial_b g^{ka}, & \frac{\partial^2 \tilde{x}^{\bar{k}}}{\partial x^{\bar{a}} \partial x^{\bar{b}}} &= 0. \end{aligned} \quad (3.6)$$

And using (2.3), (2.4), (3.3) and (3.6), the coefficients ${}^{GH} \Gamma_{AB}^C$ of ${}^{GH} \nabla^*$ are obtained from equation

$$g_*^b {}^H \Gamma_{AB}^C = ({}^{GH} \Gamma_{AB}^C) = \left(\frac{\partial \tilde{x}^M}{\partial x^A} \frac{\partial \tilde{x}^S}{\partial x^B} \frac{\partial x^C}{\partial \tilde{x}^K} {}^H \Gamma_{MS}^K + \frac{\partial x^C}{\partial \tilde{x}^K} \frac{\partial^2 \tilde{x}^K}{\partial x^A \partial x^B} \right). \quad (3.7)$$

The non-zero coefficients ${}^{GH} \Gamma_{AB}^C$ of ${}^{GH} \nabla^*$ are obtained as follows where $A, B, \dots = 1, \dots, 2n$:

$${}^{GH}\Gamma_{ab}^*{}^c = \delta_a^m \delta_b^s \delta_k^c \Gamma_{ms}^k = \Gamma_{ab}^c$$

$$\begin{aligned} {}^{GH}\Gamma_{ab}^*{}^{\bar{c}} &= \delta_a^m g^{sb} g_{ck} \Gamma_{ms}^k + g_{ck} \partial_a g^{kb} \\ &= \Gamma_{ac}^b - g^{kb} \partial_a g_{ck} \\ &= -\Gamma_{ac}^b + g^{bk} \partial_c g_{ak} - g^{bk} \partial_k g_{ac} \\ &= -\Gamma_{ac}^b + g^{bk} (\nabla_c g_{ak} + \Gamma_{ca}^t g_{tk} + \Gamma_{ck}^t g_{at}) - g^{bk} (\nabla_k g_{ac} + \Gamma_{ka}^t g_{tc} + \Gamma_{kc}^t g_{ta}) \\ &= -\Gamma_{ac}^b + g^{bk} \nabla_c g_{ak} + g^{bk} \Gamma_{ca}^t g_{tk} - g^{bk} \nabla_k g_{ac} - g^{bk} \Gamma_{ka}^t g_{tc} \\ &= -\Gamma_{ac}^b \end{aligned}$$

$$\begin{aligned} {}^{GH}\Gamma_{ab}^*{}^{\bar{c}} &= g^{ma} \delta_b^s g_{ck} \Gamma_{ms}^k + g_{ck} \partial_b g^{ka} \\ &= \Gamma_{cb}^a - g^{ka} \partial_b g_{ck} \\ &= -\Gamma_{cb}^a + g^{ak} \partial_c g_{bk} - g^{ak} \partial_k g_{cb} \\ &= -\Gamma_{cb}^a + g^{ak} (\nabla_c g_{bk} + \Gamma_{ck}^t g_{bt} + \Gamma_{cb}^t g_{tk}) - g^{ak} (\nabla_k g_{cb} + \Gamma_{kc}^t g_{tb} + \Gamma_{kb}^t g_{ct}) \\ &= -\Gamma_{cb}^a + g^{ak} \nabla_c g_{bk} + g^{ak} \Gamma_{cb}^t g_{tk} - g^{ak} \nabla_k g_{cb} - g^{ak} \Gamma_{kb}^t g_{ct} \\ &= -\Gamma_{cb}^a \end{aligned}$$

$$\begin{aligned} {}^{GH}\Gamma_{ab}^*{}^{\bar{c}} &= \delta_a^m \delta_b^s y^t \partial_k g_{tc} \Gamma_{ms}^k + \delta_a^m \delta_b^s g_{ck} (y^t \partial_t \Gamma_{ms}^k - y^t R_{tms}^k) + p_t \partial_a g^{mt} \delta_b^s g_{ck} \Gamma_{ms}^k \\ &\quad + \delta_a^m p_t \partial_b g^{st} g_{ck} \Gamma_{ms}^k + g_{ck} p_t \partial_{ab}^2 g^{tk} \\ &= y^t (\partial_k g_{tc}) \Gamma_{ab}^k + g_{ck} y^t \partial_t \Gamma_{ab}^k - g_{ck} y^t R_{tab}^k + p_t \partial_a g^{mt} g_{ck} \Gamma_{mb}^k + p_t \partial_b g^{st} g_{ck} \Gamma_{as}^k \\ &\quad + g_{ck} p_t \partial_{ab}^2 g^{tk} \\ &= y^t (\partial_k g_{tc}) \Gamma_{ab}^k + g_{ck} y^t \partial_t \Gamma_{ab}^k - g_{ck} y^t R_{tab}^k + p_t \partial_a g^{mt} g_{ck} \Gamma_{mb}^k + p_t \partial_b g^{st} g_{ck} \Gamma_{as}^k \\ &\quad - p_t (\partial_a \Gamma_{bc}^t) - g_{ck} p_t \Gamma_{bs}^t \partial_a g^{sk} - g_{ck} p_t (\partial_a \Gamma_{bs}^k) g^{ts} - g_{ck} p_t \Gamma_{bs}^k \partial_a g^{ts} \\ &= -p_t (\partial_a \Gamma_{bc}^t) + y^t (\partial_k g_{tc}) \Gamma_{ab}^k + g_{ck} y^t \partial_t \Gamma_{ab}^k - g_{ck} y^t R_{tab}^k + p_t (\partial_b g^{st}) g_{ck} \Gamma_{as}^k \\ &\quad - g_{ck} p_t \Gamma_{bs}^t \partial_c g^{sk} - g_{ck} p_t (\partial_a \Gamma_{bs}^k) g^{ts} \\ &= -p_t (\partial_a \Gamma_{bc}^t) + y^t (\partial_k g_{tc}) \Gamma_{ab}^k + g_{ck} y^t \partial_t \Gamma_{ab}^k - g_{ck} y^t R_{tab}^k - g_{ck} p_t (\partial_a \Gamma_{bs}^k) g^{ts} \\ &\quad + p_t (\partial_b g^{st}) g_{ck} \Gamma_{as}^k - g_{ck} p_t \Gamma_{bs}^t \partial_a g^{sk} \\ &= -p_t (\partial_a \Gamma_{bc}^t) + y^t (\partial_k g_{tc}) \Gamma_{ab}^k + g_{ck} (g^{ts} p_s \partial_t \Gamma_{ab}^k - p_t g^{ts} \partial_a \Gamma_{bs}^k - y^t R_{tab}^k) \\ &\quad + p_t (-\Gamma_{bm}^s g^{mt} - \Gamma_{bm}^t g^{sm}) g_{ck} \Gamma_{as}^k - g_{ck} p_t \Gamma_{bs}^t (-\Gamma_{am}^s g^{mk} - \Gamma_{am}^k g^{ms}) \\ &= -p_t (\partial_a \Gamma_{bc}^t) + p_s \Gamma_{ck}^s \Gamma_{ab}^k + g^{ts} g_{mc} p_s \Gamma_{kt}^m \Gamma_{ab}^k + g_{ck} g^{ts} p_s (\partial_t \Gamma_{ab}^k - \partial_a \Gamma_{bt}^k) \\ &\quad - g_{ck} g^{ts} p_s R_{tab}^k - g_{ck} p_s (\Gamma_{bm}^t \Gamma_{at}^k g^{ms} + \Gamma_{bm}^s \Gamma_{at}^k g^{tm}) \\ &\quad + g_{hk} p_s (\Gamma_{bt}^s \Gamma_{am}^t g^{mk} + \Gamma_{bt}^t \Gamma_{am}^s g^{mt}) \\ &= -p_t (\partial_a \Gamma_{bc}^t) + p_s \Gamma_{ck}^s \Gamma_{ab}^k + p_s \Gamma_{kc}^s \Gamma_{ab}^k + p_s \partial_c \Gamma_{ab}^s - p_s \partial_a \Gamma_{bc}^s - p_s \partial_c \Gamma_{ab}^s \\ &\quad + p_s \partial_a \Gamma_{cb}^s - p_s \Gamma_{cm}^s \Gamma_{ab}^m + p_s \Gamma_{am}^s \Gamma_{cb}^m - p_s \Gamma_{at}^s \Gamma_{bc}^t - p_s \Gamma_{bm}^s \Gamma_{ac}^m + p_s \Gamma_{bt}^s \Gamma_{ac}^t \\ &\quad + p_s \Gamma_{bt}^s \Gamma_{ac}^t \\ &= -p_s \partial_a \Gamma_{bc}^s + p_s \Gamma_{kc}^s \Gamma_{ab}^k + p_s \Gamma_{bt}^s \Gamma_{ac}^t \\ &= p_s (-\partial_a \Gamma_{bc}^s + \Gamma_{kc}^s \Gamma_{ab}^k + \Gamma_{bt}^s \Gamma_{ac}^t) \end{aligned}$$

After these results we get the following theorem

Theorem 3.1 *Let (M, g) be a pseudo Riemannian manifold with dimension n . Let ∇ be an affine connection on manifold M and ${}^H\nabla$ be the horizontal lift of ∇ to the tangent bundle TM . Then the differential of ${}^H\nabla$ by g^b , i.e., a gh-lift ${}^{GH}\nabla^*$ to the cotangent bundle T^*M coincides with the horizontal lift ${}^H\nabla^*$ to the cotangent bundle T^*M if ∇ is a Levi-Civita connection.*

4 The Torsion and Curvature Tensors of gh – lift of Levi-Civita Connection

Let \tilde{T} be torsion tensor of horizontal lift of the symmetric affine connection ∇ on M to the cotangent bundle T^*M . The torsion tensor \tilde{T} of ${}^H\nabla^*$ is determined by

$$\tilde{T}({}^V\theta, {}^V\omega) = 0, \quad \tilde{T}({}^HX, {}^V\omega) = 0, \quad \tilde{T}({}^HX, {}^HY) = -\gamma R(X, Y) \quad (4.1)$$

where R is the curvature tensor of ∇ . ${}^HX, {}^HY$ are horizontal lift of $X, Y \in \mathfrak{S}_0^1(M)$ and ${}^V\theta, {}^V\omega$ are vertical lift of $\theta, \omega \in \mathfrak{S}_1^0(M)$ to the cotangent bundle T^*M . The non zero component \tilde{T}_{AB}^C of \tilde{T} is given

$$\tilde{T}_{ab}^{\bar{c}} = -p_t R_{abc}^t \quad (4.2)$$

according to the induced coordinates (x^h, p_h) to the cotangent bundle T^*M [11].

Let \hat{T} be torsion tensor of the gh -lift ${}^{GH}\nabla^*$ of the Levi-Civita connection on M to the cotangent bundle T^*M . Using the coefficient ${}^{GH}\Gamma_{AB}^C$ of ${}^{GH}\nabla^*$ to the cotangent bundle T^*M the components \hat{T}_{AB}^C of \hat{T} are obtained with the equation

$$\hat{T}_{AB}^C = {}^{GH}\Gamma_{AB}^C - {}^{GH}\Gamma_{BA}^C \quad (4.3)$$

according to the induced coordinates (x^h, p_h) to the cotangent bundle T^*M . We have

$$\begin{aligned} \hat{T}_{ab}^c &= 0, & \hat{T}_{\bar{a}\bar{b}}^c &= 0, & \hat{T}_{ab}^{\bar{c}} &= 0, & \hat{T}_{ab}^{\bar{c}} &= -p_t R_{abc}^t \\ \hat{T}_{\bar{a}\bar{b}}^{\bar{c}} &= 0, & \hat{T}_{\bar{a}\bar{b}}^c &= 0, & \hat{T}_{\bar{a}\bar{b}}^c &= 0, & \hat{T}_{\bar{a}\bar{b}}^{\bar{c}} &= 0. \end{aligned} \quad (4.4)$$

After we have

Corollary 4.1 *The torsion tensor \hat{T} of the gh -lift ${}^{GH}\nabla^*$ of Levi-Civita connection coincides with the torsion tensor \tilde{T} of horizontal lift ${}^H\nabla^*$ of the symmetric affine connection.*

Let \tilde{R} be curvature tensor of horizontal lift of the symmetric affine connection ∇ on M to the cotangent bundle T^*M . For any $X, Y, Z \in \mathfrak{S}_0^1(M)$ and $\omega, \theta, \psi \in \mathfrak{S}_1^0(M)$, the curvature tensor \tilde{R} of ${}^H\nabla^*$ provides the conditions

$$\begin{aligned} \tilde{R}({}^V\theta, {}^V\omega) &= 0, & \tilde{R}({}^HX, {}^V\omega) &= 0 \\ \tilde{R}({}^HX, {}^HY) {}^V\psi &= -{}^V(\psi \circ R(X, Y)) \\ \tilde{R}({}^HX, {}^HY) {}^HZ &= {}^H(R(X, Y)Z) \end{aligned} \quad (4.5)$$

where R is the curvature tensor of ∇ . The non zero components \tilde{R}_{ABD}^C of \tilde{R} are given

$$\begin{aligned} \tilde{R}_{abd}^c &= R_{abd}^c \\ \tilde{R}_{abd}^{\bar{c}} &= p_s (\Gamma_{ct}^s R_{abd}^t + \Gamma_{dt}^s R_{abc}^t) \\ \tilde{R}_{\bar{a}\bar{b}\bar{d}}^{\bar{c}} &= -R_{abc}^d \end{aligned} \quad (4.6)$$

according to the induced coordinates (x^h, p_h) to the cotangent bundle T^*M .

Let \widehat{R} be curvature tensor of the gh-lift ${}^{GH}\nabla^*$ of Levi-Civita connection on M to the cotangent bundle T^*M . Using the coefficient ${}^{GH}\Gamma_{AB}^C$ of ${}^{GH}\nabla^*$ to the cotangent bundle T^*M the components \widehat{R}_{ABD}^C of \widehat{R} are obtained with the equation

$$\widehat{R}_{ABD}^C = \partial_A {}^{GH}\Gamma_{BD}^C - \partial_B {}^{GH}\Gamma_{AD}^C + {}^{GH}\Gamma_{AT}^C {}^{GH}\Gamma_{BD}^T - {}^{GH}\Gamma_{BT}^C {}^{GH}\Gamma_{AD}^T \quad (4.7)$$

according to the induced coordinates (x^h, p_h) . We have

$$\begin{aligned} \widehat{R}_{abd}^c &= R_{abd}^c \\ \widehat{R}_{abd}^{\bar{c}} &= p_s (\Gamma_{ct}^s R_{abd}^t + \Gamma_{dt}^s R_{abc}^t) \\ \widehat{R}_{ab\bar{d}}^c &= -R_{abc}^d \end{aligned} \quad (4.8)$$

After we have

Corollary 4.2 *The curvature tensor \widehat{R} of the gh-lift ${}^{GH}\nabla^*$ of Levi-Civita connection coincides with the curvature tensor \widetilde{R} of horizontal lift ${}^H\nabla^*$ of the symmetric affine connection.*

5 Geodesics of gh-lift of Levi-Civita Connection

Let \widetilde{C} be a geodesic curve of the gh-lift ${}^{GH}\nabla^*$ of Levi-Civita connection on M to the cotangent bundle T^*M . The geodesic \widetilde{C} is determined with the equations

$$\frac{d^2 x^C}{dt^2} + {}^{GH}\Gamma_{AB}^C \frac{dx^A}{dt} \frac{dx^B}{dt} = 0 \quad (5.1)$$

according to the induced coordinates $(x^c, x^{\bar{c}}) = (x^c, p_c)$.

Using the coefficient of gh-lift ${}^{GH}\nabla^*$ we obtain following equations from (5.1):

$$\begin{aligned} \frac{d^2 x^c}{dt^2} + {}^{GH}\Gamma_{ab}^{*c} \frac{dx^a}{dt} \frac{dx^b}{dt} + {}^{GH}\Gamma_{\bar{a}\bar{b}}^{*c} \frac{dx^{\bar{a}}}{dt} \frac{dx^{\bar{b}}}{dt} + {}^{GH}\Gamma_{ab}^{*c} \frac{dx^a}{dt} \frac{dx^{\bar{b}}}{dt} + {}^{GH}\Gamma_{\bar{a}\bar{b}}^{*c} \frac{dx^{\bar{a}}}{dt} \frac{dx^{\bar{b}}}{dt} &= 0 \\ \frac{d^2 x^c}{dt^2} + \Gamma_{ab}^c \frac{dx^a}{dt} \frac{dx^b}{dt} &= 0, \end{aligned} \quad (5.2)$$

$$\begin{aligned} \frac{d^2 x^{\bar{c}}}{dt^2} + {}^{GH}\Gamma_{ab}^{*\bar{c}} \frac{dx^a}{dt} \frac{dx^b}{dt} + {}^{GH}\Gamma_{\bar{a}\bar{b}}^{*\bar{c}} \frac{dx^{\bar{a}}}{dt} \frac{dx^{\bar{b}}}{dt} + {}^{GH}\Gamma_{ab}^{*\bar{c}} \frac{dx^a}{dt} \frac{dx^{\bar{b}}}{dt} + {}^{GH}\Gamma_{\bar{a}\bar{b}}^{*\bar{c}} \frac{dx^{\bar{a}}}{dt} \frac{dx^{\bar{b}}}{dt} &= 0 \\ \frac{d^2 p_c}{dt^2} + p_s (-\partial_a \Gamma_{bc}^s + \Gamma_{cm}^s \Gamma_{ab}^m + \Gamma_{bm}^s \Gamma_{ca}^m) \frac{dx^a}{dt} \frac{dx^b}{dt} - \Gamma_{cb}^a \frac{dp_a}{dt} \frac{dx^b}{dt} - \Gamma_{ac}^b \frac{dx^a}{dt} \frac{dp_b}{dt} &= 0 \\ \frac{d}{dt} \left(\frac{dp_c}{dt} - \Gamma_{cb}^a p_a \frac{dx^b}{dt} \right) - \Gamma_{ac}^m \left(\frac{dp_m}{dt} - \Gamma_{bm}^s p_s \frac{dx^b}{dt} \right) \frac{dx^a}{dt} &= 0 \\ \frac{d}{dt} \left(\frac{\delta p_c}{dt} \right) - \Gamma_{ac}^m \left(\frac{\delta p_m}{dt} \right) \frac{dx^a}{dt} &= 0 \\ \frac{\delta}{dt} \left(\frac{\delta p_c}{dt} \right) &= 0 \\ \frac{\delta^2 p_c}{dt^2} &= 0 \end{aligned} \quad (5.3)$$

where $\frac{\delta p_c}{dt} = \frac{dp_c}{dt} - \Gamma_{cb}^a p_a \frac{dx^b}{dt}$. After expressions (5.2) and (5.3) we have

Theorem 5.1 Let \tilde{C} be a geodesic according to the gh -lift ${}^{GH}\nabla^*$ of Levi-Civita connection on M to the cotangent bundle T^*M . The geodesic \tilde{C} has the equations

$$\begin{aligned} \frac{d^2x^c}{dt^2} + \Gamma_{ab}^c \frac{dx^a}{dt} \frac{dx^b}{dt} &= 0, \\ \frac{\delta^2 p_c}{dt^2} &= 0 \end{aligned}$$

according to the induced coordinates (x^c, p_c) to T^*M .

Theorem 5.2 The curve \tilde{C} on the cotangent bundle T^*M is a geodesic according to the gh -lift ${}^{GH}\nabla^*$ of Levi-Civita connection on M if the projection $C = \pi(\tilde{C})$ on M is a geodesic according to the ∇ on M and the second covariant differentiation of $p_c = p_c(t)$ along C vanishes where $\pi : T^*M \rightarrow M$ is the natural projection.

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