

Some differential properties of grand generalized Sobolev-Morrey spaces

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Received: 23.01.2020 / Revised: 03.08.2020 / Accepted: 02.10.2020

Abstract. In this paper we introduce a grand generalized Sobolev-Morrey spaces. Also, with the help of integral representations we study differential properties of functions from this spaces.

Keywords. Grand generalized Sobolev-Morrey spaces, embedding theorems, integral representation, Hölder condition.

Mathematics Subject Classification (2010): 46E35, 46E30, 26D15

1 Introduction and preliminary notes

This paper is devoted to investigations of embedding theorems for the grand generalized Sobolev-Morrey spaces

$$\bigcap_{i=0}^n L_{p^i, \varkappa, a}^{<l^i>}(G), \quad (1.1)$$

where $G \subset \mathbb{R}^n$ is a bounded domain, $1 < p^i < \infty$; $l^i \in N_0^n$ ($i = 0, 1, 2, \dots, n$), $l_j^0 \geq 0, (j = 1, 2, \dots, n)$; $l_j^i \geq 0, (i = 0, 1, 2, \dots, n), l_i^i > 0$ are non-negative integers ($i = 0, 1, 2, \dots, n$); $a \in [0, 1]$; $\varkappa \in (0, \infty)^n$. First we introduce a grand generalized Sobolev-Morrey, and on the based of the method of integral representation we study some differential properties of functions, defined on n -dimensional domains satisfying flexible φ -horn condition (see, [12]).

Note that the grand Lebesgue spaces $L_p(G)$ for a measurable set $G \subset \mathbb{R}^n$ of finite Lebesgue measure were introduced in the work of T. Iwaniec and C. Sbordone in [3]. After a vast amount of research about small Lebesgue-Morrey space, grand and small Sobolev spaces, grand grand Lebesgue-Morrey space, grand grand and small small Sobolev-Morrey spaces, grand grand Nikolskii-Morrey spaces, and grand Sobolev-Morrey spaces with dominant mixed derivatives (with different norms) has been done by many mathematicians (see, e.g. [2, 4–8, 10, 11, 13–21]).

Let $G \subset \mathbb{R}^n$ is a bounded domain, $t > 0$, and let for all $x \in \mathbb{R}^n$

$$G_{t\varkappa}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2} t^{\varkappa_j}, j = 1, 2, \dots, n \right\}.$$

Definition 1.1 We denote by $\bigcap_{i=0}^n L_{p^i, \varkappa, a}^{<l^i>}(G)$ the space of locally summable functions f on G having weak derivatives $D^i f$ ($i = 0, 1, \dots, n$) with the finite norm

$$\|f\|_{\bigcap_{i=0}^n L_{p^i, \varkappa, a}^{<l^i>}(G)} = \sum_{i=0}^n \|D^i f\|_{p^i, \varkappa, a; G}. \quad (1.2)$$

Here

$$\begin{aligned} \|f\|_{p^i, \varkappa, a; G} &= \|f\|_{L_{p^i, \varkappa, a}(G)} \\ &= \sup_{\substack{0 < t \leq d_0, \\ x \in G, \\ 0 < \varepsilon < p-1}} \left(\frac{1}{t^{|\varkappa|a}} \frac{\varepsilon}{|G_{t\varkappa}(x)|} \int_{G_{t\varkappa}(x)} |f(y)|^{p^{i-\varepsilon}} dy \right)^{\frac{1}{p^i - \varepsilon}}, \end{aligned} \quad (1.3)$$

and $d_0 = \text{diam } G$.

Note that the space $L_{p^i, \varkappa, a}(G)$ defined and studied in [13], and $\bigcap_{i=0}^n L_{p^i, \varkappa, a}(G)$, in the case $l^0 = (0, \dots, 0)$, $l^i = (0, \dots, 0, l_i, 0, \dots, 0)$, $p^i = p$ ($i = 0, 1, \dots, n$) coincides with the grand Sobolev-Morrey space $W_{p^i, \varkappa, a}^{l^i}(G)$ introduced and studied in [13].

We give some properties of the spaces $\bigcap_{i=0}^n L_{p^i, \varkappa, a}^{<l^i>}(G)$.

1. The following embedding holds:

$$\bigcap_{i=0}^n L_{p^i, \varkappa, a}^{<l^i>}(G) \hookrightarrow \bigcap_{i=0}^n L_{p^i}^{<l^i>}(G),$$

i.e.,

$$\|f\|_{\bigcap_{i=0}^n L_{p^i}^{<l^i>}(G)} \leq C \|f\|_{\bigcap_{i=0}^n L_{p^i, \varkappa, a}^{<l^i>}(G)}. \quad (1.4)$$

Norm in grand generalized Sobolev space $\bigcap_{i=0}^n L_{p^i}^{<l^i>}(G)$ defined as:

$$\|f\|_{\bigcap_{i=0}^n L_{p^i}^{<l^i>}(G)} = \sum_{i=0}^n \|D^i f\|_{p^i, G},$$

$$\|f\|_{p^i, G} = \|f\|_{L_{p^i}(G)} = \sup_{0 < \varepsilon < p^i - 1} \left(\frac{\varepsilon}{|G|} \int_G |f(x)|^{p^{i-\varepsilon}} dx \right)^{\frac{1}{p^i - \varepsilon}};$$

2. The spaces $\bigcap_{i=0}^n L_{p^i, \varkappa, a}^{<l^i>}(G)$ is complete;

3. $\|f\|_{\bigcap_{i=0}^n L_{p^i, \varkappa, 0}^{<l^i>}(G)} = \|f\|_{\bigcap_{i=0}^n L_{p^i}^{<l^i>}(G)}$.

To prove the main theorems, we need some auxiliary inequalities given in the lemma below. Let $\psi(\cdot, y, z) \in C_0^\infty(\mathbb{R}^n)$ be such that

$$S(\psi) = \text{supp } \psi \subset I_1 = \left\{ x : |x_j| < \frac{1}{2}, j = 1, 2, \dots, n \right\},$$

and let $0 < T \leq 1$, $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_j > 0 (j = 1, 2, \dots, n)$ and put

$$V = \bigcup_{0 < t \leq T} \left\{ y : \left(\frac{y}{t^\lambda} \right) \in S(\psi) \right\}.$$

Clearly, $V \subset I_{T^\lambda} = x; |x_j| < \frac{1}{2}T_j^\lambda, j = 1, 2, \dots, n$, and let U be an open set contained in the domain G ; henceforth we always assume that $U + V \subset G$, and put $G_{T^\lambda}(U) = U + I_{T^\lambda(x)} \cap G$.

Obviously, if $0 < \varkappa_j \leq \lambda_j (j = 1, 2, \dots, n)$, then $I_{T^\lambda} \subset I_{T^\varkappa}$ and therefore $U + V \subset G_{T^\varkappa}(U) = Z$.

Lemma 1.1 *Let $1 < p^i < p \leq r \leq \infty; 0 < |\varkappa| \leq \frac{|\lambda|}{1+a}; 0 < t, \eta \leq T \leq 1; 0 < \gamma < \gamma_0; \nu = (\nu_1, \nu_2, \dots, \nu_n), \nu_j \geq 0$ are integers ($j = 1, 2, \dots, n$); $\varphi \in L_{p^i, \varkappa, a}(G)$ and*

$$m^i = (l^i, \lambda) - (\nu, \lambda) - (|\lambda| - |\varkappa| - |\varkappa|a) \left(\frac{1}{p^i - \varepsilon} - \frac{1}{p - \varepsilon} \right), \quad (1.5)$$

$$(\nu, \lambda) = \sum_{j=1}^n \nu_j \lambda_j, |\lambda| = \sum_{i=1}^n \lambda_j,$$

$$R_{\eta}^i(x) = \int_0^{\eta} t^{-1-|\lambda|-(\nu, \lambda)+(l^i, \lambda)} dt \int_{\mathbb{R}^n} \varphi(x+y) \psi\left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x)\right) dy, \quad (1.6)$$

$$R_{\eta, T}^i(x) = \int_{\eta}^T t^{-1-|\lambda|-(\nu, \lambda)+(l^i, \lambda)} dt \int_{\mathbb{R}^n} \varphi(x+y) \psi\left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x)\right) dy. \quad (1.7)$$

Then

$$\sup_{\bar{x} \in U} \|R_{\eta}^i\|_{p-\varepsilon, U_{\gamma^\varkappa}(\bar{x})} \leq C^1 \|\varphi\|_{p^i, \varkappa, a, Z} \varepsilon^{-\frac{1}{p^i-\varepsilon}} \gamma^{|\varkappa| \frac{a+1}{p^i-\varepsilon}} \eta^{m^i} \quad (m^i > 0), \quad (1.8)$$

$$\begin{aligned} \sup_{\bar{x} \in U} \|R_{\eta, T}^i\|_{p-\varepsilon, U_{\gamma^\varkappa}(\bar{x})} &\leq C^2 \|\varphi\|_{p^i, \varkappa, a, Z} \varepsilon^{-\frac{1}{p^i-\varepsilon}} \gamma^{|\varkappa| \frac{a+1}{p^i-\varepsilon}} \\ &\times \begin{cases} T^{m^i} & \text{for } m^i > 0, \\ \ln \frac{T}{\eta} & \text{for } m^i = 0, \\ \eta^{m^i} & \text{for } m^i < 0. \end{cases} \end{aligned} \quad (1.9)$$

Here $U_{\gamma^\varkappa} \bar{x} = \{x : |x_j - \bar{x}_j| < \frac{1}{2} \gamma^{\varkappa_j}, j = 1, 2, \dots, n$ and C^1, C^2 are constants independent of φ, γ, η, T and ε .

Proof. Applying the generalized Minkowskii inequality for any $\bar{x} \in U$, we get

$$\|R_{\eta}^i\|_{p-\varepsilon, U_{\gamma^\varkappa}(\bar{x})} \leq C \int_0^{\eta_0} t^{-1-|\lambda|-(\nu, \lambda)+(l^i, \lambda)} \|\phi\|_{p-\varepsilon, U_{\gamma^\varkappa}(\bar{x})} dt, \quad (1.10)$$

where

$$\phi(x, t) = \int_{\mathbb{R}^n} \varphi(x+y) \Psi\left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho^1(t^\lambda, x)\right) dy. \quad (1.11)$$

From Hölder's inequality ($p \leq r$), we have

$$\|\phi(\cdot, t)\|_{p-\varepsilon, U_{\gamma^{\varkappa}}(\bar{x})} \leq \|\phi(\cdot, t)\|_{r-\varepsilon, U_{\gamma^{\varkappa}}(\bar{x})} \gamma^{|\varkappa| \left(\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon} \right)}. \quad (1.12)$$

Now we estimate the norm $\|\phi(\cdot, t)\|_{p-\varepsilon, U_{\gamma^{\varkappa}}(\bar{x})}$.

Let χ be the characteristic function of $S(\Psi)$ and let $\Psi(x, y, z)$ satisfy condition $|\Psi(x, y, z)| \leq C |\Psi_1(x)|$, for a.e. $(y, z) \in \mathbb{R}^n \times \mathbb{R}^n$.

Account into that $1 < p^i < r \leq \infty$, $s \leq r \left(\frac{1}{s} = 1 - \frac{1}{p^i-\varepsilon} + \frac{1}{r-\varepsilon} \right)$, represent the integrand function (1.11) as the form

$$\|\varphi \Psi\| = (|\varphi|^{p^i-\varepsilon} |\Psi|^{\frac{1}{s}})^{\frac{1}{p^i-\varepsilon}} (|\Psi|^{p^i-\varepsilon} \chi)^{\frac{1}{p^i-\varepsilon} - \frac{1}{r-\varepsilon}} - (|\Psi|^s)^{\frac{1}{s} - \frac{1}{r-\varepsilon}}$$

and applying Hölders inequality $\left(\frac{1}{r-\varepsilon} + \left(\frac{1}{p^i-\varepsilon} - \frac{1}{r-\varepsilon} \right) + \left(\frac{1}{s} - \frac{1}{r-\varepsilon} \right) = 1 \right)$, we have

$$\begin{aligned} \|\phi(\cdot, t)\|_{r-\varepsilon, U_{\gamma^{\varkappa}}(\bar{x})} &\leq \sup_{\bar{x} \in U_{\gamma^{\varkappa}}(\bar{x})} \left(\int_{\mathbb{R}^n} |\varphi(x+y)|^{p^i-\varepsilon} \chi\left(\frac{y}{t^\lambda}\right) dy \right)^{\frac{1}{p^i-\varepsilon} - \frac{1}{r-\varepsilon}} \\ &\times \sup_{\bar{y} \in V} \left(\int_{U_{\gamma^{\varkappa}}(\bar{x})} |\varphi(x+y)|^{p^i-\varepsilon} dx \right)^{\frac{1}{p^i-\varepsilon}} \left(\int_{\mathbb{R}^n} |\Psi_1\left(\frac{y}{t^\lambda}\right)|^s dy \right)^{\frac{1}{s}}. \end{aligned} \quad (1.13)$$

For any $x \in U$, we have

$$\begin{aligned} \int |\varphi(x+y)|^{p^i-\varepsilon} \chi\left(\frac{y}{t^\lambda}\right) dy &\leq \int_{Z_{t^\lambda}(x)} |\varphi(y)|^{p^i-\varepsilon} dy \\ &\leq \int_{Z_{t^\lambda}(x)} |\varphi(y)|^{p^i-\varepsilon} dy \leq \|\varphi\|_{p^i-\varepsilon, Z_{t^\lambda}(x)}^{p^i-\varepsilon} \leq \|\varphi\|_{p, \varkappa, a; Z}^{p^i-\varepsilon} \varepsilon^{-1} \gamma^{|\varkappa|(a+1)}. \end{aligned} \quad (1.14)$$

Also, for $y \in U$

$$\int_{U_{\gamma^{\varkappa}}(\bar{x})} |\varphi(x+y)|^{p^i-\varepsilon} dx \leq \int_{U_{\gamma^{\varkappa}}(\bar{x}+y)} |\varphi(x)|^{p^i-\varepsilon} dx \leq \|\varphi\|_{p, \varkappa, a; Z}^{p^i-\varepsilon} \varepsilon^{-1} \gamma^{|\varkappa|(a+1)}, \quad (1.15)$$

$$\int_{\mathbb{R}^n} |\Psi\left(\frac{y}{t^\lambda}\right)|^s dy = t^{|\lambda|} \|\Psi_1\|_s^s. \quad (1.16)$$

From inequalities (1.12)-(1.16) for ($r = p$) it is implies that

$$\begin{aligned} &\|\phi(\cdot, t)\|_{p-\varepsilon, U_{\gamma^{\varkappa}}(\bar{x})} \\ &\leq C_1 \|\Psi_1\|_s \|\varphi\|_{p^i, \varkappa, a; Z} \varepsilon^{-\frac{1}{p^i-\varepsilon}} \gamma^{|\varkappa| \frac{a+1}{p^i-\varepsilon}} t^{|\lambda| - (|\lambda| - |\varkappa| - |\varkappa|a) \left(\frac{1}{p^i-\varepsilon} - \frac{1}{p-\varepsilon} \right)}. \end{aligned}$$

Take into account last inequality in (1.10) for all $x \in U$, we have

$$\|R_\eta^i\|_{p-\varepsilon, U_{\gamma^{\varkappa}}(\bar{x})} \leq C^1 \|\varphi\|_{p^i, \varkappa, a; Z} \varepsilon^{-\frac{1}{p^i-\varepsilon}} \gamma^{|\varkappa| \frac{a+1}{p^i-\varepsilon}} \eta^{m^i} \quad (m^i > 0). \quad (1.17)$$

Similarly, we can prove (1.9).

This complete the proof.

2 Main results

Now we proved theorems for functions from grand generalized Sobolev-Morrey spaces

$$\bigcap_{i=0}^n L_{p^i, \mathcal{Z}, a}^{<l^i>}(G).$$

Theorem 2.1 *Let $G \subset \mathbb{R}^n$ be an open bounded set satisfying the flexible λ -horn condition (see [1]); $1 < p^i < p \leq \infty$; $|\mathcal{Z}| \leq \frac{|\lambda|}{1+a}$; $\nu = (\nu_1, \nu_2, \dots, \nu_n)$, $\nu_j \geq 0$ are integers ($j = 1, 2, \dots, n$); $m^i > 0$ ($i = 1, 2, \dots, n$), and $f \in \bigcap_{i=0}^n L_{p^i, \mathcal{Z}, a}^{<l^i>}(G)$.*

Then the following embeddings hold

$$D^\nu : \bigcap_{i=0}^n L_{p^i, \mathcal{Z}, a}^{<l^i>}(G) \hookrightarrow L_{p-\varepsilon}(G).$$

More precisely, for $f \in \bigcap_{i=0}^n L_{p^i, \mathcal{Z}, a}^{<l^i>}(G)$ there exists a generalized (weak) derivatives $D^\nu f$ and

$$\|D^\nu f\|_{p-\varepsilon, G} \leq C^1(\varepsilon) \sum_{i=0}^n T^{m^i} \|D^{l^i} f\|_{p^i, \mathcal{Z}, a; G}. \quad (2.1)$$

In particular, if $m^{i,0} = (l^i, \lambda) - (\nu, \lambda) - (|\lambda| - |\mathcal{Z}| - |\mathcal{Z}|a) \frac{1}{p^i - \varepsilon} > 0$ ($i = 0, 1, 2, \dots, n$), then $D^\nu f$ are continuous on G and

$$\sup_{x \in G} |D^\nu f(x)| \leq C^1(\varepsilon) \sum_{i=0}^n T^{m^{i,0}} \|D^{l^i} f\|_{p^i, \mathcal{Z}, a; G}, \quad (2.2)$$

where $0 < T \leq d_0$, $C^1(\varepsilon) = C^1 \varepsilon^{-\frac{1}{p^i - \varepsilon}}$ and C^1 is a constant independent of f , T and ε .

Proof. At first note that in the conditions of Theorem 2.1 there exists a generalized derivatives $D^\nu f$ on G . Indeed, from the condition $m^i > 0$ ($i = 1, \dots, n$) it follows that for $p^i < p$, $|\mathcal{Z}| \leq \frac{|\lambda|}{1+a}$, and

$$\bigcap_{i=0}^n L_{p^i, \mathcal{Z}, a}^{<l^i>}(G) \hookrightarrow \bigcap_{i=0}^n L_{p^i}^{<l^i>}(G) \hookrightarrow \bigcap_{i=0}^n L_{p^i - \varepsilon}^{<l^i>}(G) \quad (p^i - \varepsilon > 1).$$

Then $D^\nu f$ exists on G and belongs to $L_{p^i - \varepsilon}(G)$ and for almost each $x \in G$ the integral identity is hold

$$\begin{aligned} D^\nu f(x) &= f_{T^\lambda}^{(\nu)}(x) + \int_0^T \int_{\mathbb{R}^n} \sum_{i=1}^n t^{-1 - |\lambda| - (\nu, \lambda) + (l^i, \lambda)} D^{l^i} f(x + y) \\ &\quad \times M_i^{(\nu)}\left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x)\right) dy dt, \end{aligned} \quad (2.3)$$

$$\begin{aligned} f_{T^\lambda}^{(\nu)}(x) &= (-1)^{|\nu|} T^{-|\lambda| - (\nu, \lambda) + (l^0, \lambda)} \\ &\quad \times \int_{\mathbb{R}^n} D^{l^0} f(x + y) \Omega^{(\nu)}\left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x)\right) dy, \end{aligned} \quad (2.4)$$

where $0 < T \leq \min(d_0, T_0)$, $\Omega(\cdot, y, z)$ and $M_i(\cdot, y, z) \in C_0^\infty(\mathbb{R}^n)$ (see, [9]). Recall that the support of the integral representations (2.3) and (2.4) is $V(\lambda)$. Applying the Minkowskii inequality, from (2.3) and (2.4), we get

$$\|D^\nu f\|_{p-\varepsilon, G} \leq \|f_{T^\lambda}^\nu\|_{p-\varepsilon, G} + \sum_{i=1}^n \|R_T^i\|_{p-\varepsilon, G} \quad (2.5)$$

By (1.8) for $U = G$, $D^0 f = \phi$, $\Psi = \Omega(\nu)$, $\eta = T$, one has

$$\|f_{T^\lambda}^\nu\|_{p-\varepsilon, G} \leq C_1(\varepsilon) \|D^0 f\|_{p^0, \varkappa, a, G} T^{m^0} \quad (2.6)$$

and for $U = G$, $D^i = \phi$, $\Psi = M_i(\nu)$, $\eta = T$, we have

$$\|R_T^i\|_{p-\varepsilon, G} \leq C_2(\varepsilon) \|D^i f\|_{p^i, \varkappa, a; G} T^{m^i}. \quad (2.7)$$

Substituting (2.6) and (2.7) in (2.5), we get (2.5).

Let $m^{i,0} > 0$ ($i = 1, 2, \dots, n$). By (2.3), (2.4) from inequality (2.5) for $p = \infty$ and $m^i(p = \infty) = m^{i,0}$ ($i = 1, \dots, n$), we obtain

$$\|D^\nu f - f_{T^\lambda}^\nu\|_{\infty, G} \leq \sum_{i=1}^n T^{m^{i,0}} \|D^i f\|_{p^i, \varkappa, a; G}.$$

It follows that the left hand side of the last inequality tends to zero as $T \rightarrow 0$. Since $f_{T^\lambda}^\nu$ is continuous on G , the convergence in $L_\infty(G)$ coincides with usually uniform convergence. Then the limit function $D^\nu f$ is continuous on G .

Theorem 2.1 is proved.

Let ρ be an n -dimensional vector.

Theorem 2.2 *Let all the conditions of Theorem 2.1 be fulfilled. If $m^i > 0$ ($i = 1, 2, \dots, n$), then $D^\nu f$ satisfies the Hölder condition with exponent σ on G in the metric of $L_{p-\varepsilon}$. More exactly,*

$$\|\Delta(\zeta, G) D^\nu f\|_{p-\varepsilon, G} \leq C^2(\varepsilon) \|f\|_{\bigcap_{i=0}^n L_{p^i, \varkappa, a}(G)}^{<i>} |\zeta|^\sigma, \quad (2.8)$$

where $C^2(\varepsilon) = C^2 \varepsilon^{-\frac{1}{p^i - \varepsilon}}$ and C^2 is a constant independent of f and ε and σ is an arbitrary number satisfying the inequalities:

$$0 \leq \sigma \leq 1, \text{ if } \frac{m^0}{\lambda_0} > 0,$$

$$0 \leq \sigma < 1, \text{ if } \frac{m^0}{\lambda_0} = 1,$$

$$0 \leq \sigma \leq \frac{m^0}{\lambda_0}, \text{ if } \frac{m^0}{\lambda_0} < 1,$$

where $m^0 = \min_{1 \leq i \leq n} m^i$; $\lambda_0 = \max_{1 \leq i \leq n} \lambda_i$.

If $m^{i,0} > 0$ ($i = 1, 2, \dots, n$), then

$$\sup_{x \in G} |\Delta(\zeta, G) D^\nu f(x)| \leq C^2(\varepsilon) \|f\|_{\bigcap_{i=0}^n L_{p^i, \varkappa, a}(G)}^{<i>} |\zeta|^{\sigma_0}, \quad (2.9)$$

where σ_0 satisfy the same conditions as σ with $m^{i,0}$ instead of m^i .

Proof. According to Lemma 8.6 in [1], there exists a domain $G_\sigma \subset G$ ($\sigma = \xi r(x)$, $r = \rho_\lambda(x, \partial G)$, $x \in G$).

Suppose that $|\zeta|_\lambda < \sigma$, then for any $x \in G_\sigma$ the segment joining the points $x, x + \zeta$ is contained in G . Consequently, for all the points of this segment, identities (2.3) and (2.4) with the same kernels are valid. After some transformations, from (2.3) and (2.4), we get

$$\begin{aligned}
& |\Delta(\zeta, G)D^\nu f(x)| \leq T^{-|\lambda|-(\nu, \lambda)+(l^0, \lambda)} \\
& \times \int_{\mathbb{R}^n} D^{l^0} f(x+y) \left| \Omega^{(\nu)} \left(\frac{y-\zeta}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x) \right) \right. \\
& \quad \left. - \Omega^{(\nu)} \left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x) \right) \right| dy \\
& + \sum_{i=1}^n \int_0^{|\zeta|_\lambda^{\frac{1}{\lambda_0}}} t^{-1-|\lambda|-(\nu, \lambda)+(l^i, \lambda)} \int_{\mathbb{R}^n} (|D^{l^i} f(x+y+\zeta)| + |D^{l^i} f(x+y)|) \\
& \quad \times |M_i^{(\nu)} \left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x) \right)| dy dt \\
& + \int_{|\zeta|_\lambda^{\frac{1}{\lambda_0}}}^T t^{-1-|\lambda|-(\nu, \lambda)+(l^i, \lambda)} \int_{\mathbb{R}^n} |D^{l^i} f(x+y)| \left| M_i^{(\nu)} \left(\frac{y-\zeta}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x) \right) \right. \\
& \quad \left. - M_i^{(\nu)} \left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x) \right) \right| dy dt \\
& = R(x, \zeta) + \sum_{i=0}^n (R_{1,i}(x, \zeta) + R_{2,i}(x, \zeta)), \tag{2.10}
\end{aligned}$$

where $0 < T \leq d_0$. We also assume that $|\zeta|_\lambda < T$, and consequently $|\zeta|_\lambda \leq \min\{\sigma, T\}$. If $x \in G \setminus G_\sigma$, then $\Delta(\zeta, G)D^\nu f(x) = 0$. By (2.10)

$$\begin{aligned}
& \|\Delta(\zeta, G)D^\nu f\|_{p-\varepsilon, G} = \|\Delta(\zeta, G)D^\nu f\|_{p-\varepsilon, G_\sigma} \\
& \leq \|R(\cdot, \zeta)\|_{p-\varepsilon, G_\sigma} + \sum_{i=0}^n (\|R_{1,i}(\cdot, \zeta)\|_{p-\varepsilon, G_\sigma} + \|R_{2,i}(\cdot, \zeta)\|_{p-\varepsilon, G_\sigma}). \tag{2.11}
\end{aligned}$$

Note that

$$\begin{aligned}
& \left| \Omega^{(\nu)} \left(\frac{y-\zeta}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x) \right) - \Omega^{(\nu)} \left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x) \right) \right| \\
& \leq \sum_{i=0}^n T^{-\lambda_j} \int_0^{|\zeta|} \left| D_j \Omega^{(\nu)} \left(\frac{y-\xi e_\zeta}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x) \right) \right| d\xi,
\end{aligned}$$

$e_\zeta = \frac{\zeta}{|\zeta|}$. Therefore

$$\begin{aligned}
& R(x, \zeta) \leq \sum_{j=1}^n T^{-\lambda_j - |\lambda| - (\nu, \lambda) + (l^j, \lambda)} \\
& \times \int_0^{|\zeta|} d\xi \int_{\mathbb{R}^n} |D^{l^0} f(x + \xi e_\zeta + y)| \left| D_j \Omega^{(\nu)} \left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}, \rho'(T^\lambda, x) \right) \right| dy.
\end{aligned}$$

Similarly, we get

$$R_{2,i}(x, \zeta) \leq \sum_{j=1}^n \int_0^{|\zeta|} t^{-1-|\lambda|-(\nu,\lambda)-\lambda_j+(l^i,\lambda)} dt \\ \times \int_{\mathbb{R}^n} |D^l f(x + \xi e_\zeta + y)| |D_j M_i^\nu(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x))| dy.$$

Taking into account $\xi e_\zeta + G_\sigma \subset G$, by inequality (1.8) for $U = G$, $D^{l^0} f = \phi$, $\Omega^{(\nu)} = \psi$, and $p^i = p^0$ one has

$$\|R(\cdot, \zeta)\|_{p-\varepsilon, G_\delta} \leq C_1(\varepsilon) \|\zeta\| \|D^{l^0} f\|_{p^0, \mathcal{X}, a; G}. \quad (2.12)$$

By (1.9) for $U = G$, $t = T$, $D^{l^i} f = \phi$, $M_i^\nu = \Psi$, $\eta = |\zeta|^{\frac{1}{\lambda_0}}$, we have

$$\|R_{1,i}(\cdot, \zeta)\|_{p-\varepsilon, G_\sigma} \leq C_2(\varepsilon) |\zeta|^{\frac{m^i}{\lambda_0}} \|D^{l^i} f\|_{p^i, \mathcal{X}, a; G}. \quad (2.13)$$

Also from (1.10) for $U = G$, $\eta = |\zeta|^{\frac{1}{\lambda_0}}$, $D^{l^i} f = \varphi$, $M^{(\nu)} = \psi$, we have

$$\|R_{2,i}(\cdot, \zeta)\|_{p-\varepsilon, G_\sigma} \leq C_2(\varepsilon) |\zeta|^\sigma \|D^{l^i} f\|_{p^i, \mathcal{X}, a; G}. \quad (2.14)$$

It follows from (2.11) - (2.14) that

$$\|\Delta(\zeta, G) D^\nu f\|_{p-\varepsilon, G_\sigma} \leq C_3(\varepsilon) \|f\|_{\bigcap_{i=0}^n L_{p^i, \mathcal{X}, a}^{<i>(G)}} |\zeta|^\sigma.$$

Estimating $\|D^\nu f\|_{p-\varepsilon, G}$ by means of (2.1), in this case we get estimation (2.8). This complete the proof.

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