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On one nonlocal inverse boundary problem for the second-order elliptic equation

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Abstract. *An inverse boundary value problem for a second-order elliptic equation with periodic and integral condition is investigated. The problem is considered in a rectangular domain. To investigate the solvability of the inverse problem, we perform a conversion from the original problem to some auxiliary inverse problem with trivial boundary conditions. By the contraction mapping principle we prove the existence and uniqueness of solutions of the auxiliary problem. Then we make a conversion to the stated problem again and, as a result, we obtain the solvability of the inverse problem.*

Keywords. inverse boundary value problem · elliptic equation · Fourier method · classical solution

Mathematics Subject Classification (2010): 34C10, 34L15, 35G31, 35P10, 35R30, 47H10

1 Introduction

Determination of differential equations according to the supplementary information about their solutions are called inverse problems for differential equations. Inverse problems arise in different human activities areas such as seismology, mineral exploration, biology, medicine, quality control of industrial products etc., so that it makes them one of the most important problems of modern mathematics. Different inverse problems for special types of partial differential equations have been studied in many works. Let's note here, first of all, Tikhonov [10], Lavrent'ev [7, 8], Ivanov [2] and their students works. More information about this can be found in the book by Denisov [1].

We consider the equation

$$u_{tt}(x,t) + u_{xx}(x,t) = p(t)u(x,t) + f(x,t) \tag{1.1}$$

and state it an inverse boundary value problem in the domain
$D_t = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$. The inverse problem has nonlocal conditions

$$u(x,0) = \varphi(x),$$

$$u_t(x,T) = \psi(x) + \int_0^T N(t)u(x,t)dt,\ 0 \le x \le 1, \tag{1.2}$$

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Neumann boundary condition

$$u_x(1,t) = 0,\ 0 \le t \le T, \tag{1.3}$$

non- classic boundary condition

$$u_{xx}(0,t) - bu_x(0,t) + au(0,t) = 0,\ 0 \le t \le T, \tag{1.4}$$

and the additional condition

$$u(x_0,t) = h(t),\ 0 \le t \le T, \tag{1.5}$$

where $a,\ b$ are positive constants, $x_0 \in (0,1)$ is fixed number, $f(x,t), \varphi(x), \psi(x), N(t), h(t)$ are given functions, $u(x,t)$ and $p(t)$ are the unknown functions.

Definition 1.1 *By the classical solution of the inverse boundary value problem* (1.1)-(1.5) *we mean a pair of functions* $\{u(x,t), p(t)\}$ *such that* $u(x,t) \in C^2(D_T)$*,* $p(t) \in C\,[0,T]$ *and relations* (1.1)-(1.5) *are satisfied in the usual sense.*

The aim of the prescuted work is to prove the existence and uniqueness of the solution of stated inverse boundary problem (1.1)-(1.5) .

2 Preliminary

For the study of (1.1)-(1.5) firstly we reduce the considered problem to the equivalent problem:

$$y''(t) = p(t)y(t),\ 0 \le t \le T, \tag{2.1}$$

$$y(0) = 0,\ y'(T) = \int_0^T N(t)y(t)dt. \tag{2.2}$$

The following lemma is true.

Lemma 2.1 *Suppose that* $p(t) \in C\,[0,T]\ , N(t) \in C\,[0,T]\,,\ \|p(t)\|_{C[0,T]} \le R = \text{const}$. *Moreover,*

$$T^2 \left(\|N(t)\|_{C[0,\,T]} + \frac{1}{2}R \right) < 1\,. \tag{2.3}$$

Then the problem (2.1)*,* (2.2) *has a unique trivial solution.*

Proof. It is easy to see that boundary value problem (2.1), (2.2) is equivalent to the integral equation

$$y(t) = \int_0^T (tN(\tau) + p\,(\tau)G(t,\tau))y(\tau)d\tau\,, \tag{2.4}$$

where

$$G(t,\tau) = \begin{cases} -t, t \in [0,\tau]\,, \\ -\tau, t \in [\tau, T]\,. \end{cases}$$

We introduce the notation

$$Ay(t) = \int_0^T (tN(\tau) + p(\tau)G(t,\tau))y(\tau)d\tau, \tag{2.5}$$

and write integral equation (2.4) as follows

$$y(t) = Ay(t)\,. \tag{2.6}$$

We well investigate the Eq. (2.6) in the space $C\,[0,T]$. Obviously, the operator A is continuous in the space $C\,[0,T]$.

We show that the operator A is contracting in the space $C\,[0,T]$. Indeed, for arbitrary $y(t), \overline{y}(t) \in C\,[0,T]$ we have

$$\left\| Ay(t) - A\overline{y(t)} \right\|_{C[0,T]}$$

$$\leq \left(T^2 \left\| N(\tau) \right\|_{C[0,T]} + \frac{T^2}{2} \left\| p(t) \right\|_{C[0,T]} \right) \left\| y(t) - \overline{y}(t) \right\|_{C[0,T]} . \tag{2.7}$$

by (2.3) it follows from (2.7) that operator A is contracting in the space $C\,[0,T]$. Therefore, in the space $C\,[0,T]$ this operator has a unique fixed point $y(t)$, which is the solution of the equation (2.6). Thus, the integral equation (2.4) has unique solution in $C\,[0,T]$. Since boundary value problem (2.1), (2.2) also has unique solution in $C\,[0,T]$ and $y(t) \equiv 0$ is the solution of (2.1), (2.2), then the boundary value problem (2.1), (2.2) has a unique trivial solution. The proof of this lemma is complete.

Besides with inverse boundary value problem (1.1)-(1.5) we consider the following auxiliary inverse boundary value problem. It is required to determine the pair $\{u(x,t), p\,(t)\}$ of functions $u(x,t) \in C(D_T)$ and $p\,(t) \in C\,[0,T]$ from the relations (1.1)-(1.4) and

$$h''(t) + u(x_0,t) = p\,(t)h(t) + f(x_0,t)\,, 0 \leq t \leq T). \tag{2.8}$$

The following theorem holds.

Theorem 2.1 *Assume that* $f(x,t) \in C(D_T), \varphi(x), \psi(x) \in C\,[0,T]\,,\ N(t) \in C\,[0,T]$, $h(t) \in C^2\,[0,T]\,, h(t) \neq 0$, $0 \leq t \leq T$*, and the compatibility conditions*

$$\varphi(x_0) = h(0),$$

$$\psi(x_0) = h'(T) - \int_0^T N(t)h(t)dt \tag{2.9}$$

hold. Then the following assertions are valid:

1) each classical solution $\{u(x,t), p\,(t)\}$ *of problem* (1.1)-(1.5) *is also the solution of problem* (1.1)-(1.4)*,* (2.8)*;*

2) each solution $\{u(x,t), p(t)\}$ *of problem* (1.1)-(1.4)*,* (2.8) *satisfying*

$$T^2 \left(\left\| N(t) \right\|_{C[0,T]} + \frac{1}{2} \left\| p(t) \right\|_{C[0,T]} \right) < 1 \tag{2.10}$$

is a classic solution of (1.1)-(1.5)*.*

Proof. Let $\{u(x,t), p(t)\}$ be a classical solution of (1.1)-(1.5). Taking into consideration $h(t) \in C\,[0,T]$ and twice differentiating (1.5), we find:

$$u_t(x_0,t) = h'(t),\ u_{tt}(x_0,t) = h''(t),\ 0 \leq t \leq T. \tag{2.11}$$

Setting $x = x_0$ in the equation (1.1) we have

$$u_{tt}(x_0,t) + u_{xx}(x_0,t) = p(t)u(x_0,t) + f(x_0,t),\ 0 \leq t \leq T. \tag{2.12}$$

Hence by (1.5) and (2.11) we conclude that (2.8) is valid.

Now, assume that $\{u(x,t), p(t)\}$ is a solution of (1.1)-(1.4), (2.8), and the compatibility conditions (2.9) are satisfied. Then from (1.1) and (2.12) we get

$$\frac{d}{dt^2}(u(x_0,t) - h(t)) = a(t)(u(x_0,t) - h(t)),\ 0 \leq t \leq T. \tag{2.13}$$

Further, by (1.2) and the compatibility conditions (2.9) we have

$$\begin{aligned} &u(x_0,0)-h(0)=\varphi(x_0)-h(0)=0,\\ &u(x_0,T)-h'(T)-\int_0^T N(t)(u(x_0,t)-h(t))dt\\ &=u(x_0,T)-\int_0^T N(t)u(x_0,t)dt-(h'(T)\\ &-N(t)h(t)dt)=\psi(x_0)-\left(h'(T)-\int_0^T N(t)h(t)\right)dt=0. \end{aligned} \tag{2.14}$$

By virtue of Lemma 2.1 from (2.13) and (2.14), we conclude that the condition (1.5) is satisfied. Theorem 2.1 is proved.

Now we consider the following spectral problem

$$y''(x)+\lambda y(x)=0,\ 0\le x\le 1, \tag{2.15}$$

$$y'(1)=0,\ (a-\lambda)y(0)=by'(0), \tag{2.16}$$

with positive coefficients a and b. Eigenfunctions of this problem has the form

$$y_k(x)=\sqrt{2}\cos(\sqrt{\lambda_k}(1-x)),\ k=0,1,2,\ \dots,$$

with positive eigenvalues λ_k from characteristic equation

$$\mathrm{tg}\sqrt{\lambda}=\frac{a-\lambda}{b\sqrt{\lambda}}$$

(see [3]-[5]). We assign zero index to any pre-selected eigenfunction, and number all the reminded by increasing order of eigenvalues.

Lemma 2.2 (see [5]) *Starting from some number N, the following estimate holds:*

$$0\le\sqrt{\lambda_k}-\frac{\pi}{2}-\pi(n-1)<\frac{b}{\frac{\pi}{4}+\pi(k-1)}\,.$$

We compare the system $\{y_k(x)\}_{k=1}^{\infty}$ without the function $y_0(x)$ to a known system $\{v_k(x)\}_{k=1}^{\infty}$, $v_k(x)=\sqrt{2}\cos\sqrt{\mu_k}(1-x)$, where $\sqrt{\mu_k}=\frac{\pi}{2}+\pi(k-1)$, $k=1,\,2,\,\dots$, which is orthonormal basis in $L_2(0,1)$.

By the analogously way [7]-[8], we have

$$\sum_{k=N}^{\infty}\|y_k(x)-v_k(x)\|^2{}_{L_2(0,1)}<b^2\sum_{k=N}^{\infty}\frac{2}{3\left(\frac{\pi}{4}+\pi(k-1)\right)^2}\,, \tag{2.17}$$

which we get convergence of the series from the left hand side of this inequality.

Suppose that

$$\eta_k(x)=\sqrt{2}\sin(\sqrt{\lambda_k}(1-x)),\ \xi_k(x)=\sqrt{2}\sin\left(\sqrt{\mu_k}(1-x)\right).$$

Then the inequalities

$$\sum_{k=N}^{\infty}\|\eta_k(x)-\xi_k(x)\|^2{}_{L_2(0,1)}<b^2\frac{2}{3\left(\frac{\pi}{4}+\pi(k-1)\right)^2} \tag{2.18}$$

hold.

It is known [6] that the elements of the system $\{z_k(x)\}_{k=1}^{\infty}$ adjoint to the system $\{y_k(x)\}_{k=1}^{\infty}$ are defined by the equality

$$z_k(x)=\frac{\sqrt{2}}{\alpha_k}\left(\cos\left(\sqrt{\lambda_k}(1-x)\right)-\frac{\cos\left(\sqrt{\lambda_k}\right)\cos(\sqrt{\lambda_0}(1-x))}{\cos(\sqrt{\lambda_0})}\right), \tag{2.19}$$

where

$$\alpha_k=1+\frac{\cos^2(\sqrt{\lambda_k})}{b}+\frac{a\cos^2(\sqrt{\lambda_k})}{b\lambda_k}\,.$$

Here it is also known that $\{y_k(x)\}_{k=1}^{\infty}$ form a Riesz basis in the space $L_2(0,1)$.

Suppose that $g(x)\in L_2(0,1)$. Then by (2.17) we get

$$\left(\sum_{k=1}^{\infty}\left(\int_0^1 g(x)y_k(x)dx\right)\right)^{\frac{1}{2}}\le M\left\|g(x)\right\|_{L_2(0,1)} \tag{2.20}$$

where

$$M=\left(\frac{N(1+N)}{2}+b\sum_{k=N}^{\infty}\frac{2}{3\left(\frac{\pi}{4}+\pi(k-1)\right)^2}+2\right)^{\frac{1}{2}}.$$

Analogously (2.20), by (2.18), we find

$$\left(\sum_{k=1}^{\infty}\left(\int\limits_0^1 g(x)\eta_k(x)dx\right)^2\right)^{\frac{1}{2}}\le M\cdot\left\|g(x)\right\|_{L_2(0,1)}$$

Since, the system $\{y_k(x)\}_{k=1}^{\infty}$ is a Riesz basis in $L_2(0,1)$, then for any function $g(x)\in L_2(0,1)$ we have (see [9])

$$g(x)=\sum_{k-1}^{\infty}g_k y_k(x), \tag{2.21}$$

where

$$g_k=\int_0^1 g(x)z_k(x),\ k=1,\,2,\,\dots\ .$$

It's not difficult to see that

$$|g_k|\le\sqrt{2}\left(\int_0^1 g(x)y_k(x)dx+\frac{1}{\left|\cos\sqrt{\lambda_0}\right|}\cdot\frac{b\sqrt{\lambda_k}}{|a-\lambda_k|}\cdot\int_0^1|g(x)|\,dx\right).$$

From here, by virtue of (2.20), we get

$$\left(\sum_{k=1}^{\infty}g_k^2\right)^{\frac{1}{2}}\le M_0\left\|g(x)\right\|_{L_2(0,1)}, \tag{2.22}$$

where

$$M_0=2\left[M+\frac{b}{\cos\sqrt{\lambda_0}}\cdot\sup_k\left(\frac{\lambda}{|a-\lambda_k|}\right)\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}}\right].$$

Assume that $g(x) \in C\,[0,1]\,, g'(x) \in L_2(0,1)$, and

$$J(g) \equiv g(x) + \frac{b}{\cos\sqrt{\lambda_0}} \int\limits_0^1 g(x) \cos\sqrt{\lambda_0}(1-x)dx \equiv 0.$$

Then we have

$$g_k = \frac{\sqrt{2}}{\alpha_k} \int\limits_0^1 g(x) \left(\cos\left(\sqrt{\lambda_k}(1-x)\right) - \frac{\cos\sqrt{\lambda_k}}{\cos\sqrt{\lambda_0}} \cos\left(\sqrt{\lambda_0}(1-x)\right)\right) dx$$

$$= \frac{\sqrt{2}}{\alpha_k} \cdot \frac{a}{b\lambda_k} g(0) \cos\sqrt{\lambda_k} + \frac{\sqrt{2}}{\alpha_k} \cdot \frac{1}{\sqrt{\lambda_k}} \int_0^1 g'(x) \sin(\sqrt{\lambda_k}(1-x))dx \qquad (2.23)$$

Hence, by (2.21), we have

$$\left(\sum_{k=1}^{\infty} (\lambda_k g_k)^2\right)^{\frac{1}{2}} \le m_0\,|g(0)| + 2M \left\|g'(x)\right\|_{L_2(0,1)}, \qquad (2.24)$$

where

$$m_0 = \frac{2a}{b} \left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k}\right)^{\frac{1}{2}}.$$

Assume that $g(x) \in C^1\,[0,1]$, $g'''(x) \in L_2(0,1)$, $J(g) = 0$, and $g'(1) = 0$. Then from (2.23) we get

$$g_k = \frac{\sqrt{2}}{\alpha_k} \cdot \frac{1}{a-\lambda_k} \cdot \frac{1}{\sqrt{\lambda_k}} (ag(0) - bg'(0) \sin\sqrt{\lambda_k} + \frac{\sqrt{2}}{\alpha_k} \cdot \frac{1}{\lambda_k\sqrt{\lambda_k}}$$

$$\times \int_0^1 g''(x) d \sin\left(\sqrt{\lambda_k}(1-x)\right) = -\frac{\sqrt{2}}{\alpha_k} \cdot \frac{1}{a-\lambda_k} \cdot \frac{a}{\lambda_k\sqrt{\lambda_k}} g''(0)$$

$$-\frac{\sqrt{2}}{\alpha_k} \cdot \frac{1}{\lambda_k\sqrt{\lambda_k}} \int_0^1 g'''(x) \sin\left(\sqrt{\lambda_k}(1-x)\right) dx. \qquad (2.25)$$

From this we get

$$\left(\sum_{k=1}^{\infty} \left(\lambda_k\sqrt{\lambda_k}\,|f_k|\right)^2\right)^{\frac{1}{2}} \le m_1 a \left|g''(0)\right| + 2M \left\|g'''(x)\right\|_{L_2(0,1)}, \qquad (2.26)$$

where

$$m_1 = 2 \sup_k \frac{\lambda_k}{|a-\lambda_k|} \left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k}\right)^{\frac{1}{2}}.$$

New suppose that $g(x) \in C^2\,[0,1]$, $g'''(x) \in L_2(0,1)$, $J(g) = 0$, $g'(1) = 0$, and $g''(0) - bg(0) + ag(0) = 0$. Then it follows from (2.24) that

$$g_k = -\frac{\sqrt{2}}{\alpha_k} \cdot \frac{1}{a-\lambda_k} \cdot \frac{a}{\lambda_k\sqrt{\lambda_k}} g''(0) - \frac{\sqrt{2}}{\alpha_k} \cdot \frac{1}{\lambda_k\sqrt{\lambda_k}} \int_0^1 g'''(x) \sin(\sqrt{\lambda_k}(1-x))dx. \qquad (2.27)$$

Consequently, we have

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,|g_k|\right)^2\right)^{\frac{1}{2}} \le m_1 a\,|g''(0)| + 2M\,\|g'''(x)\|_{L_2(0,1)}\,. \qquad (2.28)$$

Denote by $B_{2,T}^{\frac{3}{2}}$ the set of all the functions $u(x,t)$ of the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)y_k(x),$$

considered D_T, where each function $u_k(t)$ is continuous on $[0,T]$ and

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} < +\infty.$$

We define the norm on this set as follows:

$$\|u(x,t)\|_{B_{2,T}^{\frac{3}{2}}} = \left\{\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}.$$

Denote by $E_T^{\frac{3}{2}}$ the space that consisting of the topological product

$$B_{2,T}^{\frac{3}{2}} \times C[0,T]\,.$$

The norm of the element $z=\{u,p\}\in B_{2,T}^{\frac{3}{2}}\times C[0,T]$ is defined by the formula

$$\|z\|_{E_T^{\frac{3}{2}}} = \|u(x,t)\|_{B_{2,T}^{\frac{3}{2}},} + \|p(t)\|_{C[0,T]}\,.$$

It is known that $B_{2,T}^{\frac{3}{2}}$ and $E_T^{\frac{3}{2}}$ are Banach spaces (see [6]).

3 The existence and uniqueness of the solution of the inverse boundary problem (1.1)-(1.5)

We will seek for the first component $u(x,t)$ of $\{u(x,t),p(t)\}$ in the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)y_k(x), \qquad (3.1)$$

where

$$u_k(t)=\int_0^1 u(x,t)z_k(x)dx,\ k=1,\,2,\,\ldots,$$

and

$$y_k(x)=\sqrt{2}\cos\left(\sqrt{\lambda_k}(1-x)\right),$$

By applying method of separation of variables, from (1.1), (1.2) we get

$$u_k''(t)-\lambda_k u_k(t)=F_k(t,u,p),\ k=1,\,2,\,\ldots,\ 0\le t\le T, \qquad (3.2)$$

$$u_k(0)=\varphi_k, \tag{3.3}$$

$$u_k'(T)=\psi_k+N(t)u_k(t)dt,\ k=1,\,2,\,\ldots\ . \tag{3.4}$$

where

$$F_k(t,u,p)=f_k(t)+p(t)u_k(t), f_k(t)=\int\limits_0^1 f(x,t)y_k(x)dx,$$

$$\varphi_k=\int\limits_0^1\varphi(x)z_k(x)dx, \psi_k=\int_0^1\psi(x)z_k(x)dx,\ k=1,\,2,\,\ldots\ .$$

Solving problem (3.3), (3.4) we obtain

$$u_k(t)=\frac{ch(\lambda_k(T-t)}{ch(\lambda_kT)}\cdot\varphi_k+\frac{sh(\lambda_k(T-t)}{\lambda_k ch(\lambda_kT)}\left(\psi_k+N(t)u_k(t)dt\right)$$

$$+\frac{1}{\lambda_k}\int_0^T G_k(t,\tau)F_k(\tau,u,p)d\tau, k=1,\,2,\,\ldots\ , \tag{3.5}$$

$$G_k(t,\tau)=\begin{cases}\frac{sh(\sqrt{\lambda_k}(T-(t+\tau)))-sh(\sqrt{\lambda_k}(T+t-\tau))}{2ch(\sqrt{\lambda_k}T)},\ t\in[0,\tau],\\ \frac{sh(\sqrt{\lambda_k}(T-(t+\tau)))-sh(\sqrt{\lambda_k}(T-(t-\tau)))}{2ch(\sqrt{\lambda_k}T)},\ t\in[\tau,T].\end{cases}$$

After substitution the expression (3.5) in (3.1), for definition of the component $u(x,t)$ of problem (1.1)-(1.4), (2.8) we get

$$u(x,t)=\sum_{k=1}^{\infty}\left[\frac{ch(\lambda_k(T-t)}{ch(\lambda_kT)}\varphi_k+\frac{sh(\lambda_k(T-t)}{\lambda_k ch(\lambda_kT)}\left(\,\psi_k+N(t)u_k(t)dt\right)\right.$$

$$\left.+\frac{1}{\lambda_k}\int_0^T G_k(t,\tau)F_k(\tau,u,p)d\tau\right]y_k(x)\,. \tag{3.6}$$

Now, from (2.8), by (3.1) we have

$$p(t)=[h(t)]^{-1}\left\{h''(t)-f(x_0,t)-\sqrt{2}\sum_{k=1}^{\infty}\lambda_k u_k(t)\cos\sqrt{\lambda_k}(1-x_0)\right\}. \tag{3.7}$$

In order to get the equation for the second component $p(t)$ of solution $\{u(x,t),p(t)\}$ of problem (1.1)-(1.4), (2.8) we substitute the expression (3.5) in (3.7):

$$p(t)=[h(t)]^{-1}\left\{h''(t)-f(x_0,t)-\sqrt{2}\sum_{k=1}^{\infty}\lambda_k\ \left[\frac{ch(\lambda_k(T-t)}{ch(\lambda_kT)}\varphi_k\right.\right.$$

$$+\frac{sh(\lambda_k(T-t)}{\lambda_k ch(\lambda_kT)}\left(\,\psi_k+N(t)u_k(t)dt\right)$$

$$\left.\left.+\ \frac{1}{\lambda_k}\int_0^T G_k(t,\tau)F_k(\tau,u,p)d\tau\right]\ \cos\sqrt{\lambda_k}(1-x_0)\right\}. \tag{3.8}$$

Thus, the solution of problem (1.1)-(1.4), (2.8) was reduced to the solution of the system (3.6), (3.8) respectively to unknown function $u(x,t)$ and $p(t)$.

Now we consider the operator $\Phi(u,p) = \{\Phi_1(u,p), \Phi_2(u,p)\}$ in the space $E_T^{3/2}$, where

$$\Phi_1(u,p) = \tilde{u}(x,v) \equiv \sum_{k=1}^{\infty} \tilde{u}_k(v) \cdot y_k(x)\,, \Phi_2(u,p) = \tilde{p}(v),$$

where

$$\tilde{u}_k(v),\ k = 1,\ 2,\ \ldots\ ,\ \text{and}\ \tilde{p}(v)$$

are equal to the right-hands sides parts of (3.5) and (3.8) respectively.

It is obvious that

$$\frac{ch(\lambda_k(T-t))}{ch(\lambda_k T)} < 1, \frac{sh(\lambda_k T)}{ch(\lambda_k T)} < 1, \frac{sh(\lambda_k(T+t-\tau)}{ch(\lambda_k T)} < 1,\ t \in [0,\tau]\,,$$

$$\frac{sh(\lambda_k(T-(t+\tau)}{ch(\lambda_k T)} < 1, \frac{sh(\lambda_k(T-(t-\tau))}{ch(\lambda_k T)} < 1,\ t \in [\tau, T]\,.$$

Using these relations, and simple transformations we find

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|\tilde{u}_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le \sqrt{6}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,|\varphi_k|\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{6}\left(\sum_{k=1}^{\infty}\left(\lambda_k\,|\psi_k|^2\right)^{\frac{1}{2}} + \sqrt{6}T\,\|N(T)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right.$$

$$+\sqrt{6T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\,|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} + \sqrt{6}T\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$\sqrt{6T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\,|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} + \sqrt{6}T\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \tag{3.9}$$

$$\|\tilde{p}(t)\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t) - f(x_0,t)\right\|_{C[0,T]}\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,|\varphi_k|\right)^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\left(\lambda_k\,|\psi_k|^2\right)^{\frac{1}{2}} + T\,\|N(t)\|_{C[0,T]}\right.$$

$$\times\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + \sqrt{T}\left\{\left(\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\,|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right.\right.$$

$$\left.\left.\left.\left.+T\,\|p(t)\|_{C[0,T]}\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right\}\right]\right\}. \tag{3.10}$$

Suppose that the data of (1.1)-(1.4), (2.8) satisfy the following conditions:

1. $\varphi(x) \in C^2\,[0,1]\,, \varphi'''(x) \in L_2(0,1),\ J(\varphi) = 0,\ \varphi'(1) = 0,\ \varphi''(0) - b\varphi'(0) + a\varphi(0) = 0;$
2. $\psi(x) \in C^1\,[0,1]\,, \varphi''(x) \in L_2(0,1),\ J(\psi) = 0,\ \psi^1 = 0;$

3. $f(x,t),\ f_x(x,t),\ f''_{xx}(x,t) \in L_2(D_T),\ J(f)=0,\ f_x(1,t)=0,\ 0\le t\le T;$
4. $a>0,\ b>0,\ N(t)\in C\,[0,T]\,,\ h(t)\in C^2\,[0,T]\,,\ h(t)\neq 0,\ 0\le t\le T.$
Then, by (2.25) and (2.27), from (3.9) and (3.10) we find

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|\tilde{u}_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$\le\sqrt{6}\left[m_1 a\left|\varphi''(0)\right|+2M\left\|\varphi'''(x)\right\|_{L_2(0,T)}\right]$$

$$+\sqrt{6}\left(m_1(a\left|\psi(0)\right|+b\left|\psi'(0)\right|)+\sqrt{b}M\left\|\psi''(x)\right\|_{L_2(0,\,1)}\right)$$

$$+\sqrt{b}T\,\|N(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{6T}\left[m_1(a\,\|f(0,t)\|_{C[0,T]}+b\,\|f_x(0,t)\|_{C[0,T]})+2M\,\|f_{xx}(x,t\|_{L_2(D_T)})\right]$$

$$+\sqrt{6}T\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{3.11}$$

$$\|\tilde{p}(t)\|_{C[0,T]}\le\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(x_0,t)\right\|_{C[0,T]}\right.$$

$$+\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}}\left[m_1 a\left|\varphi''(0)\right|+2M\left\|\varphi'''(x)\right\|_{L_2(0,T)}+m_1\left(a\left|\psi(0)\right|+b\left|\psi'(0)\right|\right)\right.$$

$$+\sqrt{b}M\left\|\psi''(x)\right\|_{L_2(0,\,1))}+T\,\|N(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{T}\left[m_1\left(a\,\|f(0,t)\|_{C[0,T]}+b\,\|f_x(0,t)\|_{C[0,T]}\right)+2M\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right]$$

$$\left.\left.+T\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}. \tag{3.12}$$

We introduce the following notations

$$A_1(T)=\sqrt{6}\left(m_1 a\left|\varphi''(0)\right|+2M\left\|\varphi'''(x)\right\|_{L_2(0,1)}\right)$$

$$+\sqrt{6}\left(m_1(a\left|\psi(0)\right|+b\left|\psi'(0)\right|)+\sqrt{b}M\left\|\psi''(x)\right\|_{L_2(0,1)}\right)$$

$$+\sqrt{bT}\left(m_1\left(a\,\|f(0,t)\|_{C[0,T]}+b\,\|f_x(0,t)\|_{C[0,T]}\right)+\sqrt{b}M\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right),$$

$$B_1(T)=\sqrt{6}T, D_1(T)=\sqrt{6}T\,\|N(t)\|_{C[0,T]}\,,$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(x_0,t)\right\|_{C[0,T]}\right.$$

$$+\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}}\left[m_1 a\left|\varphi''(0)\right|+2M\left\|\varphi'''(x)\right\|_{L_2(0,1)}+m_1\left(a\left|\psi(0)\right|\right.\right.$$

$$+ b\left|\psi'(0)\right|\Big) + 2M\left\|\psi''(x)\right\|_{L_2(0,1)} + \sqrt{T}\Big[m_1\left(a\left\|f(0,t)\right\|_{C[0,T]}\right.$$

$$\left.+ b\left\|f_x(0,t)\right\|_{C[0,T]}\right) + \ 2M\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}\Big)\Big)\Big\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}} T,$$

$$D_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}} T\left\|N(t)\right\|_{C[0,T]},$$

and write the estimations (3.11) and (3.12) as follows

$$\left\|\tilde{u}(x,t)\right\|_{B^{3/2}_{2,T}} \le A_1(T)$$

$$+B_1(T)\left\|p(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^{3/2}_{2,T}} + D_1(T)\left\|u(x,t)\right\|_{B^{3/2}_{2,T}}, \tag{3.13}$$

$$\left\|\tilde{p}(t)\right\|_{C[0,T]} \le A_2(T)+B_2(T)\left\|p(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^{3/2}_{2,T}}+D_2(T)\left\|u(x,t)\right\|_{B^{3/2}_{2,T}}. \tag{3.14}$$

From the inequalities (3.13) and (3.14) we conclude:

$$\left\|u(x,t)\right\|_{B^{\frac{3}{2}}_{2,T}} + \left\|\tilde{p}(t)\right\|_{C[0,T]} \le A(t)$$

$$+B(T)\left\|p(t)\right\|_{C[0,T]}\cdot\left\|u(x,t)\right\|_{B^{3/2}_{2,T}} + D(T)\left\|u(x,t)\right\|_{B^{3/2}_{2,T}}, \tag{3.15}$$

(3.15) where

$$A(T) = A_1(T) + A_2(T),\ B(T) = B_1(T) + B_2(T), D(T) = D_1(T) + D_2(T).$$

The following theorem can be proved.

Theorem 3.1 *If conditions* (1.1)-(1.4) *and the condition*

$$(B(T)(A(T)+2) + D(T))(A(T)+2) < 1 \tag{3.16}$$

hold, then problem (1.1)-(1.4)*,* (2.8) *has a unique solution in the ball* $K = K_R\left(\|z\|_{E^{3/2}_T}\right)$.

Proof. In the space $E^{3/2}_T$, we consider the equation

$$z = \Phi z \tag{3.17}$$

where $z = \{u,p\}$, the components $\Phi_i(u,p)$, $i = 1,\ 2$, of operator $\Phi(u,p)$ defined by the right sides of equations (3.6), (3.8).

Consider the operator Φ in $K = K_R$ of the space $E^{3/2}_T$. Similarly to (3.15) we get that for any $z, z_1, z_2 \in K_R$ the following inequalities hold:

$$\|z\|_{E^{3/2}_T} \le A(T) + B(T)\left\|p(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^{3/2}_{2,T}} + D(T)\left\|u(x,t)\right\|_{B^{3/2}_{2,T}}$$

$$\le A(T) + B(T)(A(T)+2)^2 + D(T)(A(T)+2)$$

$$= A(T) + (B(T)(A(T)+2) + D(T))(A(T)+2), \tag{3.18}$$

$$\left\|\Phi z_1 - \Phi z_2\right\|_{E^{3/2}_T} \le B(T)R(\left\|P_1(t) - P_2(t)\right\|_{C[0,T]} + \left\|u_1(x,t) - u_2(x,t)\right\|_{B^{3/2}_{2.T}}$$

$$+D(T)\|u_1(x,t)-u_2(x,t)\|_{B^{3/2}_{2.T}}. \tag{3.19}$$

Then by (3.16) it follows from (3.18) and (3.19) that the operator Φ acts in the sphere $K = K_R$ and is contracting. Therefore in the sphere $K = K_R$ this operator has only unique fixed point $\{u, p\}$, which is the solution of the equation (3.17), i.e. $\{u, p\}$, is the unique solution of the system (3.6), (3.8) in the sphere $K = K_R$.

Then the function $u(x,t)$, as an element of $B^{3/2}_{2.T}$, is continuous and has continuous derivatives $u_x(x,t)$and $u_{xx}(x,t)$ in D_T.

Further, from (3.5) it is obvious that $u''_k(t) \in C[0,T]$ and

$$\left(\sum_{K=1}^{\infty}\left(\sqrt{\lambda_k}\left\|u''_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$\leq \sqrt{3}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + \sqrt{3}\left(\sum_{k=1}^{\infty}\left(\sqrt{\lambda_k}\left\|f_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{3}\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},$$

or under consideration (2.24) we have

$$\left(\sum_{K=1}^{\infty}\left(\sqrt{\lambda_k}\left\|u''_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$\leq \sqrt{3}\left(m_0\left\|f(0,t)\right\|_{C[0,T]} + 2M\left\|\left\|f_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}\right)$$

$$+\sqrt{3}\left(1+\left\|p(t)\right\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\left\|u''_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}.$$

This implies that $u_{tt}(x,t)$ is continuous in D_T. It is easy to verify that equation (1.1) and conditions (1.2), (1.3), (1.4) and (2.8) are satisfied in the usual sense. Consequently, $\{u(x,t), p(t)\}$ is the solution of (1.1)-(1.4), (2.8). The proof of this theorem is complete.

With the aid of Theorem 3.1 the following theorem was proved.

Theorem 3.2 *Suppose that all conditions of Theorem 3.1, the condition*

$$T^2\left(\left\|N(t)\right\|_{C[0,T]} + \frac{1}{2}(A(T)+2)\right) < 1,$$

and the compatibility condition (2.9) *hold. If*

$$\varphi(x_0) = h(0),$$

$$\varphi(x_0) = h'(T) - N(t)h(t)dt,$$

then problem (1.1)-(1.5) *has a unique classic solution in* $K = K_R \subset E^{3/2}_T$.

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