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Multi-sublinear operators generated by multilinear Calderón-Zygmund operators on product generalized Morrey spaces

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Abstract. *In this paper, we established the boundedness for a large class of multi-sublinear operators T_m generated by multilinear Calderón-Zygmund operators on product generalized Morrey spaces $\mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n) \times \ldots \times \mathcal{M}_{p_m,\varphi_m}(\mathbb{R}^n)$. We find the sufficient conditions on $(\varphi_1, \ldots, \varphi_m, \varphi)$ which ensures the boundedness of the operators T_m from $\mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n) \times \ldots \times \mathcal{M}_{p_m,\varphi_m}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ for $1/p = 1/p_1 + \ldots + 1/p_m$. The multi-sublinear operators under consideration contain integral operators of harmonic analysis such as multi-sublinear maximal operator $\mathcal{M}_m$, multilinear Calderón-Zygmund operators $\mathcal{T}_m$, etc.*

Keywords. Multi-sublinear maximal operator; multilinear Calderón-Zygmund operator; product generalized Morrey space.

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1 Introduction

Multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The initial work on the class of multilinear Calderón-Zygmund operators was done by Coifman and Meyer in [2] and was later systematically studied by Grafakos and Torres in [9, 10].

The classical Morrey spaces, introduced by Morrey [23] in 1938, have been studied intensively by various authors and together with Lebesgue spaces play an important role in the theory of partial differential equations. Although such spaces allow to describe local properties of functions better than Lebesgue spaces, they have some unpleasant issues. It is well known that Morrey spaces are non separable and that the usual classes of nice functions are not dense in such spaces. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [11, 21, 24] introduced generalized Morrey spaces $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ (see, also [12, 13, 17, 26]). In [13] is defined the generalized Morrey spaces $\mathcal{M}_{p,\varphi}$ with normalized norm

$$\|f\|_{\mathcal{M}_{p,\varphi}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-1/p} \, \|f\|_{L_p(B(x,r))},$$

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where the function φ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$. In [11] Guliyev also studied the boundedness of the classical operators in these spaces $\mathcal{M}_{p,\varphi}$, see also [3–8, 18, 19, 25].

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by $^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. We denote by $\overrightarrow{f}$ the m-tuple $(f_1, f_2, \ldots, f_m)$, $\overrightarrow{y} = (y_1, \ldots, y_n)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$.

Let $\overrightarrow{f} \in L^{loc}_{p_1}(\mathbb{R}^n) \times \ldots \times L^{loc}_{p_m}(\mathbb{R}^n)$. The multi-sublinear maximal operator M_m is defined by

$$M_m(\overrightarrow{f})(x) = \sup_{r>0} \prod_{i=1}^{m} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_i(y_i)| dy_i.$$

In [10] Grafakos and Torres studied the multilinear Calderón-Zygmund operator which can be written for $x \notin \cap_{j=1}^{m} \mathrm{supp} f_j$ as

$$K_m(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, dy_1 dy_2 \ldots dy_m,$$

where $K(x, y_1, \ldots, y_m)$ is the kernel function defined of the diagonal $x = y_1 = \ldots y_m =$ in $(\mathbb{R}^n)^{m+1}$ satisfying

$$|K(y_0, y_1, \ldots, y_m)| \le c_1 \Big(\sum_{k,l=0}^{m} |y_k - y_l| \Big)^{-mn},$$

and whenever $2|y_j - y_j'| \le \frac{1}{2} \max\limits_{0 \le k \le m} |y_j - y_k|$,

$$|K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y_j', \ldots, y_m)| \le \frac{c_1 |y_j - y_j'|^{\epsilon}}{\Big(\sum\limits_{k,l=0}^{m} |y_k - y_l| \Big)^{mn+\epsilon}},$$

for some $\epsilon > 0$ and all $0 \le j \le m$. Grafakos and Torres [10] proved that the operator $K_m(\overrightarrow{f})$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ for $p_i > 1 (i = 1, \ldots, m)$ and $1/p = 1/p_1 + \ldots + 1/p_m$, and bounded from $L_1(\mathbb{R}^n) \times \ldots \times L_1(\mathbb{R}^n)$ to $L_{\frac{1}{m},\infty}(\mathbb{R}^n)$.

It is well known that multi-sublinear maximal operator and multilinear Calderón-Zygmund operators play an important role in harmonic analysis (see [1, 10, 22]).

Suppose that T_m represents a multilinear or a multi-sublinear operator, which satisfies that for any $\overrightarrow{f} \in L_1(\mathbb{R}^n) \times \ldots \times L_1(\mathbb{R}^n)$ with compact support and $x \notin \cap_{j=1}^{m} \mathrm{supp} f_j$

$$|T_m(\overrightarrow{f})(x)| \le c_0 \int\limits_{(\mathbb{R}^n)^m} \frac{|f_1(y_1) \ldots f_m(y_m)|}{|(x - y_1, \ldots, x - y_m)|^{mn}} dy_1 dy_2 \ldots dy_m, \tag{1.1}$$

where c_0 is independent of $\overrightarrow{f}$ and x.

The condition (1.1) is satisfied by many interesting operators in harmonic analysis, such as the multilinear Calderón–Zygmund operators, multi-sublinear maximal operator, and so on (see [10, 20] for details).

In this work, we prove the boundedness of the multi-sublinear operator T_m satisfies the condition (1.1) generated by multilinear Calderón-Zygmund operator from product generalized Morrey space $\mathcal{M}_{p_1,\varphi_1} \times \ldots \times \mathcal{M}_{p_m,\varphi_m}$ to $\mathcal{M}_{p,\varphi}$, if $1 < p_1, \ldots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$, and from the space $\mathcal{M}_{p_1,\varphi_1} \times \ldots \times \mathcal{M}_{p_m,\varphi_m}$ to the weak space $W\mathcal{M}_{1,\varphi}$, if $1 \le p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and at least one p_i equals one (Theorem 2.3). Finally, as applications we apply this result to several particular operators such as the multi-sublinear maximal operator and multilinear Calderón-Zygmund operator.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Main Results

In this section, we will discuss the boundedness properties of multi-sublinear operators T_m generated by multilinear Calderón-Zygmund operators on product generalized Morrey spaces $\mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n) \times \ldots \times \mathcal{M}_{p_m,\varphi_m}(\mathbb{R}^n)$.

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.1 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

Lemma 2.1 *[4] Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$.*

(i) If

$$\sup_{t<r<\infty} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} = \infty \quad \text{for some } t>0 \text{ and for all } x \in \mathbb{R}^n, \tag{2.1}$$

then $M_{p,\varphi}(\mathbb{R}^n) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \text{for some } \tau>0 \text{ and for all } x \in \mathbb{R}^n, \tag{2.2}$$

then $M_{p,\varphi}(\mathbb{R}^n) = \Theta$.

Remark 2.1 We denote by Ω_p the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n} \left\| \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} \right\|_{L_\infty(t,\infty)} < \infty, \quad \text{and} \quad \sup_{x\in\mathbb{R}^n} \left\| \varphi(x,r)^{-1} \right\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.1, we always assume that $\varphi \in \Omega_p$.

We will use the following statements on the boundedness of the weighted Hardy operator

$$H_w g(r) := \int_r^\infty g(t)w(t)dt, \ 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.
The following theorem was proved in [14] (see also [16]).

Theorem 2.1 *[14] Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{r>0} v_2(r) H_w g(r) \le C \operatorname*{ess\,sup}_{r>0} v_1(r) g(r) \tag{2.3}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int_r^\infty \frac{w(t)dt}{\sup_{t<s<\infty} v_1(s)} < \infty. \tag{2.4}$$

Moreover, the value $C = B$ is the best constant for (2.3).

Remark 2.2 In (2.3) – (2.4) it is assumed that $0 \cdot \infty = 0$.

In the following lemma we get Guliyev local estimate (see, for example, [11–13] in the case $m = 1$ and [15] in the case $m > 1$) for the operator T_m.

Theorem 2.2 *Let $1 \le p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \dots + 1/p_m$. Let T_m be a multi-sublinear operator which satisfies the condition* (1.1) *bounded from $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ for $p_i > 1$, $i = 1, \dots, m$, and bounded from $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $WL_p(\mathbb{R}^n)$ for $p_i \ge 1$, $i = 1, \dots, m$.*

Then, for $1 < p_1, \dots, p_m < \infty$ the inequality

$$\|T_m(\overrightarrow{f})\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \prod_{i=1}^m \int_{2r}^\infty t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x_0,t))} dt \tag{2.5}$$

holds for any ball $B(x_0,r)$ and for all $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \ldots \times L_{p_m}^{loc}(\mathbb{R}^n)$.

Moreover, if at least one p_i equals one, the inequality

$$\|T_m(\overrightarrow{f})\|_{WL_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \prod_{i=1}^m \int_{2r}^\infty t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x_0,t))} dt \tag{2.6}$$

holds for any ball $B(x_0,r)$ and for all $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \ldots \times L_{p_m}^{loc}(\mathbb{R}^n)$.

Proof. Let $1 < p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \dots + 1/p_m$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0,2r)$. We represent $\overrightarrow{f} = (f_1, \dots, f_m)$ as

$$f_j = f_j^0 + f_j^\infty, \quad f_j^0 = f_j \chi_{2B}, \quad f_j^\infty = f_j \chi_{\complement_{(2B)}}, \quad j = 1, \dots, m. \tag{2.7}$$

Then we split $T_m(\overrightarrow{f})$ as follows

$$\left|T_m(\overrightarrow{f})(x)\right| \le c_0 \left|T_m(f_1^0, \dots, f_m^0)(x)\right| + \Big| \sum_{\beta_1,\dots,\beta_m}{}' T_m(f_1^{\beta_1}, \dots, f_m^{\beta_m})(x) \Big|,$$

where $\beta_1, \ldots, \beta_m \in \{0, \infty\}$ and each term of $\sum'$ contains at least $\beta_i \neq 0$. Then,

$$\|T_m(\overrightarrow{f})\|_{L_p(B(x,r))} \leq \|T_m(f_1^0, \ldots, f_m^0)\|_{L_p(B(x,r))} + \|\sum_{\beta_1,\ldots,\beta_m}' T_m(f_1^{\beta_1}, \ldots, f_m^{\beta_m})\|_{L_p(B(x,r))} \leq I + II.$$

For I, by the boundedness of T_m from product $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ with $1/p = 1/p_1 + \ldots + 1/p_m$ for each $p_i > 1 (i = 1, \ldots, m)$, we have,

$$\|T_m(\overrightarrow{f^0})\|_{L_p(B(x,r))} \leq \|T_m(\overrightarrow{f^0})\|_{L_p(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i^0\|_{L_{p_i}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,2r))}.$$

Taking into account that

$$\|f_i\|_{L_{p_i}(B(x,2r))} \lesssim r^{\frac{n}{p_i}} \int_{2r}^{\infty} t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x,t))} dt, \;\; i = 1, \ldots, m$$

we get

$$\|T_m(\overrightarrow{f^0})\|_{L_p(B(x,r))} \lesssim r^{\frac{n}{p}} \prod_{i=1}^m \int_{2r}^{\infty} t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x,t))} dt. \tag{2.8}$$

For II, first we consider the case $\beta_1 = \cdots = \beta_m = \infty$.

When $|x - y_i| \leq r$, $|z - y_i| \geq 2r$, we have $\frac{1}{2}|z - y_i| \leq |x - y_i| \leq \frac{3}{2}|z - y_i|$, and so by the condition (1.1) we have

$$|T_m(\overrightarrow{f^\infty})(z)| \lesssim \int_{(\complement_{B(x,2r)})^m} \frac{|f_1(y_1) \cdots f_m(y_m)|}{|(x - y_1, \ldots, x - y_m)|^{mn}} d\overrightarrow{y} \lesssim \prod_{i=1}^m \int_{\complement_{B(x,2r)}} \frac{|f_i(y_i)|}{|x - y_i|^n} dy_i$$

and

$$\|T_m(\overrightarrow{f^\infty})\|_{L_p(B(x,r))} \leq \prod_{i=1}^m \int_{\complement_{B(x,2r)}} \frac{|f_i(y_i)|}{|x - y_i|^n} dy_i \; \|\chi_{B(x,r)}\|_{L_p(\mathbb{R}^n)} \lesssim r^{\frac{n}{p}} \prod_{i=1}^m \int_{\complement_{B(x,2r)}} \frac{|f_i(y_i)|}{|x - y_i|^n} dy_i.$$

By Fubini's theorem we have

$$\int_{\complement_{B(x,2r)}} \frac{|f_i(y_i)|}{|x - y_i|^n} dy_i \approx \int_{\complement_{B(x,2r)}} |f_i(y_i)| \int_{|x_0 - y_i|}^{\infty} \frac{dt}{t^{n+1}} dy_i \approx \int_{2r}^{\infty} \int_{2r \leq |x_0 - y_i| < t} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}} \lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}}.$$

Applying Hölder's inequality, we get

$$\int_{^{\complement}B(x,2r)} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \lesssim \int_{2r}^{\infty} t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x,t))} dt. \tag{2.9}$$

Moreover, for all $p_i \in [1,\infty)$, $i=1,\dots,m$ the inequality

$$\|T_m(\overrightarrow{f^{\infty}})\|_{L_p(B(x,r))} \lesssim r^{\frac{n}{p}} \prod_{i=1}^{m} \int_{2r}^{\infty} t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x,t))} dt \tag{2.10}$$

is valid.

Next we consider the case that some $\alpha_i = 0$ and other $\alpha_j = \infty$. To this end we may assume that $\alpha_1 = \alpha_2 = \infty$ and $\alpha_3 = \cdots = \alpha_m = 0$. Recall the condition (1.1) and the fact that $|x-y_i| \approx |z-y_i|$ for $z \in B(x,r)$ and $y_i \in {}^{\complement}B(x,2r)$, we have that

$$\begin{aligned}
&T(f_1^{\infty}, f_2^{\infty}, f_3^{0}, \dots, f_m^{0})(z) \\
&\lesssim \int_{^{\complement}B(x,2r)\times {}^{\complement}B(x,2r)} \frac{|f_1(y_1)||f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{mn}} dy_1 dy_2 \prod_{i=3}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i \\
&\lesssim \int_{^{\complement}B(x,2r)} \frac{|f_1(y_1)|}{|x-y_1|^n} dy_1 \int_{^{\complement}B(x,2r)} \frac{|f_2(y_2)|}{|x-y_2|^n} dy_2 \prod_{i=3}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i.
\end{aligned}$$

By the inequality (2.9) and use the Hölder's inequality for integrals, we get

$$\begin{aligned}
&\|T(f_1^{\infty}, f_2^{\infty}, f_3^{0}, \dots, f_m^{0})\|_{L_p(B(x,r))} \\
&\lesssim r^{\frac{n}{p}} \int\limits_{^{\complement}B(x,2r)} \frac{|f_1(y_1)|}{|x-y_1|^n} dy_1 \int\limits_{^{\complement}B(x,2r)} \frac{|f_2(y_2)|}{|x-y_2|^n} dy_2 \prod_{i=3}^{m} \int\limits_{B(x,2r)} |f_i(y_i)| dy_i \\
&\le r^{\frac{n}{p}} \prod_{i=3}^{m} \int_{r}^{\infty} t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x,t))} dt.
\end{aligned}$$

For the proof of the inequality (2.6), by a similar argument as in the proof of (2.5) and pay attention to the fact that $\overrightarrow{f} \to T_m(\overrightarrow{f})$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \cdots \times L_{p_m}(\mathbb{R}^n)$ to $WL_p(\mathbb{R}^n)$, we can similarly prove (2.6) and we omit the details here.

Now we give the boundedness of multi-sublinear operators generated by multilinear Calderón-Zygmund operators on product generalized Morrey space.

Theorem 2.3 *Let* $1 \le p_1, \dots, p_m < \infty$ *with* $1/p = 1/p_1 + \ldots + 1/p_m$ *and* $(\varphi_1, \dots, \varphi_m, \varphi) \in \Omega_{p_1} \times \ldots \times \Omega_{p_1} \times \Omega_p$ *satisfies the condition*

$$\prod_{i=1}^{m} \int_{r}^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_i(x,s) s^{\frac{n}{p_i}}}{t^{\frac{n}{p_i}+1}} dt \lesssim \varphi(x,r). \tag{2.11}$$

Let also T_m *be a multi-sublinear operator which satisfies the condition* (1.1) *and bounded from* $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ *to* $L_p(\mathbb{R}^n)$ *for* $p_i > 1$, $i = 1, \dots, m$, *and bounded from* $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ *to* $WL_p(\mathbb{R}^n)$ *for* $p_i \ge 1$, $i = 1, \dots, m$. *Then the operator* T_m *is bounded from product space* $\mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n) \times \ldots \times \mathcal{M}_{p_m,\varphi_m}(\mathbb{R}^n)$ *to* $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ *for* $p_i > 1$, $i = 1, \dots, m$, *and from product space* $\mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n) \times \ldots \times \mathcal{M}_{p_m,\varphi_m}(\mathbb{R}^n)$ *to* $W\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ *for at least one* p_i *equals one.*

Proof. Let $1 < p_1, \ldots, p_m < \infty$ and $\overrightarrow{f} \in \mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n) \times \ldots \times \mathcal{M}_{p_m,\varphi_m}(\mathbb{R}^n)$. By Theorems 2.1 and 2.2 we have

$$\|T_{\alpha,m}(\overrightarrow{f})\|_{\mathcal{M}_{p,\varphi}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \prod_{i=1}^m \int_r^\infty t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x,t))} dt$$
$$\lesssim \prod_{i=1}^m \sup_{x\in\mathbb{R}^n, r>0} \varphi_i(x,r)^{-1}\, r\, \|f_i\|_{L_{p_i}(B(x,r))} = \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_i}}.$$

When $p_i = 1$, $i = 1, \ldots, m$, the proof is similar and we omit the details here.

Corollary 2.1 *[15] Let* $1 \le p_1, \ldots, p_m < \infty$ *with* $1/p = 1/p_1 + \ldots + 1/p_m$. *Let also* $(\varphi_1, \ldots, \varphi_m, \varphi) \in \Omega_{p_1} \times \ldots \times \Omega_{p_1} \times \Omega_p$ *satisfies the condition* (2.11). *Then the operators* M_m *and* K_m *are bounded from product space* $\mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n) \times \ldots \times \mathcal{M}_{p_m,\varphi_m}(\mathbb{R}^n)$ *to* $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ *for* $p_i > 1$, $i = 1, \ldots, m$ *and from product space* $\mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n) \times \ldots \times \mathcal{M}_{p_m,\varphi_m}(\mathbb{R}^n)$ *to* $W\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ *for at least one* p_i *equals one.*

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