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On approximation of a weighted Lipschitz class functions by means $t_n(f;x),\ N_n^{\beta}(f;x)$ and $R_n^{\beta}(f,x)$ of Fourier series

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Abstract. *In this work the approximation properties of the functions by means $t_n(f;x),\ N_n^{\beta}(f;x)$ and $R_n^{\beta}(f,x)$ of the trigonometric Fourier series in weighted Lebesgue spaces with variable exponent are investigated.*

Keywords. Trigonometric approximation, weighted Lebesgue spaces with variable exponent, weighted Lipschitz class, modulus of continuity.

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1 Introduction, some auxiliary results and main results

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$ and $L^p(\mathbb{T})$, $1 \le p \le \infty$, the Lebesgue space of measurable functions on $\mathbb{T}$.

Let us denote by $\wp$ the class of Lebesgue measurable functions $p : \mathbb{T} \longrightarrow (1, \infty)$ such that $1 < p_* := \underset{x\in\mathbb{T}}{ess\inf} p(x) \le p^* := \underset{x\in T}{ess\sup}\, p(x) < \infty$. The conjugate exponent of $p(x)$ is shown by $p'(x) := \frac{p(x)}{p(x)-1}$. For $p \in \wp$, we define a class $L^{p(\cdot)}(\mathbb{T})$ of 2π periodic measurable functions $f : \mathbb{T} \to \mathbb{R}$ satisfying the condition

$$\int_{\mathbf{T}} |f(x)|^{p(x)}\, dx < \infty.$$

This class $L^{p(\cdot)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{T})} := \inf\{\ \lambda > 0 : \int_{\mathbf{T}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\}.$$

The spaces $L^{p(\cdot)}(\mathbb{T})$ are called generalized Lebesgue spaces with variable exponent. It is know that for $p(x) := p$ $(1 < p < \infty)$, the space $L^{p(x)}(\mathbb{T})$ coincides with the Lebesgue

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space $L^p(\mathbb{T})$. If $p^* < \infty$ then the spaces $L^{p(\cdot)}(\mathbb{T})$ represent a special case of the so-called Orlicz-Musielak spaces [32]. For the first time Lebesgue spaces with variable exponent were introduced by Orlicz [33]. Note that the generalized Lebesgue spaces with variable exponent are used in the theory of elasticity, in mechanics, especially in fluid dynamics for the modelling of electrorheological fluids, in the theory of diferential operators, and in variational calculus [7], [8], [9], [35] and [37]. Detailed information about properties of the Lebesque spaces with variable exponent can be found in [10], [30], [31], [36] and [38]. Note that, some of the fundamental problems of the approximation theory in the generalized Lebesgue spaces with variable exponent of periodic and non-periodic functions were studied and solved by Sharapudinov [39]-[43] .

A function $\omega : \mathbb{T} \to [0, \infty]$ is called a *weight function if* ω is a measurable and almost everywhere (a.e.) positive.

Let ω be a 2π periodic weight function. We denote by $L^p_\omega(\mathbb{T})$ the weighted Lebesgue space of 2π periodic measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that $f\omega^{\frac{1}{p}} \in L^p(\mathbb{T})$. For $f \in L^p_\omega(\mathbb{T})$ we set

$$\|f\|_{L^p_\omega(\mathbb{T})} := \left\|f\omega^{\frac{1}{p}}\right\|_{L^p(\mathbb{T})}.$$

$L^{p(\cdot)}_\omega(\mathbb{T})$ stands for the class of Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that $\omega f \in L^{p(\cdot)}(\mathbb{T})$. $L^{p(\cdot)}_\omega(\mathbb{T})$ is called the weighted Lebesgue space with variable exponent. The space $L^{p(\cdot)}_\omega(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}_\omega(\mathbb{T})} := \|f\omega\|_{L^{p(\cdot)}(\mathbb{T})}.$$

Its known [25] that the set of trigonometric polynomials is dense in $L^{p(\cdot)}_\omega(\mathbb{T})$, if $[\omega(x)]^{p(x)}$ is integrable on $\mathbb{T}$.

Let $\mathcal{B}$ be the class of all intervals in $\mathbb{T}$. For $B \in \mathcal{B}$ we set

$$p_B := (\frac{1}{|B|}\int_B \frac{1}{p(x)}dx)^{-1}.$$

For given $p \in \wp$ the class of weights ω satisfying the condition

$$\left\|\omega^{p(x)}\right\|_{A_{p(.)}} := \sup_{B\in\mathcal{B}} \frac{1}{|B|^{p_B}} \left\|\omega^{p(x)}\right\|_{L^1(B)} \left\|\frac{1}{\omega^{p(x)}}\right\|_{L^{(p'(\cdot)/p(\cdot))}(B)} < \infty$$

will be denoted by $A_{p(.)}$ [1].

We say that the variable exponent $p(x)$ satisfies *local log-Hölder continuity condition,* if there is a positive constant c_1 such that

$$| p(x) - p(y) | \leq \frac{c_1}{\log(\frac{1}{|x-y|})}, \tag{1.1}$$

for all $x, y \in \mathbb{T}$.

A function $p \in \wp$ is said to belong to the class $\wp^{\log}$, if the condition (1.1) is satisfied.

We denote by $E_n(f)_{L^{p(\cdot)}_\omega(\mathbb{T})}$ the best approximation of $f \in L^{p(\cdot)}_\omega(\mathbb{T})$ by trigonometric polynomials of degree not exceeding $n,$ i.e.,

$$E_n(f)_{L^{p(\cdot)}_\omega(\mathbb{T})} = \inf\{\| f - T_n \|_{L^{p(\cdot)}_\omega(\mathbb{T})} : T_n \in \varPi_n\},$$

where $\varPi_n$ denotes the class of trigonometric polynomials of degree at most n.

Let us suppose that $p \in \wp,\ \omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$, for some $p_0 \in (1, p_*)$. For $f \in L^{p(\cdot)}_{\omega}(\mathbb{T})$ we set

$$(\nu_h f)(x) := \frac{1}{2h}\int\limits_{-h}^{h} f(x+t)\,dt,\ 0 < h < \pi,\ x \in T.$$

If $p \in \wp^{\log}$, $\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ with some $p_0 \in (1, p_*)$ and $f \in L^{p(\cdot)}_{\omega}(\mathbb{T})$, then the shift operator ν_{h_i} is a bounded linear operator on $L^{p(\cdot)}_{\omega}(\mathbb{T})$ [27]:

$$\|\nu_{h_i}(f)\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} \le c_2 \|f\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})}.$$

Let $p \in \wp$ and $\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ with some $p_0 \in (1, p_*)$. The function

$$\Omega_{p(.),\omega}(\delta,\ f) := \sup_{0<h\le\delta} \|f - (\nu_h f)\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})},\ \delta > 0$$

is called the *moduli of continuity* of $f \in L^{p(\cdot)}_{\omega}(\mathbb{T})$.

It can easily be shown that $\Omega_{p(.),\omega}(\cdot, f)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0} \Omega_{p(.),\omega}(\delta, f) = 0,\ \Omega_{p(.),\omega}(\delta, f+g) \le \Omega_{p(.),\omega}(\delta, f) + \Omega_{p(.),\omega}(\delta, g),\ \delta > 0$$

for $f, g \in L^{p(\cdot)}_{\omega}(\mathbb{T})$. Note that detailed information about properties of moduli of continuity $\Omega_{p(.),\omega}(\cdot, f)$ can be found in the paper [1]. Also, moduli of this type was considered by E. A. Hadjieva [16] in Lebesgue space with Muckenhoupt $A_p,\ 1 < p < \infty$ weight.

Let $0 < \alpha \le 1$. The set of functions $f \in L^{p(\cdot)}_{\omega}(\mathbb{T})$ such that

$$\Omega_{p(.),\omega}(f, \delta) = O(\delta^{\alpha}),\ \ \delta > 0$$

is called the *weighted Lipschitz class* $Lip(\alpha, p(.), \omega)$. Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k(f)\cos kx + b_k(f)\sin kx) \tag{1.2}$$

be the Fourier series of the function $f \in L^1(\mathbb{T})$, where $\alpha_k(f)$ are $\beta_k(f)$ the Fourier coefficients of the function f. The *n-th partial sums, Cesaro means* of the series (1.2) are defined, respectively, as

$$\begin{aligned} s_n(x, f) &= \frac{a_0}{2} + \sum_{k=1}^{n} (a_k(f)\cos kx + b_k(f)\sin kx), \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x, f),\ A_k(x, f) = (a_k(f)\cos kx + b_k(f)\sin kx), \\ \sigma_n(x, f) &= \frac{1}{n+1}\sum_{m=0}^{n} s_m(x, f). \end{aligned}$$

Let $\sum\limits_{m=0}^{\infty} a_n$ be an infinite series and let $\{s_n\}$ its n^{th} partial sum . Let $\{t_n\}$ be a sequence of (N, p, q) means of the sequence $\{s_n\}$. We define transform (N, p, q) of $\{s_n(f; x)\}$ by [44]

$$t_n(x) := t_n(f,x) := \frac{1}{r_n}\sum_{m=0}^{n} p_{n-m}q_m s_m(f;x),$$

where

$$r_n := \sum_{m=0}^{n} p_m q_{n-m} \neq 0, n \geq 0, \text{ and } p_{-1} = q_{-1} = r_{-1} = 0$$

Let $\{p_n\}$ be a real sequence, where $p_0 > 0,\ p_n \geq 0$ for $n > 0$. As in [4] we define

$$p_m^\beta = \sum_{\nu=0}^{m} A_{m-\nu} p_\nu;\ \ P_n^\beta = \sum_{m=0}^{n} p_m^\beta,\ \ P_{-i}^\beta = p_{-i} = 0,\ i \geq 1,$$

where

$$A_0^\beta = 1;\ A_n^\beta = \frac{(\beta+1)(\beta+2)(\beta+3)...(\beta+n)}{n!},\ \beta > -1,\ n = 1,2,3,... .$$

In proof of the main result we will use the notations

$$\Delta\beta_n := \beta_n - \beta_{n+1},\ \ \Delta_m\beta(n,m) := \beta(n,m) - \beta(n,m+1)$$

Considering [34] we can write the following equality

$$p_m^\beta - p_{m+1}^\beta = \sum_{\nu=0}^{m} A_{m-\nu}^{\beta-1} p_\nu - \sum_{\nu=0}^{m-1} A_{m+1-\nu}^{\beta-1} p_\nu = \sum_{\nu=0}^{m-1} A_{m+1-\nu}^{\beta-1} \Delta p_{\nu-1}$$

We define the sequence $\left\{N_n^\beta\right\}$ of the $\left\{\overline{N},p_n^\beta\right\}$ means of the sequence $\{s_n(f;x)\}$ by

$$N_n^\beta(x,f) = \frac{1}{P_n^\beta}\sum_{m=0}^{n} p_m^\beta s_m(x,f).$$

Also,

$$R_n^\beta(x,f) = \frac{1}{P_n^\beta}\sum_{m=0}^{n} p_{n-m}^\beta s_m(x,f).$$

defines the $\left(N,p_n^\beta\right)$ means of $\{s_n(f;x)\}$.

In this work we study the approximation of the functions by trigonometric polynomials $t_n(f;x),\ R_n^\beta(f,x)$ and $N_n^\beta(f;x)$ in weighted Lebesgue spaces with variable exponent. The results obtained in this work, are generalization of the results [13] and [44] to more general summability and weighted Lebesgue spaces with variable exponent. Similar problems about approximations of the functions by trigonometric polynomials in the different spaces have been investigated by several authors (see, for example, [1]- [3], [5],[6], [11]-[25], [28], [34], and [44]-[46]).

Note that, in the proof of the main results we use the method as in the proof of [44].

We shall use $c_1, c_2, ...$to denote the constants, depending in general on the parameters given in the brackets and independent of n.We also will use the relation $f = O(g)$ which means that $f \leq cg$ for a constant c nondependent of f and g.

Our main results are the following:

Theorem 1.1. *Let* $p \in \wp,\ \omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ with some $p_0 \in (1, p_*)$ *and the conditions*

$$\text{(i)} \quad n^2 q_n = O(r_n), \tag{1.3}$$

$$\text{(ii)} \sum_{m=0}^{n-1} m^{2-\alpha} \left| \Delta_m (p_{n-m} q_m \right| = O(r_n n^{-\alpha}), \tag{1.4}$$

are satisfied , then if $f \in Lip(\alpha, p(.), \omega),\ 0 < \alpha \leq 1$ *the estimate*

$$\|t_n(\cdot, f) - f\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} = O(n^{-\alpha}).$$

holds.

Theorem 1.2. *Let* $p \in \wp,\ \omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ *with some* $p_0 \in (1, p_*)$ *and let* $\left\{p_n^{\beta}\right\}$ *be a monotonic sequence such that*

$$(n+1)\, p_n^{\beta} = O(P_n^{\beta}). \tag{1.5}$$

Then for every $f \in Lip(\alpha, p\,(.)\,, \omega),\ 0 < \alpha \leq 1$ *the estimate*

$$\left\| f - R_n^{\beta}(\cdot, f) \right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} = O(n^{-\alpha}),\ n = 1, 2, ...$$

holds.

Theorem 1.3. *Let* $p \in \wp,\ \omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ with some $p_0 \in (1, p_*)$ *and let* $\left\{p_n^{\beta}\right\}$ *be a sequence of positive real numbers such that*

$$\sum_{m=0}^{n-1} \left| \frac{P_m^{\beta}}{m+1} - \frac{P_{m+1}^{\beta}}{m+2} \right| = O\left(\frac{P_n^{\beta}}{n+1} \right). \tag{1.6}$$

Then for every $f \in Lip(\alpha, p(.), \omega),\ 0 < \alpha \leq 1$ *the estimate*

$$\left\| f - N_n^{\beta}(\cdot, f) \right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} = O(n^{-\alpha}),\ n = 1, 2, ...$$

holds.

In the proof of the main result we need the following lemmas.

Lemma 1.1.[27] *Let* $p \in \wp,\ \omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}$ with some $p_0 \in (1, p_*)$. *Then for* $f \in L^{p(\cdot)}_{\omega}(\mathbb{T})$, *the estimate*

$$\|f - \sigma_n(\cdot, f)\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} = O(n \Omega_{p(.),\omega} \left(\frac{1}{n},\ f \right)),\ n = 1, 2, ..$$

holds.

Lemma 1.2.[44] *Let* $\left\{p_n^{\beta}\right\}$ *be a monotonic sequence of positive numbers. Then,*

$$\sum_{m=1}^{n} m^{-\alpha} p_{n-m} = O\left(n^{-\alpha} P_n^{\beta} \right)$$

for $0 < \alpha < 1$.

2 Proofs of the main results

Proof of Theorem 1.1. By definition of $t_n(f;x)$ and $\sigma_n(f;x)$ we have [44, p.1581]

$$\begin{aligned} t_n(x,f)-f(x) &= \frac{1}{r_n}\sum_{m=0}^{n} p_{n-m}q_m(s_m(x,f)-f(x)) \\ &= \frac{1}{r_n}\left\{\sum_{m=0}^{n-1}\Delta_m(p_{n-m}q_m)\sum_{k=0}^{m}(s_k(x,f)-f(x))\right\} \\ &+\frac{1}{r_n}\left\{p_0q_0\sum_{k=0}^{n}(s_k(x,f)-f(x))\right\} \\ &= \frac{1}{r_n}\left\{\sum_{m=0}^{n-1}(m+1)\Delta_m(p_{n-m}q_m)(\sigma_m(x;f)-f(x))\right\}. \\ &+\frac{1}{r_n}\left\{(n+1)p_0q_n(\sigma_n(x,f)-f(x))\right\}. \end{aligned} \tag{2.1}$$

Now using (2.1), conditions (1.3), (1.4) and Lemma 1.1 we obtain

$$\begin{aligned} &\|t_n(\cdot,f)-f\|_{L^{p(\cdot)}_\omega(\mathbb{T})} \\ &= O\left(\frac{1}{r_n}\right)\left\{\sum_{m=0}^{n-1}(m+1)\left|\Delta_m(p_{n-m}q_m)\right|\|\sigma_m(\cdot,f)-f\|_{L^{p(\cdot)}_\omega(\mathbb{T})}\right\} \\ &+O\left(\frac{1}{r_n}\right)(n+1)p_0q_n\|\sigma_n(\cdot,f)-f\|_{L^{p(\cdot)}_\omega(\mathbb{T})} \\ &= O\left(\frac{1}{r_n}\right)\left\{\sum_{m=0}^{n-1}(m+1)\left|\Delta_m(p_{n-m}q_m)\right|(m^{1-\alpha})+(n+1)p_0q_n(n^{1-\alpha})\right\} \\ &= O\left[\frac{1}{r_n}\sum_{m=2}^{n-1}(m+1)\left|\Delta_m(p_{n-m}q_m)(m^{1-\alpha})\right|+O(n^{-\alpha})\right] \\ &= O\left[\frac{1}{r_n}\sum_{m=2}^{n-1}m^{2-\alpha}\left|\Delta_m(p_{n-m}q_m)\right|+(n^{-\alpha})\right]=O(n^{-\alpha}). \end{aligned}$$

which completes the proof.

Proof of Theorem 1.2. Case 1. We suppose that $0<\alpha<1$. It is clear that

$$f(x)-R_n^\beta(x,f)=\frac{1}{P_n^\beta}\sum_{m=o}^{n}p_{n-m}^\beta\left\{f(x)-s_m(x,f)\right\}.$$

According to [26] the relation

$$\|f-s_n(\cdot,f)\|_{L^{p(\cdot)}_\omega(\mathbb{T})}=O\left(\Omega_{M,\omega}\left(\frac{1}{n},\ f\right)\right) \tag{2.2}$$

holds. By (2.2) , Lemma 1.2 and condition (1.5) we find

$$\begin{aligned}\left\| f-R_n^{\beta}(\cdot,f)\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} &\leq \frac{1}{P_n^{\beta}}\sum_{m=o}^{\lambda(n)} p_{n-m}^{\beta}\left\| f-s_m(\cdot,f)\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\ &= \frac{1}{P_n^{\beta}}\sum_{m=1}^{n} p_{n-m}^{\beta} O\left(m^{-\alpha}\right)\left\| f-s_m(\cdot,f)\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\ &\quad +\frac{p_n^{\beta}}{P_n^{\beta}}\left\| f-s_0(\cdot,f)\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\ &= \frac{1}{P_n^{\beta}} O\left(n^{-\alpha}P_n^{\beta}\right)+O\left(\frac{1}{n+1}\right)=O\left(n^{-\alpha}\right).\end{aligned}$$

Case 2. Let $\alpha=1$. Since

$$R_n^{\beta}(x,f)=\frac{1}{P_n^{\beta}}\sum_{m=o}^{n} P_{n-m}^{\beta}A_m(x,f)$$

using Abel's transformation, we have

$$\begin{aligned}& s_n(x,f)-R_n^{\beta}(x,f) \\ &= \frac{1}{P_n^{\beta}}\sum_{m=1}^{n}(P_n^{\beta}-P_{n-m}^{\beta})A_m(x,f) \\ &= \frac{1}{P_n^{\beta}}\sum_{m=1}^{n}\left[\frac{P_n^{\beta}-P_{n-m}^{\beta}}{m}-\frac{P_n^{\beta}-P_{n-(m+1)}^{\beta}}{m}\right]\left(\sum_{k=1}^{m}kA_k(x,f)\right) \\ &\quad +\frac{1}{n+1}\sum_{k=1}^{m}kA_k(x,f). \qquad (2.3)\end{aligned}$$

Then, taking into account (2.3) we obtain

$$\begin{aligned}& \left\| s_n(\cdot,f)-R_n^{\beta}(\cdot,f)\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\ &\leq \frac{1}{P_n^{\beta}}\sum_{m=1}^{n}\left|\frac{P_n^{\beta}-P_{n-m}^{\beta}}{m}-\frac{P_n^{\beta}-P_{n-(\mathbf{m}+1)}^{\beta}}{m}\right|\left\|\sum_{k=1}^{m}kA_k(\cdot,f)\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\ &\quad +\frac{1}{n+1}\left\|\sum_{k=1}^{n}kA_k(\cdot,f)\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})}. \qquad (2.4)\end{aligned}$$

It is clear that if the Fourier series of f is

$$f(x)\sim\sum_{k=0}^{m}A_k(x,f),$$

then $\widetilde{f'}$ has the Fourier series

$$\widetilde{f'}(x)\sim\sum_{k=0}^{m}kA_k(x,f),$$

where $\widetilde{f'}$ is the conjugate function of $f' \in L^{p(\cdot)}_{\omega}(\mathbb{T})$ [19]. Using, boundedness of the partial sums and the conjugation operator in the space $L^{p(\cdot)}_{\omega}(\mathbb{T})$ [19] , we get

$$\frac{1}{n+1}\left\|\sum_{k=1}^{n} kA_k(\cdot,f)\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} = \frac{1}{n+1}\left\|s_n(\cdot,\widetilde{f'})\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} = O(n^{-1}). \qquad (2.5)$$

Thus, (2.4) and (2.5) yield

$$\begin{aligned}&\left\|s_n(\cdot,f)-R^{\beta}_n(\cdot,f)\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})}\\ &\le \frac{1}{P^{\beta}_n}\sum_{m=1}^{n}\left|\frac{P^{\beta}_n-P^{\beta}_{n-m}}{m}-\frac{P^{\beta}_n-P^{\beta}_{n-(\mathbf{m}+1)}}{m}\right| O(1)+O(n^{-1})\\ &= O(\frac{1}{P^{\beta}_n})\sum_{m=1}^{n}\left|\frac{P^{\beta}_n-P^{\beta}_{n-m}}{m}-\frac{P^{\beta}_n-P^{\beta}_{n-(\mathbf{m}+1)}}{m}\right| + O(n^{-1}). \qquad (2.6)\end{aligned}$$

According to [43] the following relations holds :

$$\sum_{m=1}^{n}\left|\frac{P^{\beta}_n-P^{\beta}_{n-m}}{m}-\frac{P^{\beta}_n-P^{\beta}_{n-(\mathbf{m}+1)}}{m}\right| = \frac{1}{n+1}O(P^{\beta}_n).$$

The last inequality and (2.6) imply that

$$\left\|s_n(\cdot,f)-R^{\beta}_n(\cdot,f)\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} = O(n^{-1}). \qquad (2.7)$$

By (2.7) and (2.2)

$$\begin{aligned}&\left\|f-R^{\beta}_n(\cdot,f)\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})}\\ &\le \|f-s_n(\cdot,f\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} + \left\|s_n(\cdot,f)-R^{\beta}_n(\cdot,f)\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} = O(n^{-1}).\end{aligned}$$

which completes the proof.

Proof of Theorem 1.3. Case 1. We suppose that $0<\alpha<1$. Since

$$N^{\beta}_n(f;x) = \frac{1}{P^{\beta}_n}\sum_{m=0}^{n} p^{\beta}_m s_m(f;x)$$

we can write

$$f(x)-N^{\beta}_n(f;x) = \frac{1}{P^{\beta}_n}\sum_{m=0}^{n} p^{\beta}_m\left\{f(x)-s_m(f;x).\right\}. \qquad (2.8)$$

Use of (2.8) and (2.2) gives us

$$\left\|f-N_n^{\beta}(\cdot,f)\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \leq \frac{1}{P_n^{\beta}}\sum_{m=0}^{n} p_m^{\beta}\left\|f-s_m(\cdot,f).\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})}$$
$$= O(\frac{1}{P_n^{\beta}})\sum_{m=1}^{n} p_m^{\beta} m^{-\alpha} + \frac{p_0^{\beta}}{P_n^{\beta}}\left\|f-s_0(\cdot,f).\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})}$$
$$= O(\frac{1}{P_n^{\beta}})\sum_{m=1}^{n} p_m^{\beta} m^{-\alpha}. \tag{2.9}$$

Considering [44, p.1583]

$$\sum_{m=1}^{n} p_m^{\beta} m^{-\alpha} = O(n^{-\alpha}P_n^{\beta}). \tag{2.10}$$

Taking into account the relations (2.9) and (2.10) we have

$$\left\|f-N_n^{\beta}(\cdot,f)\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} = O(n^{-\alpha}).$$

Case 2. Let $\alpha = 1$. Note that by Abel's transformation

$$N_n^{\beta}(f;x) = \frac{1}{P_n^{\beta}}\left[\sum_{m=0}^{n-1} P_m^{\beta}\left\{s_m(x,f)-s_{m+1}(x,f)+P_n^{\beta}s_n(x,f)\right\}\right]$$
$$= \frac{1}{P_n^{\beta}}\sum_{m=0}^{n-1} P_m^{\beta}(-A_{m+1}(x,f)) + s_n(x,f). \tag{2.11}$$

Taking account of (2.11)

$$N_n^{\beta}(x,f) - s_n(x,f) = -\frac{1}{P_n^{\beta}}\sum_{m=0}^{n-1} P_m^{\beta}A_{m+1}(x,f). \tag{2.12}$$

By Abel's transformation [44, p.1584] the following equality holds:

$$\sum_{m=0}^{n-1} P_m^{\beta}A_{m+1}(x,f)$$
$$= \sum_{m=0}^{n-1} \frac{P_m^{\beta}}{m+1}(m+1)A_{m+1}(x,f)$$
$$= \sum_{m=0}^{n-1}\left[\frac{P_m^{\beta}}{m+1} - \frac{P_{m-1}^{\beta}}{m+2}\right]\left[\sum_{k=0}^{m}(k+1)A_{k+1}(x,f)\right]$$
$$+ \frac{P_n^{\beta}}{n+1}\sum_{k=0}^{n-1}(k+1)A_{k+1}(x,f).$$

Using the last equality, condition (1.6) and (2.5) we reach

$$\begin{aligned}
&\left\| \sum_{m=0}^{n-1} P_m^{\beta} A_{m+1}(\cdot, f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\
&\leq \sum_{m=0}^{n-1} \left| \frac{P_m^{\beta}}{m+1} - \frac{P_{m-1}^{\beta}}{m+2} \right| \left\| \sum_{k=0}^{m} (k+1) A_{k+1}(\cdot, f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\
&\quad + \frac{P_n^{\beta}}{n+1} \left\| \sum_{k=0}^{n-1} (k+1) A_{k+1}(\cdot, f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\
&= O(1) \sum_{m=0}^{n-1} \left| \frac{P_m^{\beta}}{m+1} - \frac{P_{m-1}^{\beta}}{m+2} \right| + O\left(\frac{P_n^{\beta}}{n} \right) = O\left(\frac{P_n^{\beta}}{n} \right). \qquad (2.13)
\end{aligned}$$

Hence, by (2.12) and (2.13),

$$\begin{aligned}
&\left\| N_n^{\beta}(\cdot, f) - s_n(\cdot, f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\
&= \frac{1}{P_n^{\beta}} \left\| \sum_{m=0}^{n-1} P_m^{\beta} A_{m+1}(\cdot, f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} = \frac{1}{P_n^{\beta}} O\left(\frac{P_n^{\beta}}{n} \right) = O(n^{-1}). \qquad (2.14)
\end{aligned}$$

Now, the relations (2.14) and (2.2) imply the desired inequality

$$\begin{aligned}
&\left\| f - N_n^{\beta}(\cdot, f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\
&\leq \| f - s_n(\cdot, f) \|_{L_\omega^{p(\cdot)}(\mathbb{T})} + \left\| N_n^{\beta}(\cdot, f) - s_n(\cdot, f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} = O(n^{-1}).
\end{aligned}$$

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