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ξ -Double Strongly Summable Sequences of order θ

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Abstract. The primary aim of this work is to define the concept of strongly $[V_{\xi}^{''}, \Lambda, \bar{\eta}, t]$-summable of order θ defined by using Orlicz function and the uniformly $(\bar{\eta}, \xi)$-statistical convergence of order θ. Additionally, some inclusion theorems are proved.

Keywords. Double sequence spaces, Orlicz function, double summable, double statistical convergent, order θ

Mathematics Subject Classification (2010): Primary 40A99; Secondary 40A05

1 Introduction

In recent years, there has been an increasing interest on summability methods of sequences of real numbers. One of these summability methods which has attracted the attention of many researchers is invariant summability method. However, generalized double strongly summable sequences has not been studied so far. Thus, this paper is an attempt to fill this gap in the existing literature.

Before we present the main results of this paper, we will give some notions that will be needed in the future.

Let l_∞ and c be the Banach spaces of bounded and convergent $y = (y_r)$ with the usual norm $\|y\| = \sup_r |y_r|$, respectively. Also, let ξ be an injection of the set of positive integers $\mathbf{N}$ into itself. We say that a continuous linear functional ν on l_∞ a $\xi-$mean if

i $\nu(y) \geq 0$ when the sequence $y = (y_r)$ is such that $y_r \geq 0$ for all r,
ii $\nu(e) = 1$ where $e = (1, 1, 1, \ldots)$, and
iii $\nu(y) = \nu(y_{\xi(r)})$ for all $y \in l_\infty$.

It is clear that $c \subset V_\xi$. In the case ξ is the translation mapping $\xi\,(p) = p+1$, the $\xi-$mean is often called a Banach limit and V_ξ, the bounded sequences of all whose ξ-mean are equal. It can be demonstrate that

$$V_\xi = \left\{ y \in l_\infty : \lim_q t_{q,p}(y) = \gamma \text{ uniformly in } p,\ \gamma = \xi - \lim y \right\},$$

where

$$t_{q,p}(y) = \frac{y_p + y_{\xi(p)} + \cdots + y_{\xi^q(p)}}{q+1},\ t_{-1,p}(y) = 0.$$

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(see, [18]) . We say that a bounded sequence $y = (y_r)$ is ξ-convergent if and only if $y \in V_\xi$ such that $\xi^q(p) \neq p$ for all $p \geq 0$, $q \geq 1$.

An Orlicz function is a function $\Lambda : [0,\infty) \to [0,\infty)$, which is continuous, convex, non-decreasing with $\Lambda(0) = 0$ and $\Lambda(y) > 0$ for $y > 0$, and $\Lambda(y) \to \infty$ as $y \to \infty$, (see, [3]).

Later on the notion of Orlicz function was used by Mursaleen, Khan, Chishti [8], Parashar and B. Choudhary [13], Savaş [17] and other authors.

Let ω'' denote the set of all double sequences of real numbers, and also let us consider the following definition.

Definition 1.1 (Pringsheim, [10]) *A double sequence $y = (y_{r,s})$ has Pringsheim limit γ (denoted by $P - \lim y = \gamma$) provided that given $\epsilon > 0$ there exists $\widetilde{N} \in \mathbf{N}$ such that $|y_{r,s} - \gamma| < \epsilon$ whenever $r, s > \widetilde{N}$. We shall describe such an y more briefly as "P-convergent", and denote the class of all P-convergent sequences by c''.*

Definition 1.2 ([16]) *Let $\eta = (\eta_v)$ and $\rho = (\rho_v)$ be two non-decreasing sequences of positive real numbers both of which tends to ∞ as u and v approach ∞, respectively. Also let $\eta_{u+1} \leq \eta_u + 1, \eta_1 = 0$ and $\rho_{v+1} \leq \rho_v + 1, \rho_1 = 0$. We write the double de la Valee-Poussin mean by*

$$t_{u,v}(y) = \frac{1}{\eta_u \rho_v} \sum_{(r,s)\in \bar{I}_{uv}} y_{r,s}.$$

A sequence $y = (y_{r,s})$ is said to be (V'', η, ρ)-summable to a number γ, if $t_{uv}(y) \to \gamma$ as $u, v \to \infty$ in the Pringsheim sense. During this paper we often denoted $\bar{\eta}_{uv}$ by $\eta_u\rho_v$, $(\xi^r(q), \xi^s(p))$ by $\xi^{r,s}(q,p)$, and $(r \in I_u, s \in I_v)$ by $(r,s) \in \bar{I}_{uv}$.

Definition 1.3 *A bounded double sequence $y = (y_{r,s})$ of real number is said to be $(\bar{\eta}, \xi)$-convergent to γ if,*

$$P - \lim_{uv} T^{uv}_{q,p} = \gamma \text{ uniformly in } (q,p),$$

where

$$T^{uv}_{q,p} = \frac{1}{\bar{\eta}_{uv}} \sum_{(r,s)\in \bar{I}_{uv}} y_{\xi^{r,s}(q,p)}.$$

Whenever this appears, we write $(\bar{\eta}, \xi) - \lim y = \gamma$. We shall also denote the set of all $(\bar{\eta}, \xi)-$convergent sequences by $V''_{(\bar{\eta},\xi)}$. If $\xi(q) = q + 1$, $\xi(p) = p + 1$, and $\bar{\eta}_{uv} = uv$ in the above then $(\bar{\eta}, \xi)$-convergence reduces to the almost convergence of double sequences which can be found in [6].

Definition 1.4 *Let Λ be an Orlicz function and $\theta \in (0, 1]$ be any real number and $t = (t_{r,s})$ be any factorable double sequence of strictly positive real numbers. A sequence y is strongly $[V''_\xi, \Lambda, \bar{\eta}, t]-$summable of order θ, if there is a number γ such that*

$$[V''_\xi, \Lambda, \bar{\eta}, t]^\theta = \{y \in \omega'' : P - \lim_{uv} \frac{1}{\bar{\eta}^\theta_{uv}} \sum_{(r,s)\in \bar{I}_{uv}} \left[\Lambda \left(\frac{|y_{\xi^{r,s}(q,p)} - \gamma|}{\rho} \right) \right]^{t_{r,s}} = 0$$

uniformly in (q,p), for some $\rho > 0$ and some $\gamma > 0\}$.

If we consider several assignment of M, $\bar{\eta}$, and t in the above sequence spaces, it is obvious to get the following:

1 If $\Lambda(y) = y$, $\bar{\eta}_{uv} = uv$, and $t_{r,s} = 1$ for all (r, s), then $[V_{\xi}^{''}, \Lambda, \bar{\eta}, t]^{\theta} = [V_{\xi}^{''}]^{\theta}$.
2 If $t_{r,s} = 1$ for every (r, s), then $[V_{\xi}^{''}, \Lambda, \bar{\eta}, t]^{\theta} = [V_{\xi}^{''}, \Lambda, \bar{\eta}]^{\theta}$.
3 If $t_{r,s} = 1$ for every (r, s) and $\bar{\eta}_{uv} = uv$, then $[V_{\xi}^{''}, \Lambda, \bar{\eta}, t]^{\theta} = [V_{\xi}^{''}, \Lambda]^{\theta}$.
4 If $\Lambda(y) = y$ and $t_{r,s} = t$ for all (r, s), then $[V_{\xi}^{''}, \Lambda, \bar{\eta}, t]^{\theta} = [V_{\xi}^{''}, \bar{\eta}]_t^{\theta}$.
5 If $\bar{\eta}_{uv} = uv$, then $[V_{\xi}^{''}, \Lambda, \bar{\eta}, t]^{\theta} = [V_{\xi}^{''}, \Lambda, t]^{\theta}$.
6 If $\bar{\eta}_{uv} = uv$, $\theta = 1$, then $[V_{\xi}^{''}, \Lambda, \bar{\eta}, t]^{\theta} = [V_{\xi}^{''}, \Lambda, t]$, which was studied in [15].
7 If $\Lambda(y) = y$, $\bar{\eta}_{uv} = uv$, $\xi(q) = q + 1$, and $\xi(p) = p + 1$, then $[V_{\xi}^{''}, \Lambda, \bar{\eta}, t]^{\theta} = [\hat{c}^{''}, t]^{\theta}$.
8 If $\xi(q) = q + 1$, $\xi(p) = p + 1$ and $t_{r,s} = 1$, then $[V_{\xi}^{''}, \Lambda, \bar{\eta}, t] = [\hat{c}^{''}, \Lambda, \bar{\eta}]$.
9 If $\Lambda(y) = y$, $\bar{\eta}_{uv} = uv$, $\xi(q) = q + 1$, $\xi(p) = p + 1$, and $t_{r,s} = 1$ for all (r, s) $[V_{\xi}^{''}, \Lambda, \bar{\eta}, t]^{\theta} = [\hat{c}^{''}]^{\theta}$.
10 If $\Lambda(y) = y$, $\bar{\eta}_{uv} = uv$, $\xi(q) = q + 1$, $\xi(p) = p + 1$, $\theta = 1$ and $t_{r,s} = 1$ for all (r, s) $[V_{\xi}^{''}, \Lambda, \bar{\eta}, t] = [\hat{c}^{''}]$, $[\hat{c}^{''}]$ was studied in [6].

We first recall the following.

A real number sequence y is said to be statistically convergent to the number γ if for every $\varepsilon > 0$

$$\lim_{z} \frac{1}{z} |\{r < z : |y_r - \gamma| \geq \epsilon\}| = 0.$$

In such a case, we write $S - \lim y = \gamma$ or $y_r \to \gamma(S)$. Fast [1] first put forward the idea of statistical convergence, but the fast development was started after the papers of Fridy [2] and Šalát, [11]. Now statistical convergence has become one of the most famous research area in the field of Summability Theory.

The next definition was presented by Mursaleen in [9]. A sequence y is said to be η-statistically convergent or S_η-convergent to γ, if for every $\epsilon > 0$

$$\lim_{n} \frac{1}{\eta_n} |\{r \in I_n : |y_r - \gamma| \geq \epsilon\}| = 0.$$

In such a case, we write $S_\eta - \lim y = \gamma$ or $y_r \to \gamma(S_\eta)$.

Further, concepts of uniformly η-statistical convergence considered by Savaş [12] as follows: A sequence y is said to be uniformly η-statistically convergent or $\hat{S}_\eta$-convergent to γ, if for every $\epsilon > 0$

$$\lim_{n} \frac{1}{\eta_n} \max_{q} |\{r \in I_n : |y_{r+q} - \gamma| \geq \epsilon\}| = 0.$$

In such a case, we write $\hat{S}_\eta - \lim y = \gamma$ or $y_r \to \gamma(\hat{S}_\eta)$.

For double sequence $y = (y_{r,s})$, double statistical convergence was studied by Mursaleen and Edely [7] as follows: A real double sequence $y = (y_{r,s})$ is said to be statistically convergent to γ, if for each $\epsilon > 0$

$$P - \lim_{u,v} \frac{1}{uv} |\{(r, s) : r \leq u \text{ and } s \leq v, |y_{r,s} - \gamma| \geq \epsilon\}| = 0.$$

Definition 1.5 *A double sequence $y = (y_{r,s})$ is said to be uniformly $\hat{S}_{(\bar{\eta},\xi)}$-convergent of order θ or uniformly $(\bar{\eta}, \xi)$-statistical convergent of order θ to γ, if for every $\epsilon > 0$*

$$P - \lim_{u,v} \frac{1}{\bar{\eta}_{uv}^{\theta}} \max_{q,p} |\{(r, s) \in \bar{I}_{uv} : |y_{\xi^{r,s}(q,p)} - \gamma| > \epsilon\}| = 0.$$

Let $(\hat{S}^{\theta}_{(\bar{\eta},\xi)})$ denote the set of sequences $y = (y_{r,s})$ which are uniformly $(\bar{\eta},\xi)$-statistical convergent of order θ to γ.

In such a case, we write $\hat{S}^{\theta}_{(\bar{\eta},\xi)} - \lim y = \gamma$ or $y_{r,s} \to \gamma(\hat{S}^{\theta}_{(\bar{\eta},\xi)})$ and $\hat{S}^{\theta}_{(\bar{\eta},\xi)} = \{y : \exists \xi \in \mathbb{R}, \hat{S}^{\theta}_{(\bar{\eta},\xi)} - \lim y = \gamma\}$.

Remark 1.1 If we take $\theta = 1$, the set $(\hat{S}^{\theta}_{(\bar{\eta},\xi)})$ reduces to $(\hat{S}_{(\bar{\eta},\xi)})$, (see, [14]). If we take $\bar{\eta}_{uv} = uv$ the above definition reduce to $(\hat{S}^{\theta}_{\xi})$, the set of sequences $y = (y_{r,s})$ which are uniformly double $\xi-$statistically convergent of order θ to γ and when $\theta = 1$, $(\hat{S}^{\theta}_{\xi})$ reduces to $(\hat{S}^{2}_{\xi})$ which was consider in [15]. If ξ is the translation in both dimension $(\hat{S}^{\theta}_{\xi})$ reduces to set of the uniformly double almost statistically convergent sequences of order θ to γ.

2 Main Results

In this part, we present some theorems.

Theorem 2.1 *If* $0 < \theta \leq \rho \leq 1$, *then* $\hat{S}^{\theta}_{(\bar{\eta},\xi)} \subset \hat{S}^{\rho}_{(\bar{\eta},\xi)}$.

Proof. Let $0 < \theta \leq \rho \leq 1$. Then

$$\begin{aligned} &\frac{1}{\bar{\eta}^{\rho}_{uv}} \max_{q,p} \left|\left\{(r,s) \in I_{uv} : \left|y_{\xi^{r,s}(q,p)} - \gamma\right| \geq \epsilon\right\}\right| \\ &\leq \frac{1}{\bar{\eta}^{\theta}_{uv}} \max_{q,p} \left|\left\{(r,s) \in I_{uv} : \left|y_{\xi^{r,s}(q,p)} - \gamma\right| \geq \epsilon\right\}\right| \end{aligned}$$

for every $\epsilon > 0$ and finally we have that $\hat{S}^{\theta}_{(\bar{\eta},\xi)} \subset \hat{S}^{\rho}_{(\bar{\eta},\xi)}$. This proves the result.

Theorem 2.2 $\hat{S}^{\theta}_{\xi} - \lim y = \gamma \Rightarrow \hat{S}^{\theta}_{(\bar{\eta},\xi)} - \lim y = \gamma$ *if*

$$P - \liminf_{uv} \frac{\bar{\eta}^{\theta}_{uv}}{(uv)^{\theta}} > 0.$$

Proof. For given $\epsilon > 0$ we have $\{(r,s) \in \bar{I}_{uv} : |y_{\xi^{r,s}(q,p)} - \gamma| \geq \epsilon\} \subset \{r \leq u\&s \leq v : |y_{\xi^{r,s}(q,p)} - \gamma| \geq \epsilon\}$. Therefore

$$\begin{aligned} &\frac{1}{(uv)^{\theta}} \max_{q,p} |\{r \leq u, s \leq v : |y_{\xi^{r,s}(q,p)} - \gamma| \geq \epsilon\}| \\ &\geq \frac{1}{(uv)^{\theta}} \max_{q,p} |\{(r,s) \in \bar{I}_{uv} : |y_{\xi^{r,s}(q,p)} - \gamma| \geq \epsilon\}| \\ &\geq \frac{\bar{\eta}^{\theta}_{uv}}{(uv)^{\theta}} \frac{1}{\bar{\eta}^{\theta}_{u,v}} \max_{q,p} |\{(r,s) \in \bar{I}_{uv} : |y_{\xi^{r,s}(q,p)} - \gamma| \geq \epsilon\}|. \end{aligned}$$

As $(u,v) \to \infty$ it follows that $y_{r,s} \to \gamma[\hat{S}^{\theta}_{\xi}] \Rightarrow y_{r,s} \to \gamma[\hat{S}^{\theta}_{(\bar{\eta},\xi)}]$.

We now have the following corollary;

Corollary 2.1 *Let* θ *be fixed real numbers for which* $0 < \theta \leq 1$ *and* $0 < t < \infty$, *then* $[V''_{\xi}, \bar{\eta}]^{\theta}_{t} \subset \hat{S}^{\theta}_{(\bar{\eta},\xi)}$.

Theorem 2.3 *Let* $0 < \theta \leq \rho \leq 1$ *and let* t *be a positive real number, then* $[V_{\xi}^{''}, \bar{\eta}]_t^{\theta} \subseteq [V_{\xi}^{''}, \bar{\eta}]_t^{\rho}$.

Proof. Let $y = (y_{r,s}) \in [V_{\xi}^{''}, \bar{\eta}]_t^{\theta}$. Then given θ and ρ such that $0 < \theta \leq \rho \leq 1$ and a positive real number t we write

$$\frac{1}{\bar{\eta}_{mn}^{\rho}} \sum_{(r,s)\in I_{uv}} \left|y_{\xi^{r,s}(q,p)} - \gamma\right|^t \leq \frac{1}{\bar{\eta}_{mn}^{\theta}} \sum_{(r,s)\in I_{uv}} \left|y_{\xi^{r,s}(q,p)} - \gamma\right|^t$$

and we get $[V_{\xi}^{''}, \bar{\eta}]_t^{\theta} \subseteq [V_{\xi}^{''}, \bar{\eta}]_t^{\rho}$.

The next corollary is obtained from the above theorem.

Corollary 2.2 *For* $0 < \theta \leq \rho \leq 1$ *and a positive real number* t*,*

1 If $\theta = \rho$, then $[V_{\xi}^{''}, \bar{\eta}]_t^{\theta} = [V_{\xi}^{''}, \bar{\eta}]_t^{\rho}$.
2 $[V_{\xi}^{''}, \bar{\eta}]_t^{\theta} \subseteq [V_{\xi}^{''}, \bar{\eta}]_t$ for each $\theta \in (0, 1]$ and $0 < t < \infty$.

Theorem 2.4 *Let* θ *and* ρ *be fixed real numbers such that* $0 < \theta \leq \rho \leq 1$ *and* $0 < t < \infty$*, then* $[V_{\xi}^{''}, \bar{\eta}]_t^{\theta} \subset \hat{S}_{(\bar{\eta},\xi)}^{\rho}$.

Proof. For any sequence $y = (y_{rs})$ and $\varepsilon > 0$, we let

$$\begin{aligned}
&\sum_{(r,s)\in I_{uv}} \left|y_{\xi^{r,s}(q,p)} - \gamma\right|^t \\
&= \sum_{\substack{(r,s)\in I_{uv} \\ \left|y_{\xi^{r,s}(q,p)} - \gamma\right| \geq \epsilon}} \left|y_{\xi^{r,s}(q,p)} - \gamma\right|^t + \sum_{\substack{(r,s)\in I_{uv} \\ \left|y_{\xi^{r,s}(q,p)} - \gamma\right| < \epsilon}} \left|y_{\xi^{r,s}(q,p)} - \gamma\right|^t \\
&\geq \sum_{\substack{(r,s)\in I_{uv} \\ \left|y_{\xi^{r,s}(q,p)} - \gamma\right| \geq \epsilon}} \left|y_{\xi^{r,s}(q,p)} - \gamma\right|^t \\
&\geq \max_{q,p} \left|\left\{(r,s) \in I_{uv} : \left|y_{\xi^{r,s}(q,p)} - \gamma\right| \geq \epsilon\right\}\right| .\epsilon^t
\end{aligned}$$

and so that

$$\begin{aligned}
\frac{1}{\bar{\eta}_{uv}^{\theta}} \sum_{(r,s)\in I_{uv}} |y_{r,s} - \gamma|^t &\geq \frac{1}{\bar{\eta}_{uv}^{\theta}} \max_{q,p} |\{(r,s) \in I_{uv} : |y_{r,s} - \gamma| \geq \epsilon\}| .\epsilon^t \\
&\geq \frac{1}{\bar{\eta}_{uv}^{\rho}} \max_{q,p} |\{(r,s) \in I_{uv} : |y_{r,s} - \gamma| \geq \epsilon\}| .\epsilon^t,
\end{aligned}$$

this shows that if $y = (y_{r,s}) \in [V_{\xi}^{''}, \bar{\eta}]_t^{\theta}$, then it is $y = (y_{r,s}) \in \hat{S}_{(\bar{\eta},\xi)}^{\rho}$.
This completes the proof.

For $\bar{\eta}_{uv} = uv$ and $t = 1$ in above theorem, we are granted the following:

Corollary 2.3 $y_{r,s} \to \gamma[V_{\xi}^{''}]^{\theta}$ *implies* $y_{r,s} \to \gamma(\hat{S}_{\xi}^{\theta})$.

In the next theorems we shall assume that $t = (t_{r,s})$ is bounded and $0 < h = \inf_{rs} t_{rs} \leq t_{rs} \leq \sup_{rs} t_{rs} = H < \infty$.

Theorem 2.5 *Let $\theta, \rho \in (0,1]$ be real numbers such that $\theta \leq \rho$ and let Λ be an Orlicz function, then $[V_\xi^{''}, \Lambda, \bar{\eta}, t]^\theta \subset \hat{S}^\rho_{(\bar{\eta},\xi)}$.*

Proof. Let $y \in [V_\xi^{''}, \Lambda, \bar{\eta}, t]$. Then there exists $\rho > 0$ such that

$$\frac{1}{\bar{\eta}_{uv}} \sum_{(r,s)\in \bar{I}_{uv}} \left[\Lambda\left(\frac{y_{\xi^{r,s}(q,p)-\gamma}}{\rho}\right)\right]^{t_{r,s}} \to 0,$$

as $(u,v) \to \infty$ in the Pringsheim sense uniformly in (q,p). If $\varepsilon > 0$ and let $\varepsilon_1 = \frac{\varepsilon}{\rho}$, then we obtain

$$\begin{aligned}
&\frac{1}{\bar{\eta}^\theta_{uv}} \sum_{(r,s)\in \bar{I}_{u,v}} \left[\Lambda\left(\frac{y_{\xi^{r,s}(q,p)} - \gamma}{\rho}\right)\right]^{t_{r,s}} \\
&= \frac{1}{\bar{\eta}^\theta_{u,v}} \sum_{(r,s)\in \bar{I}_{uv} \& |y_{\xi^{r,s}(q,p)}| \geq \epsilon} \left[\Lambda\left(\frac{y_{\xi^{r,s}(q,p)} - \gamma}{\rho}\right)\right]^{t_{r,s}} \\
&+ \frac{1}{\bar{\eta}^\theta_{uv}} \sum_{(r,s)\in \bar{I}_{uv} \& |y_{\xi^{r,s}(q,p)}| < \epsilon} \left[\Lambda\left(\frac{y_{\xi^{r,s}(q,p)} - \gamma}{\rho}\right)\right]^{t_{r,s}} \\
&\geq \frac{1}{\bar{\eta}^\rho_{uv}} \sum_{(r,s)\in \bar{I}_{uv} \& |y_{\xi^{r,s}(q,p)}| \geq \epsilon} \left[\Lambda\left(\frac{y_{\xi^{r,s}(q,p)} - \gamma}{\rho}\right)\right]^{t_{r,s}} \\
&\geq \frac{1}{\bar{\eta}^\rho_{uv}} \sum_{(r,s)\in \bar{I}_{uv} \& |y_{\xi^{r,s}(q,p)}| \geq \epsilon} [\Lambda(\epsilon_1)]^{t_{r,s}} \\
&\geq \frac{1}{\bar{\eta}^\rho_{uv}} \sum_{(r,s)\in \bar{I}_{uv}} \min\{[\Lambda(\epsilon_1)]^{\inf t_{r,s}}, [\Lambda(\epsilon_1)]^H\} \\
&\geq \frac{1}{\bar{\eta}^\rho_{uv}} \max_{q,p} |\{(r,s) \in \bar{I}_{uv} : |y_{\xi^{r,s}(q,p)} - \gamma| \geq \epsilon\}| \min\{[\Lambda(\epsilon_1)]^h, [\Lambda(\epsilon_1)]^H\}.
\end{aligned}$$

Hence $y \in \hat{S}^\rho_{(\bar{\eta},\xi)}$.

Corollary 2.4 *Let $\theta \in (0,1]$ and let Λ be an Orlicz function, then $[V_\xi^{''}, \Lambda, \bar{\eta}, t]^\theta \subset \hat{S}^\theta_{(\bar{\eta},\xi)}$.*

Theorem 2.6 *Let Λ be an Orlicz function and let $y = (y_{r,s})$ be a bounded sequence, then $\hat{S}^\theta_{(\bar{\eta},\xi)} \subset [V_\xi^{''}, M, \bar{\eta}, t]^\theta$.*

Proof. Assume that y is bounded. Then there exists an integer $\widetilde{K}$ such that $\left|y_{\xi^{r,s}(q,p)}\right| < \widetilde{K}$. Thus

$$
\begin{aligned}
&\frac{1}{\bar{\eta}_{uv}^{\theta}} \sum_{(r,s)\in \bar{I}_{uv}} \left[\Lambda\left(\frac{|y_{\xi^{r,s}(q,p)} - \gamma|}{\rho} \right) \right]^{t_{r,s}} \\
&= \frac{1}{\bar{\eta}_{uv}^{\theta}} \sum_{(r,s)\in \bar{I}_{uv} \& |y_{\xi^{r,s}(q,p)} - \gamma| \geq \epsilon} \left[\Lambda\left(\frac{|y_{\xi^{r,s}(q,p)} - \gamma|}{\rho} \right) \right]^{t_{r,s}} \\
&+ \frac{1}{\bar{\eta}_{uv}^{\theta}} \sum_{(r,s)\in \bar{I}_{uv} \& |y_{\xi^{r,s}(q,p)} - \gamma| < \epsilon} \left[\Lambda\left(\frac{|y_{\xi^{r,s}(q,p)} - \gamma|}{\rho} \right) \right]^{t_{r,s}} \\
&\leq \frac{1}{\bar{\eta}_{uv}^{\theta}} \sum_{(r,s)\in \bar{I}_{uv} \& |y_{\xi^{r,s}(q,p)} - \gamma| \geq \epsilon} \max\{\widetilde{K}^h, \widetilde{K}^H\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\bar{\eta}_{uv}^{\theta}} \sum_{(r,s)\in \bar{I}_{uv} \& |y_{\xi^{r,s}(q,p)} - \gamma| < \epsilon} \left[\Lambda\left(\frac{\epsilon}{\rho} \right) \right]^{t_{r,s}} \\
&\leq \max\{\widetilde{K}^h, \widetilde{K}^H\} \frac{1}{\bar{\eta}_{uv}^{\theta}} \max_{q,p} |\{(r,s) \in \bar{I}_{uv} : |y_{\xi^{r,s}(q,p)} - \gamma| \geq \epsilon\}| \\
&+ \max\left\{ \left[\Lambda\left(\frac{\epsilon}{\rho} \right) \right]^h, \left[\Lambda\left(\frac{\epsilon}{\rho} \right) \right]^H \right\}.
\end{aligned}
$$

Hence, $y \in [V_{\xi}^{''}, \Lambda, \bar{\eta}, t]^{\theta}$. This completes the proof.

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