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Randić Type Hadi Index of Graphs

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Abstract. *In this paper, we calculate the Randić type hadi index of some standard graphs, double graphs, subdivision graphs, complements and line graphs. Also we compute the index for the chemical structure graphene.*

Keywords. Randić type hadi index · double graphs · subdivision graph · graphene.

Mathematics Subject Classification (2010): 05C07, 05C30, 05C50

1 Introduction

In the last few centuries, the graph topological indices have become very popular due to their applications in several areas including chemistry and networks. The most famous indices are Zagreb, Randić, Wiener, harmonic, GA, ABC indices and their variants. The Randić type hadi index as one of those topological indices is defined as

$$RH(G) = \sum_{uv \in E(G)} \frac{1}{2^{d_u + d_v}}.$$

The subdivision graph $S(G)$ of a simple graph G, as one of the derived graphs, is defined as the new graph obtained by adding an extra vertex into each edge of G. The subgraphs have been studied in literature, [3] and [7].

All versions of Zagreb indices and coindices of subdivision graphs of certain graph types were studied in [6]. The Zagreb indices and multiplicative Zagreb indices of subdivision graphs of double graphs were studied in [8]. In [7], the Zagreb indices and multiplicative Zagreb indices of double graphs of subdivision graphs have been determined. Some graph

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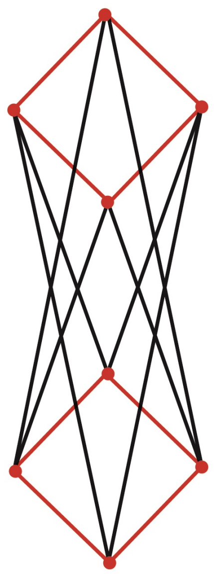
indices including the Randić index were studied in [2].

In [4] and [5], a generalization of subdivision graphs called as the r-subdivision graph of a graph G, which is denoted by $S^r(G)$, was introduces and studied as the new graph obtained from G by replacing each of its edges by a path of length $r+1$; or equivalently by inserting r additional vertices into each edge of G. Clearly, in the case of $r=1$, the obtained 1-subdivision graph is just the classical subdivision graph. In [1], topological indices of the derived graphs of subdivision graphs were studied.

For a graph G with vertex set $V(G) = \{v_1,\ v_2,\ \cdots,\ v_n\}$, we take another copy of G with vertices labelled by $\{v_1,\ v_2,\ \cdots,\ v_n\}$, this time, where v_i corresponds to v_i for each i. If we connect v_i to the neighbours of v_i for each i, we obtain a new graph called the double graph of G. It is denoted by $D(G)$. The double graph of the subdivision graph of a graph G was studied in [7].



D(C_4)

Fig. 1 Double graph of C_4

The rest of this paper is planned as follows: In Section 2, The Randić type hadi indices of some well-known graph classes are calculated. The same index is calculated for the double graphs in Section 3. Some types of complements and their Randić type hadi indices were studied in Section 4. In Section 5, Randić type hadi indices of subdivision graphs, line graphs and line graphs of subdivision graphs are determined. Finally in Section 6, an application of the Randić type hadi index is demonstrated by calculating it for graphene. Similar calculations can be made for other chemical compounds following the same combinatorial calculations.

2 The Randić type hadi index of some standard graphs

There are some classes of graphs which are more frequently used than others such as path graph P_n, cycle graph C_n, complete graphs K_n, complete bipartite graphs $K_{r,s}$, star graphs

$K_{1,n-1}$, double star graphs $S_{n,n}$, crown graphs S_n^0, friendship graphs F_n^3. Although they look like just special classes of graphs, in most theories, they are used either as examples to illustrate the theory or give the extremal cases. Now we calculate the Randić type hadi indices of these important graph classes:

Theorem 2.1 *The Randić type hadi indices of some well-known graph classes are*

$$RH(G) = \begin{cases} \frac{n(n-1)}{2^{2n-1}} & \text{if } G = K_n \\ \frac{n-1}{2^n} & \text{if } G = K_{1,n-1} \\ \frac{n(n-1)}{2^{2n-2}} & \text{if } G = S_n^0 \\ \frac{n}{16} & \text{if } G = C_n,, \\ \frac{n+1}{16} & \text{if } G = P_n, n \\ \frac{2n(n-1)}{2^{4n-4}} & \text{if } G = K_{n\times 2} \\ \frac{2n-2}{2^{n+1}} + \frac{1}{2^{2n}} & \text{if } G = S_{n,n} \\ \frac{2n}{2^{2n+2}} + \frac{n}{16} & \text{if } G = F_n^3. \end{cases} \tag{2.1}$$

Proof. Here we only consider the proof for the complete graph. The same proof can be applied for the remaining graph structures.

In a complete graph K_n, we have n vertices each having the same vertex degree $n-1$ and we have $n(n-1)/2$ edges between the vertices. By the definition of Randić type hadi index, we get

$$\begin{aligned} RH(K_n) &= \frac{n(n-1)}{2} \cdot \frac{1}{2^{n-1+n-1}} \\ &= \frac{n(n-1)}{2^{2n-1}}. \end{aligned}$$

3 Randić type hadi index of double graphs

Double graph of a given graph has been used in some calculations related to the symmetrical structures such as molecules. Therefore, their study helps to do some calculations with large graphs having some special type of symmetry in terms of smaller graphs. Now we calculate the Randić type hadi index of the double graph of any graph. First we need to detemine the degree sequence of the double graph:

Theorem 3.1 *Let a simple connected graph G have the degree sequence*

$$DS(G) = \{1^{(a_1)},\ 2^{(a_2)},\ 3^{(a_3)},\ 4^{(a_4)},\ \cdots, \varDelta^{(a_\varDelta)}\}.$$

Then the double graph $D(G)$ has the degree sequence

$$DS(D(G)) = \{2^{(2a_1)},\ 4^{(2a_2)},\ 6^{(2a_3)},\ 8^{(2a_4)},\ \cdots, (2\varDelta)^{(2a_\varDelta)}\}.$$

Proof. Let a vertex v of G had degree $d_G v$ in G. Obviously, v has $d_G v$ adjacent vertices in G. By the definition of the double graph, v will also be connected to the $d_G v$ vertices in the copy of G. So it will have degree $2d_G v$ in the double graph. Hence the degree of each vertex in G will be doubled in $D(G)$. Secondly, to form the double graph, another identical copy of G is taken and all the vertices in this copy will have the same degrees with their copy vertices in G. Therefore, the multiplicities in $DS(D(G))$ will also be twice the multiplicities in $DS(G)$. So the proof follows.

To calculate the Randić type hadi index of the double graph in general, we need to find the types of the edges in $D(G)$. For each edge uv in G of type $(d_G u, d_G v)$, there will be two identical edges uv and $u'v'$ in $D(G)$ of type $(2d_G u, 2d_G v)$ by Theorem 3.1. Further u will be connected to v' and v will be connected to u' to form the double graph by definition and these connection edges between two copies of G will also have the same type $(2d_G u, 2d_G v)$. Hence for each edge uv in G of type $(d_G u, d_G v)$, there will be four edges of type $(2d_G u, 2d_G v)$. Hence we state

Theorem 3.2 *For a simple, connected graph G, the Randić type hadi index of the double graph $D(G)$ satisfies the inequality*

$$RH(D(G)) \leq 4(RH(G))^2.$$

Proof.

$$\begin{aligned} RH(D(G)) &= \textstyle\sum_{uv\in E(D(G))} \frac{1}{2^{d_{D(G)}u+d_{D(G)}v}} \\ &= 4\textstyle\sum_{uv\in E(G)} \frac{1}{2^{2d_G u+2d_G v}} \\ &= 4\textstyle\sum_{uv\in E(G)} \left(\frac{1}{2^{d_G u+d_G v}}\right)^2 \\ &\leq 4\left(\textstyle\sum_{uv\in E(G)} \frac{1}{2^{d_G u+d_G v}}\right)^2 \\ &= 4\,(RH(G))^2 . \end{aligned}$$

Theorem 3.3 *The Randić type hadi indices of the double graphs of path, cycle, complete and star graphs are*

$$RH(D(G)) = \begin{cases} \frac{n+5}{64} & \textit{if}\, G = P_n, \\ \frac{8n-12}{2^{2(n+1)}} & \textit{if}\, G = C_n, \\ \frac{2n(n-1)}{2^{4(n-1)}} & \textit{if}\, G = K_n, \\ \frac{4n-4}{2^{2n}} & \textit{if}\, G = K_{1,n-1}. \end{cases}$$

Proof. Consider the path graph P_n. The double graph of P_n will have two sets of vertices. One set consists of four vertices of degree 2 and the other set will have the remaining $2n-4$ vertices of degree 4. Further, there is a total of $4n-4$ edges, 8 of them having end vertex degrees 2 and 4, we say that they are of type $(2,4)$ and $4n-12$ of them are of type $(4,4)$. Hence by the definition of Randić type hadi index, we get

$$RH(D(P_n)) = 8(\tfrac{1}{2^6}) + \left((4n-12)\tfrac{1}{2^{4+4}}\right)$$

and after easy computations, we find the result as $\frac{n+5}{64}$ for the double graph of path graph. The others can be proven by a similar method.

4 Randić type hadi index of complements

Definition 4.1 *The complement of a graph G is a graph denoted by $\overline{G}$ on the same vertex set such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.*

Theorem 4.1 *The Randić type hadi index of some complement graphs are as follows:*

$$RH(\overline{G}) = \begin{cases} 0 & \text{if } G = K_n, \\ RH(K_{n-1}) & \text{if } G = K_{1,n-1}, \\ \frac{n}{4} & \text{if } G = K_{n\times 2}. \end{cases} \tag{4.1}$$

Proof. We consider the complement of a star graph $(K_{1,n-1})$ on n vertices. In the complement of the star graph, one vertex will become isolated and remaining vertices form the complete graph. Thus we get the index value as the same as complete graph K_{n-1}.

5 Randić type hadi index of subdivision graphs, line graphs and line graphs of subdivision graphs

In this section, we calculate the Randić type hadi index of subdivision graphs and line graphs. The proofs are similar to the ones in previous sections are therefore omitted. First we determine the Randić type hadi index of some subdivision graphs:

Theorem 5.1 *The Randić type hadi index of subdivision graphs of path, cycle, complete and star graphs are*

$$RH(S(G)) = \begin{cases} \frac{n}{8} & \text{if } G = P_n, \\ \frac{n}{8} & \text{if } G = C_n, \\ \frac{n^2-n}{2^{n+1}} & \text{if } G = K_n, \\ (n-1)[\frac{1}{2^{n+1}} + \frac{1}{8}] & \text{if } G = K_{1,n-1}. \end{cases} \tag{5.1}$$

Secondly, we calculate the Randić type hadi indices of the tadpole graph, the subdivision graph of the tadpole graph, the line graph of the tadpole graph and the line graph of the subdivision graph of the tadpole graph:

Theorem 5.2

$$RH(G) = \begin{cases} \frac{2n+2k-1}{32} & \text{if } G = T_{n,k} \text{ for } n \geq 3, k \geq 3 \\ \frac{4n+4k-1}{32} & \text{if } G = S(T_{n,k}) \text{ for } n \geq 3, k \geq 3 \\ \frac{4n+4k-7}{64} & \text{if } G = L(T_{n,k}) \text{ for } n \geq 3, k \geq 3 \\ \frac{8n+8k-7}{64} & \text{if } G = L(S(T_{n,k})) \text{ for } n \geq 3, k \geq 3. \end{cases} \tag{5.2}$$

Randić type hadi index of the wheel graph, its subdivision graph, its line graph and also the line graph of its subdivision graph are as follows:

Theorem 5.3

$$RH(G) = \begin{cases} (n-1)[\frac{1}{2^{n+2}} + \frac{1}{64}] & \text{if } G = W_n \\ (n-1)[\frac{1}{2^{n+1}} + \frac{3}{32}] & \text{if } G = S(W_n) \\ \frac{n-1}{256} + \frac{2n-2}{2^{n+4}} + \frac{n^2-3n+2}{2^{2n}} & \text{if } G = L(W_n) \\ \frac{8n+8k-7}{64} & \text{if } G = L(S(W_n)). \end{cases} \quad (5.3)$$

Let $G = P_n \times P_2$ be the ladder graph. Then the Randić type hadi indices of the ladder graph, its subdivision graph, its line graph and the line graph of its subdivision graph are as follows:

Theorem 5.4

$$RH(G) = \begin{cases} \frac{3n+8}{64} & \text{if } G = P_n \times P_2 \\ \frac{3n+2}{16} & \text{if } G = S(P_n \times P_2) \\ \frac{n-1}{256} + \frac{2n-2}{2^{n+4}} + \frac{n^2-3n+2}{2^{2n}} & \text{if } G = L(P_n \times P_2) \\ \frac{8n+8k-7}{64} & \text{if } G = L(S(P_n \times P_2)). \end{cases} \quad (5.4)$$

6 Randić type hadi index of graphene

Graphene is the honeycomb lattice formed by carbon atoms, see Fig. 6. Graphene is attracted researchers due to its applications in electronical goods including semiconductors, electronics, battery energy and composites. Here we compute the Randić type hadi index of

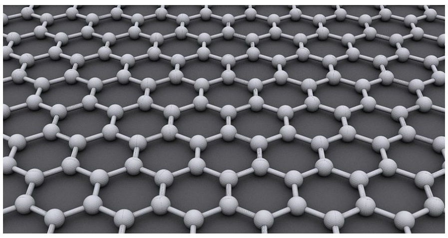


Fig. 2 Graphene

the graphene.

Theorem 6.1 *Randić type hadi index of graphene with x rows of benzene rings such that y benzene rings are placed in each row is given by*

$$RH(graphene) = \begin{cases} \frac{3xy+7x+6y+7}{64} & \text{if } x \neq 1 \\ \frac{9y+15}{64} & \text{if } x = 1. \end{cases} \quad (6.1)$$

Proof. Here we consider the chemical structure called graphene with x rows and y benzene rings in each row. Graphene contains $x+4$ edges between the vertices of degree 2 each in addition to $2x+4y-4$ edges between the vertices of degree 2 and 3. Also there are $3xy-2y-x-1$ edges between the vertices of common degree 3. Thus

Case l. Randić type hadi index of graphene for $x \neq 1$ is

$$RH(graphene) = (x+4)\frac{1}{2^4} + (2x+4y-4)\frac{1}{2^5} + (3xy-2y-x-1)\frac{1}{2^6}$$
$$= \frac{3xy+7x+6y+7}{64}.$$

Case 2. For $x=1$, there exist 6 edges between the vertices with degree 2 each, $4y-4$ edges between the vertices of degrees 2 and 3 and $y-1$ edges between the vertices of degree 3 each. Thus,

$$RH(graphene) = 6\frac{1}{2^4} + (4y-4)\frac{1}{2^5} + (y-1)\frac{1}{2^6}$$
$$= \frac{9y+15}{64}.$$

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