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# Bounded weak solutions for implicit Hadamard fractional differential equations on reflexive Banach spaces

**Saïd Abbas · Mouffak Benchohra · Jamal-Eddine Lazreg · Yong Zhou**

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**Abstract.** *In this article, we present some results concerning the existence of bounded weak solutions for some functional implicit differential equations of Hadamard fractional derivative. The main results are proved by applying Mönch's fixed point theorem associated with the technique of measure of weak noncompactness and the diagonalization method.*

**Keywords.** Fractional differential equation, Pettis–Hadamard integral of fractional order, Pettis–Hadamard fractional derivative, implicit, bounded weak solution, fixed point, unbounded domain, diagonalization process.

**Mathematics Subject Classification (2010):** 26A33

## 1 Introduction

Fractional differential equations have been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [23, 32]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to monographs of Abbas *et al.* [4, 5], Kilbas *et al.* [25] and Zhou [35]. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hadamard fractional derivative; see [1–3, 7, 33]. Implicit functional differential equations have been considered by many authors [6, 12, 27, 34].

Saïd Abbas
Department of Mathematics, University of Saïda–Dr. Moulay Tahar, P.O. Box 138, EN-Nasr, 20000 Saïda, Algeria
E-mail: abbasmsaid@yahoo.fr, said.abbas@univ-saida.dz

Mouffak Benchohra
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail: benchohra@yahoo.com

Jamal-Eddine Lazreg
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
E-mail: lazregjamal@yahoo.fr

Yong Zhou
Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, P.R. China
E-mail: yzhou@xtu.edu.cn

The measure of weak noncompactness was introduced by De Blasi [19]. The strong measure of noncompactness was developed first by Banaŝ and Goebel [11] and subsequently developed and used in many papers; see for example, Akhmerov *et al.* [8], Alvàrez [9], Benchohra *et al.* [17], Guo *et al.* [21], and the references therein. In [17,29] the authors considered some existence results by applying the techniques of the measure of noncompactness. Recently, several researchers obtained other results by application of the technique of measure of weak noncompactness; see [5,14,15], and the references therein.

In [10,13,16], the authors used the diagonalization method to prove some existence of bounded solutions for several classes of scalar fractional differential equations ion the half line. In this paper, we discuss the existence of bounded weak solutions for the following implicit Hadamard fractional differential equation of the form

$$\begin{cases} ({}^H D_1^r u)(t) = f(t, u(t), ({}^H D_1^r u)(t));\ t \in J := [1, \infty), \\ \\ ({}^H I_1^{1-r} u)(t)|_{t=1} = \phi,\ u \text{ is bounded on } J, \end{cases} \tag{1.1}$$

where $\phi \in E,\ f : J \times E \times E \to E$ is a given continuous function, $E$ is a real (or complex) reflexive Banach space with norm $\|\cdot\|_E$ and dual $E^*$, such that $E$ is the dual of a weakly compactly generated Banach space $X$, ${}^H I_1^r$ is the left-sided mixed Hadamard integral of order $r \in (0, 1]$, and ${}^H D_1^r$ is the Hadamard fractional derivative of order $r$.

Our goal in this work is to give some existence results for implicit Hadamard fractional differential equations on an unbounded domain by applying the diagonalization method. This paper initiates the use of measure of weak noncompactness and the diagonalization process for the study of more general fractional differential equations.

## 2 Preliminaries

Let $I_n := [1, n],\ n \in \mathbb{N}^*$ and $C_n := C(I_n)$ be the Banach space of all continuous functions $v$ from $I_n$ into $E$ with the supremum (uniform) norm

$$\|v\|_n := \sup_{t \in I_n} \|v(t)\|_E.$$

As usual, $AC(I_n)$ denotes the space of absolutely continuous functions from $I_n$ into $E$. By $C_{r,\ln}(I_n)$, we denote the weighted space of continuous functions defined by

$$C_{r,\ln}(I_n) = \{w(t) : (\ln t)^r w(t) \in C_n\}$$

with norm

$$\|w\|_{C_{r,\ln}} := \sup_{t \in I_n} \|(\ln t)^r w(t)\|_E.$$

In the following we denote $\|w\|_{C_{r,\ln}}$ by $\|w\|_C$. Let $(E, w) = (E, \sigma(E, E^*))$ be the Banach space $E$ with its weak topology.

**Definition 2.1** *A Banach space $X$ is called weakly compactly generated (WCG, for short) if it contains a weakly compact set whose linear span is dense in $X$.*

**Definition 2.2** *A function $h : E \to E$ is said to be weakly sequentially continuous if $h$ takes each weakly convergent sequence in $E$ to a weakly convergent sequence in $E$ (i.e., for any $(u_n)$ in $E$ with $u_n \to u$ in $(E, w)$ then $h(u_n) \to h(u)$ in $(E, w)$).*

**Definition 2.3** *[30] The function $u : I_n \to E$ is said to be Pettis integrable on $I$ if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s))ds$ for all $\phi \in E^*$, where the integral on the right hand side is assumed to exist in the sense of Lebesgue, (by definition, $u_J = \int_J u(s)ds$).*

Let $P(I_n, E)$ be the space of all $E-$valued Pettis integrable functions on $I$, and $L^1(I_n, \mathbb{R})$ be the Banach space of Lebesgue integrable functions $u : I_n \to \mathbb{R}$. Define the class $P_1(I_n, E)$ by

$$P_1(I_n, E) = \{u \in P(I_n, E) : \varphi(u) \in L^1(I_n, \mathbb{R});\ \text{for every } \varphi \in E^*\}.$$

The space $P_1(I_n, E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi \in E^*,\ \|\varphi\| \le 1} \int_1^n |\varphi(u(x))| d\lambda x,$$

where $\lambda$ stands for a Lebesgue measure on $I_n$.

The following result is due to Pettis (see [[30], Theorem 3.4 and Corollary 3.41]).

**Proposition 2.1** *[30] If $u \in P_1(I, E)$ and $h$ is a measurable and essentially bounded real-valued function, then $uh \in P_1(J, E)$.*

For all that follows, the symbol "$\int$" denotes the Pettis integral.

**Theorem 2.1** *[31] A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

**Theorem 2.2** *[26] Let $D$ be a weakly compact subset of $C(I_n, E)$. Then $D(t)$ is weakly compact subset of $E$ for each $t \in I_n$.*

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [22,25] for a more detailed analysis.

**Definition 2.4** *[22,25] The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1(I_n, E)$, is defined as*

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

*provided the integral exists, where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by*

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt;\ \xi > 0.$$

*Example 1* Let $0 < q < 1$. Then

$${}^H I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q},\ \text{for a.e. } t \in [1, e].$$

**Remark 2.1** Let $g \in P_1(I_n, E)$. For every $\varphi \in E^*$, we have

$$\varphi({}^H I_1^q g)(x) = ({}^H I_1^q \varphi g)(x),\ \text{for a.e. } x \in I_n.$$

Analogous to the Riemann–Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way. Set

$$\delta = x\frac{d}{dx},\ q > 0,\ n = [q] + 1,$$

where $[q]$ is the integer part of $q$, and

$$AC_\delta^n := \{u : I_n \to E : \delta^{n-1}[u(x)] \in AC(I_n)\}.$$

**Definition 2.5** *[22,25] The Hadamard fractional derivative of order $q$ applied to the function $w \in AC_\delta^n$ is defined as*

$$({}^H D_1^q w)(x) = \delta^n({}^H I_1^{n-q} w)(x).$$

*Example 2* Let $0 < q < 1$. Then

$${}^H D_1^q \ln t = \frac{1}{\Gamma(2-q)}(\ln t)^{1-q},\ \text{for a.e. } t \in [1, e].$$

It has been proved (see e.g. Kilbas [[24], Theorem 4.8]) that in the space $L^1(I, E)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$({}^H D_1^q)({}^H I_1^q w)(x) = w(x).$$

From Theorem 2.3 of [25], we have

$$({}^H I_1^q)({}^H D_1^q w)(x) = w(x) - \frac{({}^H I_1^{1-q} w)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

**Lemma 2.1** *Let $h : I_n \to E$ be a continuous function. Then the equation*

$$({}^H D_1^q w)(t) = h(t),$$

*has solutions $w \in L^1(I_n, E)$ defined by*

$$w(t) = \frac{({}^H I_1^{1-q} u)(1)}{\Gamma(q)}(\ln t)^{q-1} + ({}^H I_1^q h)(t).$$

From the above lemma and Lemma 1 of [6], we have the following lemma.

**Lemma 2.2** *Let $f(t, u, z) : I_n \times E \times E \to E$ be a continuous function. Then problem (1.1) is equivalent to the problem of obtaining the solution of the equation*

$$g(t) = f\left(t, \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_1^r g)(t), g(t)\right),$$

*and if $g(\cdot) \in C_n$ is the solution of this equation, then*

$$u(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_1^r g)(t).$$

**Definition 2.6** *[19] Let $E$ be a Banach space, $\Omega_E$ the bounded subsets of $E$ and $B_1$ the unit ball of $E$. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by*

$$\beta(X) = \inf\{\epsilon > 0 : \textit{there exists a weakly compact } \Omega \subset E \text{ such that } X \subset \epsilon B_1 + \Omega\}.$$

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$,
(b) $\beta(A) = 0 \Leftrightarrow\ A$ is relatively weakly compact,
(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,
(d) $\beta(\overline{A}^{\omega}) = \beta(A)$, ($\overline{A}^{\omega}$ denotes the weak closure of $A$),
(e) $\beta(A + B) \leq \beta(A) + \beta(B)$,
(f) $\beta(\lambda A) = |\lambda|\beta(A)$,
(g) $\beta(conv(A)) = \beta(A)$,

The next result follows directly from the Hahn–Banach theorem.

**Proposition 2.2** *Let $E$ be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set $V$ of functions $v : I \to E$ let us denote by

$$V(t) = \{v(t) : v \in V\};\ t \in I,$$

and

$$V(I) = \{v(t) : v \in V,\ t \in I\}.$$

**Lemma 2.3** *[21] Let $H \subset C$ be a bounded and equicontinuous subset. Then the function $t \to \beta(H(t)$ is continuous on $I$, and*

$$\beta_C(H) = \max_{t \in I} \beta(H(t)),$$

*and*

$$\beta\left(\int_I u(s)ds\right) \leq \int_I \beta(H(s))ds,$$

*where $H(s) = \{u(s) : u \in H,\ s \in I\}$, and $\beta_C$ is the De Blasi measure of weak noncompactness defined on the bounded sets of $C$.*

For our purpose we will need the following fixed point theorem:

**Theorem 2.3** *[28] Let $Q$ be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(J, E)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly-sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \ \textit{ is relatively weakly compact}, \tag{2.1}$$

*holds for every subset $V \subset Q$, then the operator $T$ has a fixed point.*

## 3 Existence Results

Let us start by defining what we mean by a weak solution of the problem (1.1).

**Definition 3.1** *By a bounded weak solution of the problem (1.1) we mean a measurable bounded function that satisfies the condition $({}^H I_1^{1-r} u)(t)|_{t=1} = \phi$, and the equation $({}^H D_1^r u)(t) = f(t, u(t), ({}^H D_1^r u)(t))$ on $J$.*

The following hypotheses will be used in the sequel.

$(H_1)$ For a.e. $t \in I_n$, the functions $v \to f(t, v, \cdot)$ and $w \to f(t, \cdot, w)$ are weakly sequentially continuous.

$(H_2)$ For each $v, w \in E$, the function $t \to f(t, v, w)$ is Pettis integrable on $I_n$.
$(H_3)$ There exists $p_n \in C(I_n, [0, \infty))$ such that for all $\varphi \in E^*$, we have

$$|\varphi(f(t,u,v))| \leq p_n(t)\|\varphi\|, \text{ for a.e. } t \in I_n, \text{ and each } u, v \in E.$$

$(H_4)$ For each bounded and measurable set $B \subset E$ and for each $t \in I_n$, we have

$$\beta(f(t, B, {}^H D_1^r B) \leq (\ln t)^{1-r} p_n(t) \beta(B),$$

where ${}^H D_1^r B = \{{}^H D_1^r w : w \in B\}$.

Set

$$p_n^* = \sup_{t \in I_n} p_n(t).$$

**Theorem 3.1** *Assume that the hypotheses* $(H_1) - (H_4)$ *hold. If*

$$L_n := \frac{p_n^* \ln n}{\Gamma(1+r)} < 1, \tag{3.1}$$

*then the problem (1.1) has at least one bounded weak solution defined on* $J$.

**Proof.** The proof will be given in two parts. Fix $n \in \mathbb{N}^*$ and consider the problem

$$\begin{cases} ({}^H D_1^r u)(t) = f(t, u(t), ({}^H D_1^r u)(t)); \ t \in I_n, \\ \\ ({}^H I_1^{1-r} u)(t)|_{t=1} = \phi. \end{cases} \tag{3.2}$$

**Part 1:** We begin by showing that (3.2) has a solution $u_n \in C_n$ with $|u_n| \leq R_n$ for each $t \in I_n$, where

$$R_n > \frac{p_n^* \ln n}{\Gamma(1+r)}.$$

Transform the integral equation (3.2) into a fixed point equation. Consider the operator $N : C_n \to C_n$ defined by:

$$(Nu)(t) = \frac{\phi}{\Gamma(r)} (\ln t)^{r-1} + \int_1^t \left( \ln \frac{t}{s} \right)^{r-1} \frac{g(s)}{s\Gamma(r)} ds, \tag{3.3}$$

where $g(\cdot) \in C_n$ with

$$g(t) = f\left(t, \frac{\phi}{\Gamma(r)} (\ln t)^{r-1} + ({}^H I_1^r g)(t), g(t)\right).$$

First notice that, the hypotheses imply that $t \mapsto \left(\ln \frac{t}{s}\right)^{r-1} \frac{g(s)}{s}$, for a.e. $t \in I_n$, is Pettis integrable, and for each $u \in C$, the function

$$t \mapsto f\left(t, \frac{\phi}{\Gamma(r)} (\ln t)^{r-1} + ({}^H I_1^r g)(t), g(t)\right)$$

is Pettis integrable over $I_n$. Thus, the operator $N$ is well defined. Consider the set

$$\begin{aligned} Q = \Big\{ & u \in C_n : \|u\|_n \leq R_n \text{ and } \|(\ln t_2)^{1-r} u(t_2) - (\ln t_1)^{1-r} u(t_1)\|_E \\ & \leq \frac{p_n^*}{\Gamma(1+r)} (\ln n)^{1-r} \left| \ln \frac{t_2}{t_1} \right|^r \\ & + \frac{p_n^*}{\Gamma(r)} \int_1^{t_1} \left| (\ln t_2)^{1-r} \left( \ln \frac{t_2}{s} \right)^{r-1} - (\ln t_1)^{1-r} \left( \ln \frac{t_1}{s} \right)^{r-1} \right| ds, \ t_1, \ t_2 \in I_n \Big\}. \end{aligned}$$

Clearly, the subset $Q$ is closed, convex end equicontinuous. We shall show that the operator $N$ satisfies all the assumptions of Theorem 2.3. The proof will be given in several steps.

**Step 1:** *$N$ maps $Q$ into itself.*
Let $u \in Q,\ t \in I_n$ and assume that $(Nu)(t) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|(\ln t)^{1-r}(Nu)(t)\|_E = \left|\varphi(|(\ln t)^{1-r}(Nu)(t))\right|$. Thus

$$\|(\ln t)^{1-r}(Nu)(t)\|_E = \left|\varphi\left(\frac{\phi}{\Gamma(r)} + \frac{(\ln t)^{1-r}}{\Gamma(r)}\int_1^t \left(\ln\frac{t}{s}\right)^{r-1}\frac{g(s)}{s}ds\right)\right|,$$

where $g(\cdot) \in C_n$ with

$$g(t) = f\left(t, \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_1^r g)(t), g(t)\right).$$

Then

$$\begin{aligned}
\|(\ln t)^{1-r}(Nu)(t)\|_E &\leq \frac{(\ln t)^{1-r}}{\Gamma(r)}\int_1^t \left(\ln\frac{t}{s}\right)^{r-1}\frac{|\varphi(g(s))|}{s}ds \\
&\leq \frac{p_n^*(\ln n)^{1-r}}{\Gamma(r)}\int_1^t \left(\ln\frac{t}{s}\right)^{r-1}\frac{ds}{s} \\
&\leq \frac{p_n^*\ln n}{\Gamma(1+r)} \\
&\leq R.
\end{aligned}$$

Next, let $t_1, t_2 \in I$ such that $t_1 < t_2$ and let $u \in Q$, with

$$(\ln t_2)^{1-r}(Nu)(t_2) - (\ln t_1)^{1-r}(Nu)(t_1) \neq 0.$$

Then there exists $\varphi \in E^*$ such that

$$\|(\ln t_2)^{1-r}(Nu)(t_2) - (\ln t_1)^{1-r}(Nu)(t_1)\|_E = \left|\varphi((\ln t_2)^{1-r}(Nu)(t_2) - (\ln t_1)^{1-r}(Nu)(t_1))\right|,$$

and $\|\varphi\| = 1$. Then

$$\begin{aligned}
&\|(\ln t_2)^{1-r}(Nu)(t_2) - (\ln t_1)^{1-r}(Nu)(t_1)\|_E \\
&= \left|\varphi((\ln t_2)^{1-r}(Nu)(t_2) - (\ln t_1)^{1-r}(Nu)(t_1))\right| \\
&\leq \left|\varphi\left((\ln t_2)^{1-r}\int_1^{t_2}\left(\ln\frac{t_2}{s}\right)^{r-1}\frac{g(s)}{s\Gamma(r)}ds - (\ln t_1)^{1-r}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{r-1}\frac{g(s)}{s\Gamma(r)}ds\right)\right|,
\end{aligned}$$

where $g(\cdot) \in C_n$ with

$$g(t) = f\left(t, \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_1^r g)(t), g(t)\right).$$

Then

$$
\begin{aligned}
&\|(\ln t_2)^{1-r}(Nu)(t_2)-(\ln t_1)^{1-r}(Nu)(t_1)\|_E\\
&\leq (\ln t_2)^{1-r}\int_{t_1}^{t_2}\left|\ln\frac{t_2}{s}\right|^{r-1}\frac{|\varphi(g(s))|}{s\Gamma(r)}ds\\
&\quad+\int_1^{t_1}\left|(\ln t_2)^{1-r}\left(\ln\frac{t_2}{s}\right)^{r-1}-(\ln t_1)^{1-r}\left(\ln\frac{t_1}{s}\right)^{r-1}\right|\frac{|\varphi(g(s))|}{s\Gamma(r)}ds\\
&\leq (\ln t_2)^{1-r}\int_{t_1}^{t_2}\left|\ln\frac{t_2}{s}\right|^{r-1}\frac{p_n(s)}{\Gamma(r)}ds\\
&\quad+\int_1^{t_1}\left|(\ln t_2)^{1-r}\left(\ln\frac{t_2}{s}\right)^{r-1}-(\ln t_1)^{1-r}\left(\ln\frac{t_1}{s}\right)^{r-1}\right|\frac{p_n(s)}{\Gamma(r)}ds.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
&\|(\ln t_2)^{1-r}(Nu)(t_2)-(\ln t_1)^{1-r}(Nu)(t_1)\|_E\\
&\leq \frac{p_n^*}{\Gamma(1+r)}(\ln n)^{1-r}\left|\ln\frac{t_2}{t_1}\right|^{r}\\
&\quad+\frac{p_n^*}{\Gamma(r)}\int_1^{t_1}\left|(\ln t_2)^{1-r}\left(\ln\frac{t_2}{s}\right)^{r-1}-(\ln t_1)^{1-r}\left(\ln\frac{t_1}{s}\right)^{r-1}\right|ds.
\end{aligned}
$$

Hence $N(Q)\subset Q$.

**Step 2:** *$N$ is weakly-sequentially continuous.*

Let $(u_m)$ be a sequence in $Q$ and let $(u_m(t))\to u(t)$ in $(E,\omega)$ for each $t\in I_n$. Fix $t\in I_n$, since $f$ satisfies the assumption $(H_1)$, we have $f(t,u_m(t),^H D_1u_m(t))$ converges weakly to $f(t,u(t),^H D_1u(t))$. Hence the Lebesgue dominated convergence theorem for Pettis integral (see [20]) implies $(Nu_m)(t)$ converges weakly to $(Nu)(t)$ in $(E,\omega)$, for each $t\in I_n$. Thus, $N(u_m)\to N(u)$. Hence, $N:Q\to Q$ is weakly-sequentially continuous.

**Step 3:** *The implication (2.1) holds.*

Let $V$ be a subset of $Q$ such that $\overline{V}=\overline{conv}(N(V)\cup\{0\})$. Obviously

$$V(t)\subset\overline{conv}(NV)(t))\cup\{0\}),\ \forall t\in I_n.$$

Further, as $V$ is bounded and equicontinuous, by Lemma 3 in [18] the function $t\to v(t)=\beta(V(t))$ is continuous on $I_n$. From $(H_3)$, $(H_4)$, Lemma 2.3 and the properties of the measure $\beta$, for any $t\in I$, we have

$$
\begin{aligned}
(\ln t)^{1-r}v(t)&\leq\beta((\ln t)^{1-r}(NV)(t)\cup\{0\})\\
&\leq\beta((\ln t)^{1-r}(NV)(t))\\
&\leq\frac{(\ln n)^{1-r}}{\Gamma(r)}\int_1^t\left|\ln\frac{t}{s}\right|^{r-1}\frac{p_n(s)\beta(V(s))}{s}ds\\
&\leq\frac{(\ln n)^{1-r}}{\Gamma(r)}\int_1^t\left|\ln\frac{t}{s}\right|^{r-1}\frac{(\ln s)^{1-r}p_n(s)v(s)}{s}ds\\
&\leq\frac{p_n^*\ln n}{\Gamma(1+r)}\|v\|_{C_n}.
\end{aligned}
$$

Thus

$$\|v\|_{C_n}\leq L_n\|v\|_{C_n}.$$

From (3.1), we get $\|v\|_{C_n} = 0$, that is $v(t) = \beta(V(t)) = 0$, for each $t \in I_n$ and then by Theorem 2 in [26], $V$ is weakly relatively compact in $C_n$. Applying now Theorem 2.3, we conclude that $N$ has a fixed point which is a weak solution of the problem (3.2).

**Part 2:** The diagonalization process.

Now, we use the following diagonalization process. For $k \in \mathbb{N}^*$ let

$$\begin{cases} w_k(t) = u_{n_k}(t);\ t \in [1, n_k], \\ w_k(t) = u_{n_k}(n_k);\ t \in [n_k, \infty). \end{cases}$$

Here $\{n_k\}_{k\in\mathbb{N}^*}$ is a sequence of numbers satisfying

$$1 < n_1 < n_2 < \ldots n_k < \ldots \uparrow \infty.$$

Let $S = \{w_k\}_{k=1}^{\infty}$. Notice that

$$|w_{n_k}(t)| \leq R_n : \text{ for } t \in [1, n_1],\ k \in \mathbb{N}^*.$$

Also, if $k \in \mathbb{N}^*$ and $t \in [1, n_1]$, we have

$$w_{n_k}(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + \int_1^{n_1} \left(\ln \frac{t}{s}\right)^{r-1} \frac{g(s)}{s\Gamma(r)} ds,$$

where $g(\cdot) \in C_{n_1}$ with

$$g(t) = f\left(t, \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_1^r g)(t), g(t)\right).$$

Thus, for $k \in \mathbb{N}^*$ and $t, x \in [1, n_1]$, we have

$$|(\ln t)^{1-\gamma} w_{n_k}(t) - (\ln x)^{1-\gamma} w_{n_k}(x)| \leq \int_1^{n_1} \left| \left(\ln \frac{t}{s}\right)^{r-1} - \left(\ln \frac{x}{s}\right)^{r-1} \right| \frac{g(s)}{s\Gamma(r)} ds.$$

Hence

$$|(\ln t)^{1-\gamma} w_{n_k}(t) - (\ln x)^{1-\gamma} w_{n_k}(x)| \leq \frac{p_1^*}{\Gamma(r)} \int_1^{n_1} \left| \left(\ln \frac{t}{s}\right)^{r-1} - \left(\ln \frac{x}{s}\right)^{r-1} \right| \frac{ds}{s}.$$

From the above and Step 3, Arzelà–Ascoli theorem guarantees that there is a subsequence $P_1^*$ of $\mathbb{N}^*$ and a function $z_1 \in C([1, n_1], E)$ with $w_{n_k} \to z_1$ as $k \to \infty$ in $C([1, n_1], E)$ through $P_1^*$. Let $P_1 = P_1^* - \{2\}$.

Notice that

$$|w_{n_k}(t)| \leq R_n : \text{ for } t \in [1, n_2],\ k \in \mathbb{N}^*.$$

Also, if $k \in \mathbb{N}^*$ and $t, x \in [1, n_2]$, we have

$$|(\ln t)^{1-\gamma} w_{n_k}(t) - (\ln x)^{1-\gamma} w_{n_k}(x)| \leq \frac{p_2^*}{\Gamma(r)} \int_1^{n_2} \left| \left(\ln \frac{t}{s}\right)^{r-1} - \left(\ln \frac{x}{s}\right)^{r-1} \right| \frac{ds}{s}.$$

The Arzelà–Ascoli theorem guarantees that there is a subsequence $P_2^*$ of $P_1$ and a function $z_2 \in C([1, n_2], E)$ with $w_{n_k} \to z_2$ as $k \to \infty$ in $C([1, n_2], E)$ through $P_2^*$. Note that $z_1 = z_2$ on $[1, n_1]$ since $P_2^* \subset P_1$. Let $P_2 = P_2^* - \{3\}$. Proceed inductively to obtain for $m = 4, 5, \ldots$ a subsequence $P_m^*$ of $P_{m-1}$ and a function $z_m \in C([1, n_m], E)$ with $w_{n_k} \to z_m$ as $k \to \infty$ in $C([1, n_m], E)$ through $P_m^*$. Let $P_m = P_m^* - \{m+1\}$.

Define a function $u$ as follows. Fix $t \in (1,\infty)$ and let $m \in \mathbb{N}^*$ with $t \leq n_m$. Then define $u(t) = z_m(t)$. Then $u \in C((1,\infty),E))$ and $|u(t)| \leq R_n : \; for\ t \in [1,\infty)$.
Again fix $t \in (1,\infty)$ and let $m \in \mathbb{N}^*$ with $t \leq n_m$. Then for $n \in \mathbb{N}_m$ we have

$$u_{n_k}(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + \int_1^{n_m} \left(\ln \frac{t}{s}\right)^{r-1} \frac{g(s)}{s\Gamma(r)} ds,$$

where $g(\cdot) \in C_{n_m}$ with

$$g(t) = f\left(t, \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + ({}^H I_1^r g)(t), g(t)\right).$$

Let $n_k \to \infty$ through $\mathbb{N}_m$ to obtain

$$z_m(t) = \frac{\phi}{\Gamma(r)}(\ln t)^{r-1} + \int_1^{n_m} \left(\ln \frac{t}{s}\right)^{r-1} \frac{g(s)}{s\Gamma(r)} ds.$$

We can use this method for each $t \in [1, n_m]$ and for each $m \in \mathbb{N}^*$. Thus

$$({}^H D_1^r u)(t) = f(t, u(t), ({}^H D_1^r u)(t)); \; for\ t \in [1, n_m]$$

for each $m \in \mathbb{N}^*$ and the constructed function $u$ is a bounded weak solution of the problem (1.1).

## 4 An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_m, \ldots), \sum_{m=1}^{\infty} |u_m| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{m=1}^{\infty} |u_m|.$$

We consider the following problem of implicit Hadamard fractional differential equation

$$\begin{cases} ({}^H D_1^{\frac{1}{2}} u_n)(t) = f_m(t, u(t), ({}^H D_1^{\frac{1}{2}} u)(t)), \;\; t \in [1,\infty), \\ ({}^H I_1^{\frac{1}{2}} u)(t)|_{t=1} = 0, \; u \text{ is bounded on } [1,\infty), \end{cases} \tag{4.1}$$

where

$$f_m(t, u(t), ({}^H D_1^{\frac{1}{2}} u)(t)) = \frac{c_n t^2}{2(1+t^2)(1 + \|u(t)\|_E + \|{}^H D_1^{\frac{1}{2}} u(t)\|_E)} \left(e^{-7} + e^{-t-5}\right) u_m(t),$$

for each $t \in [1, n]; \; n \in \mathbb{N}^* - \{1\}$, with

$$u = (u_1, u_2, \ldots, u_m, \ldots), \; and\ c_n := \frac{e^4}{8 \ln n} \Gamma\left(\frac{1}{2}\right); \; n \in \mathbb{N}^* - \{1\}.$$

Set

$$f = (f_1, f_2, \ldots, f_m, \ldots).$$

For each $u \in E$ and $t \in [1,n]$, we have

$$|f_m(t,u(t),({}^HD_1^{\frac{1}{2}}u)(t))| \leq \frac{c_nt^2}{2(1+t^2)}\left(e^{-7}+e^{-t-5}\right)\frac{|u_m(t)|}{1+\|u(t)\|_E+\|({}^HD_1^{\frac{1}{2}}u)(t))\|_E}.$$

This gives

$$\|f(t,u(t),({}^HD_1^{\frac{1}{2}})(t))\|_E \leq \frac{c_nt^2}{2(1+t^2)}\left(e^{-7}+e^{-t-5}\right).$$

Thus, for all $\varphi \in E^* = l^\infty$, we have

$$|\varphi(f(t,u(t),({}^HD_1^{\frac{1}{2}})(t)))| \leq \frac{c_nt^2}{2(1+t^2)}\left(e^{-7}+e^{-t-5}\right)\|\varphi\|; \text{ for a.e. } t \in [1,n] \text{ and each } u \in E.$$

Hence, the hypothesis $(H_3)$ is satisfied with

$$p_n(t) = \frac{c_nt^2}{2(1+t^2)}\left(e^{-7}+e^{-t-5}\right).$$

So; $p_n^* = c_ne^{-4}$. Condition (3.1) holds, indeed,

$$\frac{p_n^*\ln n}{\Gamma(1+r)} = \frac{c_n\ln n}{e^4\Gamma(\frac{3}{2})} = \frac{1}{4} < 1.$$

Simple computations show that all conditions of Theorem 3.1 are satisfied. It follows that the problem (4.1) has at least one bounded weak solution on $[1,\infty)$.

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