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On regular solvability of a boundary value problem for a fourth order operator differential equation on a finite segment

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Abstract. *The paper is devoted to the study a regular solvability of some boundary value problems for a fourth order operator-differential equations on a finite segment. Using the theorem on the estimations of inter mediate derivatives, sufficient conditions for regular solvability of the given problem are obtained. Sufficient conditions for regular solvability of a boundary value problem are expressed by means of the norms of the operators of intermediate derivatives.*

Keywords. Operator · operator-differential equations · boundary conditions · regular solvability · Fredholm solvability · Sobolev space · Hilbert space · norm · scalar product · self-adjoint operators · vector-function · isomorphism.

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1 Introduction. Problem statement

Let H be a separable Hilbert space with a scalar product, $(x, y)_H$, where $x, y \in H$.

Denote by $\tilde{H} = L_2([0,1]; H)$ a Hilbert space of all vector-functions f determined on the segment $[0,1]$ with the values in H, that have the norm

$$\|f\|_{L_2([0,1];H)} = \left(\int_0^1 \|f(t)\|_H^2 \, dt \right)^{1/2}.$$

Let A be a self-adjoint positive-definite operator in space H with the domain of definition $D(A)$. We define the operator A^p $(p \geq 0)$ with domain of definition $H_p = D(A^p)$ that are Hilbert spaces with regard to the scalar product

$$(x, y)_{H_p} = (A^p x, A^p y)_H, \; x, y \in D(A^p).$$

For $p = 0$ we assume $H_0 = H, \; (x, y)_{H_0} = (x, y), x, y \in H$.

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We define the Hilbert space

$$W_2^4([0,1];H) = \left\{u(t); u^{(4)} \in L_2([0,1];H),\ A^4 u \in L_2([0,1];H)\right\}$$

with the scalar product

$$(u,v)_{W_2^4([0,1];H)} = \int_0^1 \left(u^{(4)}(t), v^{(4)}(t)\right)_H dt + \int_0^1 \left(A^4 u(t), A^4 v(t)\right)_H dt$$

and with the norm

$$(u,v)_{W_2^4([0,1];H)} = \left\|u^{(4)}\right\|^2_{L_2([0,1];H)} + \left\|A^4 u\right\|^2_{L_2([0,1];H)}.$$

From the theorem on traces it follows that [10] if $u(\cdot) \in W_2^4([0,1];H)$, then $u^{(k)}(0) \in H_{4-k-\frac{1}{2}}$, $k = 1,2,3$. The derivatives $u^{(k)}(t) \equiv \frac{d^k u(t)}{dt^k}$, $k = 1,2,3$ are understood in the sense of distributions theory.

In separable Hilbert space H we consider the boundary value problem

$$Lu = \frac{d^4 u(t)}{dt^4} + A^4 u(t) + \sum_{j=0}^{4} A_{4-j} u^{(j)}(t) = f(t),\ t \in [0,1], \tag{1.1}$$

$$u(0) = u'(0) = 0,\ \ u(1) = u'(1) = 0. \tag{1.2}$$

Here $f(t)$ and $u(t)$ are the vector-functions determined almost everywhere in $[0,1]$ with the values in H. We will assume that the coefficients of the equation (1.1) satisfy the following conditions:

1) A is a self-adjoint, positive-definite operator in the space H.

2) Let A_j be the linear operators such that $B_j = A_j A^{-j}$, $j = \overline{0,4}$ and let B_j are bounded operators in H.

Definition 1.1 *If the vector-function* $u(t) \in W_2^4([0,1];H)$ *satisfies equation* (1.1) *almost everywhere in* $[0,1]$*, then we call it a regular solution of equation* (1.1)

Definition 1.2 *If for any* $f(t) \in L_2([0,1];H)$ *there exists a regular solution of equation* (1.1) *for which boundary conditions* (1.2) *are fulfilled in the sense*

$$\lim_{t\to+0} \|u(t)\|_{H_{7/2}} = 0,\ \lim_{t\to+0} \left\|u'(t)\right\|_{H_{5/2}} = 0, \lim_{t\to1-0} \|u(t)\|_{H_{7/2}} = 0,\ \lim_{t\to1-0} \left\|u'(t)\right\|_{H_{5/2}} = 0,$$

then the vector-function $u(t)$ *is said to be a regular solution of the boundary value problem* (1.1)*,* (1.2)

Definition 1.3 *If for all* $f \in L_2([0,1];H)$ *boundary value problem* (1.1)*,* (1.2) *has a regular solution and the estimation*

$$\|u\|_{W_2^4([0,1];H)} \le c\,\|f\|_{L_2([0,1];H)}$$

is fulfilled, then boundary value problem (1.1)*,* (1.2) *is called uniquely solvable or regularly solvable.*

In the space $\overset{\circ}{W}{}_2^4\left([0,1];H\right)$ we define the following operators

$$L_0u(t)=\frac{d^4u(t)}{dt}+A^4u(t),\quad u\in \overset{\circ}{W}{}_2^4\left([0,1];H\right)$$

$$L_1u(t)=\sum_{j=0}^{4}A_{4-j}u^{(j)}(t),\quad u\in \overset{\circ}{W}{}_2^4\left([0,1];H\right).$$

In [15] it is shown that the operator L_0 realizes an isomorphism of the space $\overset{\circ}{W}{}_2^4\left([0,1];H\right)$ onto the space $L_2\left([0,1];H\right)$ and $L_0^{-1}:L_2\left([0,1];H\right)\to W_2^4\left([0,1];H\right)$ is a bounded operator.

Note that solvability of various problems for higher order differential equations was studied in [1,2,4,7,8,11,13,14,15] and others.

In [12], a new approach to the solvability of boundary value problems for operator-differential equations was developed. This approach consists of finding exact values or upper bounds of the norms of intermediate derivatives operators participating on the perturbed part of operators participating on the perturbed part of operator-differential equations.These estimations allow to establish exact conditions of solvability boundary value problems for these equations in the terms of operator coefficients.

There estimations are of independent mathematical interest as well. Such estimations, for example, are closely connected with the problems of the best approximation of differentiation operator by bounded operators. In this direction we can note the papers [3,6,9,10] etc.

To complete the statement of the proof of the basic theorem of the present work we give formulation of the theorem on intermediate derivatives.

Theorem on intermediate derivatives [5]. Let condition 1) be fulfilled, and $A\geq 0$. Then for all $u\in \overset{\circ}{W}{}_2^4\left([0,1];H\right)$ we have the following inequalities

$$\left\|A^4u\right\|_{L_2([0,1];H)}\leq c_0\left\|L_0u\right\|_{L_2([0,1];H)}, \tag{1.3}$$

$$\left\|A^3u'\right\|_{L_2([0,1];H)}\leq c_1\left\|L_0u\right\|_{L_2([0,1];H)}, \tag{1.4}$$

$$\left\|A^2u''\right\|_{L_2([0,1];H)}\leq c_2\left\|L_0u\right\|_{L_2([0,1];H)}, \tag{1.5}$$

$$\left\|Au'''\right\|_{L_2([0,1];H)}\leq c_3\left\|L_0u\right\|_{L_2([0,1];H)}, \tag{1.6}$$

$$\left\|\frac{d^4u}{dt^4}\right\|_{L_2([0,1];H)}\leq c_4\left\|L_0u\right\|_{L_2([0,1];H)}, \tag{1.7}$$

where

$$c_0=c_4=1,\ c_2=\frac{1}{2},\ c_1=\frac{1}{\sqrt{2}},\ c_3=\sqrt{3}+1+\frac{1}{2}\sqrt{10\left(2+\sqrt{3}\right)}. \tag{1.8}$$

2 Main results

One of the main results of the present paper is the following theorem.

Theorem 2.1 *Let A be a self-adjoint, positive-definite operator, the operators $B_j = A_j A^{-j}$, $j = \overline{0,4}$ be bounded operators in the space H. Assume that in addition to these conditions, the following algebraic condition is also fulfilled:*

$$h = \sum_{j=0}^{4} c_j \left\| B_{4-j} \right\| < 1,$$

where the coefficients $c_j, j = \overline{0,4}$ are determined by equality (1.8). *Then problem* (1.1), (1.2) *is regularly solvable.*

Proof. As noted above, the operator $L_0 : \overset{\circ}{W}{}_2^4 ([0,1]; H) \to L_2 ([0,1]; H)$ is an isomorphism, i.e. it maps the space $\overset{\circ}{W}{}_2^4 ([0,1]; H)$ into $L_2 ([0,1]; H)$ and there exists a bounded inverse operator $L_0^{-1} : L_2 ([0,1]; H) \to \overset{\circ}{W}_2 ([0,1]; H)$.

Show that the operator L_1 boundedly acts from the space $\overset{\circ}{W}{}_2^4 ([0,1]; H)$ to the space $L_2 ([0,1]; H)$.

Indeed, for all $u \in \overset{\circ}{W}{}_2^4 ([0,1]; H)$ we have:

$$\begin{aligned}
\|L_1 u\|_{L_2([0,1];H)} &= \left\| \sum_{j=0}^{4} A_{4-j} u^{(j)} \right\|_{L_2([0,1];H)} \\
&= \left\| \sum_{j=0}^{4} A_{4-j} A^{-(4-j)} A^{4-j} u^{(j)} \right\|_{L_2([0,1];H)} \\
&\le \sum_{j=0}^{4} \left\| A_{4-j} A^{-(4-j)} \right\| \cdot \left\| A^{4-j} u^{(j)} \right\|_{L_2([0,1];H)} \\
&= \sum_{j=0}^{4} \left\| B_{4-j} \right\| \cdot \left\| A^{4-j} u^{(j)} \right\|_{L_2([0,1];H)}.
\end{aligned}$$

By the theorem on intermediate derivatives, we have the inequality

$$\left\| A^{4-j} u^{(j)} \right\|_{L_2([0,1];H)} \le c_j \|u\|_{W_2^4([0,1];H)}, j = \overline{0,4}. \tag{2.1}$$

Then we get

$$\|L_1 u\|_{L_2([0,1];H)} \le \sum_{j=0}^{4} \left\| B_{4-j} \right\| \cdot c_j \|u\|_{W_2^4([0,1];H)} \le \text{const} \cdot \|u\|_{W_2^4([0,1];H)}.$$

Thus, we get the boundedness of the operator

$$L_1 : \overset{\circ}{W}{}_2^4 ([0,1]; H) \to L_2 ([0,1]; H).$$

We represent problem (1.1), (1.2) in the form of the operator equation

$$L_0 u + L_1 u = f.$$

Here $u \in \overset{\circ}{W}{}_2^4\left([0,1];H\right)$, $f \in L_2\left([0,1];H\right)$.

Since $L_0 : \overset{\circ}{W}{}_2^4\left([0,1];H\right) \to L_2\left([0,1];H\right)$ is an isomorphism, we have

$L_0\left(\overset{\circ}{W}{}_2^4\left([0,1];H\right)\right) = L_2\left([0,1];H\right).$

Denote $u = L_0^{-1}v$. Thus, in the space $L_2\left([0,1];H\right)$ we get the operator equation

$$v + L_1 L_0^{-1} v = f, \quad \text{where} \;\; v,\, f \in L_2\left([0,1];H\right). \tag{2.2}$$

For any $v(t) \in L_2\left([0,1];H\right)$ we have

$$\begin{aligned}\left\|L_1 L_0^{-1} v\right\|_{L_2([0,1];H)} &= \left\|L_1 u\right\|_{L_2([0,1];H)} = \left\|\sum_{j=0}^{4} A_{4-j} u^{(j)}\right\|_{L_2([0,1];H)} \\ &\le \sum_{j=0}^{4}\left\|A_{4-j}A^{-(4-j)}\right\| \cdot \left\|A^{4-j}u^{(j)}\right\|_{L_2([0,1];H)} \\ &\le \sum_{j=1}^{4}\left\|B_{4-j}\right\| \cdot \left\|A^{4-j}u^{(j)}\right\|_{L_2([0,1];H)}.\end{aligned} \tag{2.3}$$

If we use inequality (2.1) for the operators of intermediate derivarives, where the conditions $c_j, j=\overline{0,4}$ are determined by equalities (1.8), we obtain:

$$\left\|L_1 L_0^{-1}\right\|_{L_2([0,1];H)} \le \Big(\sum_{j=0}^{4} c_j \left\|B_{4-j}\right\|\Big) \cdot \|v\|_{L_2([0,1];H)} = h\,\|v\|_{L_2([0,1];H)}.$$

Since $0<h<1$, we get that the operator $E + L_1 L_0^{-1}$ has a bounded inverse determined everywhere in the space $L_2\left([0,1];H\right)$. Then from equation (2.2) we get $v = \left(E + L_1 L_0^{-1}\right) f$, $u = L_0^{-1} v = L_0^{-1}\left(E + L_1 L_0^{-1}\right)^{-1} f$. Then we have:

$$\|u\|_{W_2^4([0,1];H)} \le \left\|L_0^{-1}\left(E + L_1 L_0^{-1}\right)^{-1} f\right\|_{L_2([0,1];H)}$$

$$\le \left\|L_0^{-1}\right\|_{L_2([0,1];H)\to \overset{\circ}{W}{}_2^4([0,1];H)} \cdot \left\|\left(E + L_1 L_0^{-1}\right)^{-1}\right\|_{L_2([0,1];H)\to L_1([0,1];H)} \|f\|_{L_2([0,1];H)}$$

$$\le \left\|L_0^{-1}\right\|_{L_2([0,1];H)\to \overset{\circ}{W}{}_2^4([0,1];H)} \cdot \frac{1}{1-h}\|f\|_{L_2([0,1];H)} = const\,\|f\|_{L_2([0,1];H)}.$$

Hence it follows regular solvability of boundary value problem (1.1), (1.2).

Theorem 2.1 is proved.

Now let us consider the following boundary value problem

$$\frac{\alpha^4 u(t)}{dt} + A^4 u(t) + \sum_{j=0}^{n} A_{4-j} u^{(j)}(t) = 0, \quad t \in [0,1], \tag{2.4}$$

$$u(0) = \varphi_0, \; u'(0) = \varphi_1 \;\; u(1) = \psi_0, \; u'(1) = \psi_1. \tag{2.5}$$

Definition 2.1 *If for $\varphi_0 \in H_{7/2}, \varphi_1 \in H_{5/2}$, $\psi_0 \in H_{7/2}, \psi_1 \in H_{5/2}$, equation* (2.4) *has a regular solution $u(t)$ and this solution satisfies boundary condition* (2.5) *in the sense*

$$\lim_{t \to +0} \|u(t) - \varphi_0\|_{H_{7/2}} = 0, \quad \lim_{t \to +0} \left\|u'(t) - \varphi_1\right\|_{H_{5/2}} = 0,$$

$$\lim_{t \to 1-0} \|u(t) - \psi_0\|_{H_{7/2}} = 0, \quad \lim_{t \to 1-0} \left\|u'(t) - \psi_1\right\|_{H_{5/2}} = 0,$$

then the function $u(t)$ is called a regular solution of problem (2.4)-(2.5).

Definition 2.2 *If for all, $\varphi_0 \in H_{7/2}, \varphi_1 \in H_{5/2}$, $\psi_0 \in H_{7/2}, \psi_1 \in H_{5/2}$, problem* (2.4)-(2.5) *has a regular solution $u(t)$ and for this solution the inequality*

$$\|u\|_{W_2^4([0,1];H)} \leq c\left(\|\varphi_0\|_{H_{7/2}} + \|\varphi_1\|_{5/2} + \|\psi_0\|_{H7/2} + \|\psi_1\|_{H_{5/2}}\right)$$

holds the problem (2.4)-(2.5) *is said to be regularly solvable.*

Theorem 2.2 *Let the coefficients of equation* (2.4) *satisfy all conditions of theorem 2.1, i.e. A be a self-adjoint, positive-definite operator, the operators $B_j = A_j A^{-j}$ $\left(j = \overline{0,4}\right)$ be bounded in the space H and the algebraic condition $h = \sum\limits_{j=0}^{4} c_j \|B_{4-j}\| < 1$ be fulfilled. Then problem* (2.4)-(2.5) *is regularly solvable.*

Proof. For $A_j = 0$ $(j = \overline{0,4})$ the theorem was proved in [15]. Now we prove the theorem for $A_j \neq 0$.

We look for the regular solution of problem (2.4)-(2.5) in the form $u(t) = v(t) + u_0(t)$. Here $u \in \overset{0}{W}{}_2^{4}([0,1];H)$, $v(0) = v'(0) = 0$, $v(1) = v'(1) = 0$ and

$$u_0(t) = e^{w_1 tA} x_1 + e^{w_2 tA} x_2 + e^{w_3 (t-1)A} x_3 + e^{w_4 (t-1)A} x_4.$$

Unknown vectors $x_1, x_2, x_3, x_4 \in H_{7/4}$ are determined from the boundary conditions

$$u_0(0) = \varphi_0, u_0'(0) = \varphi_1, \ u_0(1) = \psi_0, \ u_0'(1) = \psi_1. \tag{2.6}$$

Here w_1, w_2, w_3, w_4 are the roots of the equation $w^4 + 1 = 0$. It is easy to see that $w_1 = -\frac{1}{\sqrt{2}}(1+i)$,$w_2 = -\frac{1}{\sqrt{2}}(1-i)$, $w_3 = \frac{1}{\sqrt{2}}(1+i)$,$w_3 = \frac{1}{\sqrt{2}}(1-i)$.

As can be seen $Rew_1 < 0$, $Rew_2 < 0$, $Rew_3 > 0$, $Rew_4 > 0$. Then the function $u_0(t) \in W_2^4([0,1];H)$ is a regular solution of the problem

$$\frac{du_0(t)}{dt^4} + Au_0(t) = 0, \ \ t \in [0,1], \tag{2.7}$$

$$u_0(0) = \varphi_0, \ u_0'(0) = \varphi_1, \ u_0(1) = \psi_0, \ u_0'(1) = \psi_1 \tag{2.8}$$

and satisfies the estimation

$$\|u_0\|_{W_2^4([0,1];H)} \leq \text{const}\left(\|\varphi_0\|_{H_{7/2}} + \|\varphi_1\|_{H_{5/2}} + \|\psi_0\|_{H_{7/2}} + \|\psi_1\|_{H_{5/2}}\right). \tag{2.9}$$

Since $v(t) = u(t) - u_0(t)$, we get

$$v(0) = u(0) - u_0(0) = 0, \ v'(0) = u'(0) - u_0'(0) = 0,$$

$$v(1) = u(1) - u_0(1) = 0, \ v'(1) = u'(1) - u_0'(1) = 0.$$

If we take into account these conditions, then for $v(t)$ we get the following problem

$$\frac{d^4(u_0(t)+v(t))}{dt^4}+A^4(u(t)+v(t))+\sum_{j=0}^{4}A_{4-j}\left(u_0(t)+v(t)\right)=0, \tag{2.10}$$

$$v(0)=0, \quad v'(0)=0, \quad v(1)=v'(1)=0. \tag{2.11}$$

Hence we get

$$\left(\frac{d^4u_0(t)}{dt^4}+A^4u_0(t)\right)+\left(\frac{d^4v(t)}{dt^4}+A^4v(t)\right)+\sum_{j=0}^{4}A_{4-j}v^{(j)}(t)=-\sum_{j=0}^{4}A_ju_0^{(j)}(t),$$

$$v(0)=\ v'(0)=0, \quad v(1)=v'(1)=0.$$

Since

$$\frac{d^4u_0(t)}{dt^4}+A^4u_0(t)=0, \quad t\in[0,1]$$

we finally get the following boundary value problem with respect to the function $v(t)$

$$\frac{d^4v(t)}{dt^4}+A^4v(t)+\sum_{j=0}^{4}A_{4-j}v^{(j)}(t)=g(t), \ \ t\in[0,1], \tag{2.12}$$

$$v(0)=\ v'(0)=0, \quad v(1)=v'(1)=0, \tag{2.13}$$

where $g(t)=-\sum\limits_{j=0}^{4}A_{4-j}u_0^{(j)}(t)$.

Show that $g\in L_2\left([0,1];H\right)$. Indeed,

$$\|g\|_{L_2([0,1];H)}=\left\|-\sum_{j=0}^{4}A_{4-j}u_0^{(j)}\right\|_{L_2([0,1];H)}$$

$$=\left\|-\sum_{j=0}^{4}\left(A_{4-j}A^{-(4-j)}\right)\cdot\left(A^{-(4-j)}u_0^{(j)}\right)\right\|_{L_2([0,1];H)}$$

$$\le\sum_{j=0}^{4}\left\|A_{4-j}A^{-j}\right\|\cdot\left\|A^{4-j}u_0^{(j)}\right\|_{L_2([0,1];H)}$$

$$\le\sum_{j=0}^{4}\|B_{4-j}\|\cdot\left\|A^{4-j}u_0^{(j)}\right\|_{L_2([0,1];H)}.$$

On the other hand, since $u_0(t)$ is a regular solution of problem (2.6)-(2.7) we have inequality (2.8). Then for the function $g(t)$ we get

$$\|g\|_{L_2([0,1];H)}\le\sum_{j=0}^{4}\|B_{4-j}\|\cdot\left\|A^{4-j}u_0^{(j)}\right\|_{L_2([0,1];H)}\le\sum_{j=0}^{4}\|B_{4-j}\|\ \|u_0\|_{W_2^4([0,1];H)}$$

$$\le const\left(\|\varphi_0\|_{H_{7/2}}+\|\varphi_1\|_{H_{5/2}}+\|\psi_0\|_{H_{7/2}}+\|\psi_1\|_{H_{5/2}}\right). \tag{2.14}$$

Since $\varphi_0,\ \psi_0 \in H_{7/2},\ \varphi_1, \psi_1 \in H_{5/2}$, we have $g \in L_2([0,1];H)$. Then by Theorem 2.1, problem (2.14)-(2.12) is regularly solvable, the function $v_0(t)$ is a regular solution of problem (2.14)-(2.12) and we have the following inequality

$$\|v\|_{W_2^4([0,1];H)} \le const\, \|g\|_{L_2([0,1];H)}\,.$$

By inequality (2.8) we get

$$\|v\|_{W_2^4([0,1];H)} \le \mathrm{const}\left(\|\varphi_0\|_{H_{7/2}} + \|\varphi_1\|_{H_{5/2}} + \|\psi_0\|_{H_{7/2}} + \|\psi_1\|_{H_{5/2}}\right).$$

We finally have

$$\|u\|_{W_2^4([0,1];H)} \le \|v\|_{W_2^4([0,1];H)} + \|u_0\|_{W_2^4([0,1];H)}$$

$$\le \mathrm{const}\left(\|\varphi_0\|_{H_{7/2}} + \|\varphi_1\|_{H_{5/2}} + \|\psi_0\|_{H_{7/2}} + \|\psi_1\|_{H_{5/2}}\right).$$

Thus, we get that problem (2.4)-(2.5) is regularly solvable and the function $u(t) = v(t) + u_0(t)$ is a regular solution of this problem. Theorem 2.2 is proved.

Now we consider a boundary value problem for a nonhomogeneous equation with nonhomogeneous boundary conditions and find solvability conditions of this problem.

We consider the following problem:

$$\frac{d^4u(t)}{dt^4} + A^4u(t) + \sum_{j=0}^{4} A_{4-j}u^{(j)}(t) = f(t), \quad t \in [0,1], \tag{2.15}$$

$$u(0) = \varphi_0, \quad u'(0) = \varphi_1, \quad u(1) = \psi_0, \quad u'(1) = \psi_1. \tag{2.16}$$

Definition 2.3 *If for any $f \in L_2([0,1];H)$ and for any $\varphi_0 \in H_{7/2}$, $\varphi_1 \in H_{5/2}$, $\psi_0 \in H_{7/2}$, $\psi_1 \in H_{5/2}$ there exists a function $u \in W_2^4([0,1];H)$ that satisfies equation* (2.15) *almost everywhere on the segment [0,1] and satisfies boundary condition* (2.16) *in the sense of convergence*

$$\lim_{t\to+0} \|u(t) - \varphi_0\|_{H_{7/2}} = 0, \lim_{t\to+0} \left\|u'(t) - \varphi_1\right\|_{H_{5/2}} = 0,$$

$$\lim_{t\to1-0} \|u(t) - \psi_0\|_{H_{7/2}} = 0, \lim_{t\to1-0} \|u(t) - \psi_1\|_{H_{5/2}} = 0,$$

and the inequality

$$\|u\|_{W_2^4([0,1];H)} \le \mathrm{const}\left(\|f\|_{L_2([0,1];H)} + \|\varphi_0\|_{H_{7/2}} + \|\varphi_1\|_{H_{5/2}} + \|\psi_0\|_{H_{7/2}} + \|\psi_1\|_{H_{5/2}}\right)$$

is fulfilled, then boundary value problem (2.15), (2.16) *is called a regularly solvable problem and $u(t)$ is said to be a regular solution of boundary value problem* (2.15)-(2.16).

We have the following theorem.

Theorem 2.3 *Let the coefficients of equation* (2.15) *satisfy the conditions of Theorem 2.1. Then boundary value problem* (2.15)-(2.16) *is a regularly solvable problem.*

Proof. We look for the solution of problem (2.15)-(2.16) in the form $u(t) = w(t) + u_0(t)$. Here $w(t)$ is a regular solution of the problem

$$\frac{d^4 w(t)}{dt^4} + A^4 w(t) + \sum_{j=0}^{4} A_{4-j} w^{(j)}(t) = f(t), \quad t \in [0,1], \tag{2.17}$$

$$w(0) = w(1) = 0, \quad w'(1) = w'(1) = 0, \tag{2.18}$$

the function $u_0(t)$ is a regular solution of the problem

$$\frac{d^4 u_0(t)}{dt^4} + A^4 u_0(t) = 0, \tag{2.19}$$

$$u_0(0) = \varphi_0, \; u_0'(0) = \varphi_1, \quad u_0(1) = \psi_0, \; u_0'(1) = \psi_1. \tag{2.20}$$

According to conditions 1), 2) of Theorem 2.1 indicated in the introduction boundary value problems (2.17)-(2.18) and (2.19)-(2.20) are regularly solvable. Then we have the inequalities

$$\|w\|_{W_2^4([0,1];H)} \le \text{const} \, \|f\|_{L_2([0,1];H)} ,$$

$$\|u_0\|_{W_2^4([0,1];H)} \le \text{const} \left(\|\varphi_0\|_{H_{7/2}} + \|\varphi_1\|_{H_{5/2}} + \|\psi_0\|_{H_{7/2}} + \|\psi_1\|_{H_{5/2}} \right).$$

Substituting $u(t) = w(t) + u_0(t)$ in equation (2.15), we obtain the following boundary value problem with respect to the function $w(t)$:

$$\frac{d^4 w(t) + u_0(t)}{dt^4} + A^4 \left(w(t) + u_0(t)\right) + \sum_{j=0}^{4} A^{4-j} \left(w(t) + u_0(t)\right)^{(j)} = f(t),$$

$$w(0) = w'(0) = 0, \quad w(1) = w'(1) = 0$$

or

$$\frac{d^4 w(t)}{dt^4} + A^4 w(t) + \sum_{j=0}^{4} A^{4-j} w^{(j)}(t) + \left(\frac{d^4 u_0(t)}{dt^4} + A^4 u_0(t) \right) = f(t) - \sum_{j=0}^{4} A^{4-j} u_0^{(j)}(t),$$

$$w(0) = w'(0) = 0, \quad w(1) = w'(1) = 0.$$

In view of $\frac{d^4 u_0(t)}{dt^4} + A^4 u_0(t) = 0$, we get

$$\frac{d^4 w(t)}{dt^4} + A^4 w(t) + \sum_{j=0}^{4} A^{4-j} w^{(j)}(t) = g(t), \tag{2.21}$$

$$w(0) = w'(0) = 0, \quad w(1) = w'(1) = 0, \tag{2.22}$$

where $g(t) = f(t) - \sum\limits_{j=0}^{4} A^{4-j} u_0{}^{(j)}(t)$. We show that $g \in L_2([0,1];H)$. Indeed,

$$\|g\|_{L_2([0,1];H)} \le \|f\|_{L_2([0,1];H)} + \sum_{j=0}^{4} \left\| A^{4-j} u_0^{(j)} \right\|_{L_2([0,1];H)}$$

$$\le \|f\|_{L_2([0,1];H)} + \sum_{j=0}^{4} \|B_{4-j}\| \cdot \left\| A^{4-j} u_0^{(j)} \right\|_{L_2([0,1];H)} .$$

Since $u_0(t)$ is a regular solution of a homogeneous equation under inhomogeneous boundary conditions, (problem (2.19)-(2.20)) we have the following inequality:

$$\|u_0\|_{W_2^4([0,1];H)} \leq \text{const}\left(\|\varphi_0\|_{H_{7/2}} + \|\varphi_1\|_{H_{5/2}} + \|\psi_0\|_{H_{7/2}} + \|\psi_1\|_{H_{5/2}}\right).$$

Then we get

$$\|g\|_{L_2([0,1];H)} \leq \|f\|_{L_2([0,1];H)} + \text{const}\left(\|\varphi_0\|_{H_{7/2}} + \|\varphi_1\|_{H_{5/2}} + \|\psi_0\|_{H_{5/2}} + \|\psi_1\|_{H_{5/2}}\right),$$

i.e. $g \in L_2\left([0,1];H\right)$.
By the conditions of the theorem, problem (2.21)-(2.22) is regularly solvable and $w(t)$ is a regular solution of this problem. Therefore, we have an estimate:

$$\|w\|_{W_2^4([0,1];H)} \leq c \cdot \|g\|_{L_2([0,1];H)}.$$

As a result, finally, for the solution $u(t)$ of boundary value problem (2.15)-(2.16) we get the estimation:

$$\|u\|_{W_2^4([0,1];H)} \leq \|w\|_{W_2^4([0,1];H)} + \|u_0\|_{W_2^4([0,1];H)}$$
$$\leq c \cdot \left(\|f\|_{L_2([0,1];H)} + \|\varphi_0\|_{H_{7/2}} + \|\varphi_1\|_{H_{7/2}} + \|\psi_0\|_{H_{7/2}} + \|\psi_1\|_{H_{5/2}}\right).$$

Theorem 2.3 is proved.

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