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Sufficient conditions for uniform exponential stability of some classes of dynamic equations on time scales and applications

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Abstract. *This paper focuses on the problem of exponential stability of certain classes of dynamic perturbed systems on time scales using time scale versions of some Gronwall type inequalities. We prove under certain conditions on the nonlinear perturbations that the resulting perturbed nonlinear initial value problem still acquir uniformly exponentially stable, if the associated time-varying linear system has already owned this property. Furthermore, an example is given to illustrate the applicability of the obtained results.*

Keywords. Dynamic equations, time scale, Gronwall integral inequality, exponential stability.

Mathematics Subject Classification (2010): 74H55, 93D20, 26E70, 35A23

1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [17] in his Ph. D. thesis in 1988 in order to unify continuous and discrete analysis. A great deal of work has been done since 1988, unifying the theory of differential equations and the theory of difference equations by establishing the corresponding results in time scale setting. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. During the last decades, time scale methods have rapidly been developed, and have received a lot of attention by several authors, not only to unify continuous and discrete processes, but also help reveal diversities in the corresponding results. The analysis

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of nonlinear perturbations of linear systems is not only important for its own sake but also has a broad range of applications.

One of the analytic methods of the perturbation theory was referred to integral inequalities to quest some type of stability. Latterly, there have been several papers [2,3,5,8–12, 14–16,19–22], studying various types of stability of dynamical time scale systems.

In this paper we investigate uniform exponential stability for nonlinear perturbed systems on time scales by using the Gronwall- Bellman-Bihari type integral inequality.

The paper is organized as follows: in Section 2, provides a brief review of the time scale theory and integral inequalities which play an important role in our analysis. In Section 3, contains the statements and proofs of our main results.Section 4 shows the applicability of the theoretical results by numerical example.

First, we will briey mention some basic definitions and results of time scale calculus for reader's convenience, as they are detailed in the books of M. Bohner and A. Peterson [6,7].

2 Preliminaries

2.1 Time scale calculus

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, \infty)$ is the given subset of $\mathbb{R}$ and $\mathbb{T}$ is an arbitrary time scale. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) =: \sup\{s \in \mathbb{T} : s < t\}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$; we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$; then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and left-sided limits exists(finite) at left-dense points in $\mathbb{T}$ and denotes by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T},\mathbb{R})$. The set $\mathbb{T}^k$ which is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)\, p(t) \neq 0$ for all $t \in \mathbb{T}^k$. $\Re = \Re(\mathbb{T}, \mathbb{R})$ denotes the set of all regressive and rd-continuous functions, we define the set of all positively regressive functions by

$$\Re^+ = \Re^+(\mathbb{T}, \mathbb{R}) = \{p \in \Re : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Also we define the interval $[a, b]$ means the set $\{t \in \mathbb{T} : a \leq t \leq b\}$ for the points $a < b$ in $\mathbb{T}$. If $b = +\infty$, we denote $\mathbb{T}_a^+ = [a, +\infty[$.

Definition 2.1 *If $p \in \Re(\mathbb{T}, \mathbb{R})$, then we define the generalized exponential function $e_p(t, t_0)$ by*

$$e_p(t, t_0) = \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \text{ for } t, t_0 \in \mathbb{T},$$

where $\xi_h(z)$ is the cylinder transformation given by

$$\xi_h(z) = \frac{1}{h}\log(1 + zh), \quad \text{if } h \neq 0,$$
$$\xi_0(z) = z, \quad \text{if } h = 0,$$

Corollary 2.1 *[6] Let $p \in \Re$ and $t, t_0, s \in \mathbb{T}$, then*

(i) $e_0(t, t_0) \equiv 1$ *and* $e_p(t, t) \equiv 1$,
(ii) $e_p(\sigma(t), t_0) = (1 + \mu(t)\, p(t))\, e_p(t, t_0)$,
(iii) $e_p(t, t_0)\, e_p(t_0, s) = e_p(t, s)$,
(iv) $e_p(t, t_0) = \frac{1}{e_p(t_0, t)}$,
(v) *If* $p \in \Re^+$ *then* $e_p(t, t_0) > 0$ *for all* $t \in \mathbb{T}$.

Definition 2.2 *A function $w : \mathbb{R}_+ \to \mathbb{R}_+$ belong to the class $\widehat{H}$ if*

(H_1) $w(u)$ *is nondecreasing and continuous for $u \geq 0$ and positive for $u > 0$,*
(H_2) *there exists a continuous function ϕ on $\mathbb{R}_+$ with $w(\alpha u) \leq \phi(\alpha) w(u)$ for $\alpha > 0, u \geq 0$.*

Definition 2.3 *[13] A function $w : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathfrak{F}$, if it satisfies the following conditions*

$$w(x) > 0 \ \text{ is nondecreasing and continuous for } x \geq 0,$$
$$\frac{1}{a} w(x) \leq w\left(\frac{x}{a}\right) \quad \text{for } a > 0.$$

Lemma 2.1 *[2] Suppose that $y, f, g, h, m \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$. If*

$$y(t) \leq f(t) + g(t) \int_a^t \{h(s)y(s) + m(s)\} \Delta s \text{ for all } t \in \mathbb{T}_a^+,$$

then

$$y(t) \leq f(t) + g(t) \int_a^t \{h(s)f(s) + m(s)\} \exp\left[\int_{\sigma(s)}^t h(\tau)g(\tau)\Delta\tau\right] \Delta s \text{ for all } t \in \mathbb{T}_a^+.$$

Lemma 2.2 *[1] Suppose that g is continuous and nondecreasing, p is rd-continuous and nonnegative, and y is rd-continuous. Let w be the solution of*

$$w^{\Delta} = p(t)g(w(t)), \ w(t_0) = \beta$$

and suppose there is a bijective function G with $(G \circ w)^{\Delta} = p$. Then

$$y(t) \leq \beta + \int_{t_0}^t p(\tau)g(y(\tau))\Delta\tau \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq G^{-1}\left[G(\beta) + \int_{t_0}^t p(\tau)\,\Delta\tau\right] \text{ for all } t \in \mathbb{T}.$$

2.2 Stability definitions

For our purpose, we will assume that the time scale $\mathbb{T}$ is unbounded above, i.e., $\sup \mathbb{T} = +\infty$. Let $t_0 \in \mathbb{T}$ and $t \in \mathbb{T}_{t_0}^+$. Let us consider time scale dynamic equations of the form

$$\begin{aligned} x^{\Delta}(t) &= f(t, x(t)), \\ x(t_0) &= x_0. \end{aligned} \tag{2.1}$$

where $x : \mathbb{T}_{t_0}^+ \to \mathbb{R}^n$ is the state vector and $f : \mathbb{T}_{t_0}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a rd-continuous vector-valued function. It is assumed that the conditions for the existence of a unique solution of system (2.1) are satisfied. For the existence, uniqueness and extensibility of its solutions, one can refer to [6]. Designate any solution of (2.1) with the initial state (t_0, x_0) by $x(t) = x(t, t_0, x_0)$. The Euclidean norm of an $n \times 1$ vector x(t) is defined to be a real-valued function of t and is denoted by $\|x(t)\| = \sqrt{x(t)^T x(t)}$.

Definition 2.4 *[7] A mapping $A : \mathbb{T} \to M_n(\mathbb{R})$ is called regressive if for each $t \in \mathbb{T}^k$ the $n \times n$ matrix $I_n + \mu(t)A$ is invertible, where I_n is the identity matrix.*

The class of all regressive and rd-continuous functions A from $\mathbb{T}$ to $M_n(\mathbb{R})$ is denoted by $C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$.

Definition 2.5 *[18] Let $t_0 \in \mathbb{T}$. The unique matrix-valued solution of the IVP*

$$X^{\Delta} = A(t)X \qquad X(t_0) = I_n, \tag{2.2}$$

where $A \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$, is called the matrix exponential function and it denoted by $\phi_A(t, t_0)$.

Accordingly, the matrix function $\phi_A(t, t_0)$ possesses the following two properties:

$$\phi_A^{\Delta}(t, t_0) = A(t)\phi_A(t, t_0),$$
$$\phi_A(t_0, t_0) = I_n.$$

This matrix function is referred to as the state transition matrix, and our assumption in the nature of $A(t)$ turns out that the state transition matrix exists and is unique.

Theorem 2.1 *[6] Suppose $A, B \in C_{rd}\mathcal{R}(\mathbb{T}; M_n(\mathbb{R}))$ are matrix-valued functions on $\mathbb{T}$, then*

(i) $\phi_A(t, r)\phi_A(r, s) = \phi_A(t, s)$ for all r, s, $t \in \mathbb{T}$,

(ii) $\phi_A(\sigma(t), s) = (I + \mu(t)A(t))\phi_A(t, s)$,

(iii) If $\mathbb{T} = \mathbb{R}$ and A is constant, then $\phi_A(t, s) = e_A(t, s) = e^{A(t-s)}$,

(iv)If $\mathbb{T} = h\mathbb{Z}$, with $h > 0$, and A is constant, then $\phi_A(t, s) = (I + hA)^{\frac{(t-s)}{h}}$.

Definition 2.6 *[12] The system of dynamic equations (2.1) is said to be uniformly exponentially stable if there exist constants $\gamma \geq 1$ (independent of t_0) , $\lambda > 0$ $(-\lambda \in \Re^+)$ such that*

$$\|x(t)\| \leq \gamma\|x_0\|e_{-\lambda}(t, t_0).$$

Definition 2.7 *[9] The system of dynamic equations (2.1) is said to be h-stable if there exist a nonincreasing bounded rd-continuous function $h : \mathbb{T}_{t_0}^+ \to \mathbb{R}$, a constant $\gamma \geq 1$ such that, any solution $x(t) = x(t, t_0, x_0)$ of equation (2.1) satisfies*

$$\|x(t)\| \leq \gamma\|x_0\|h(t)\,(h(t_0))^{-1} \text{ for all } t \in \mathbb{T}_{t_0}^+,$$

(here $(h(t))^{-1} = \frac{1}{h(t)}$).

Now, we give the following characterization in terms of the transition matrix for system (2.2).

Theorem 2.2 *[12] The system of dynamic equations (2.2) is uniformly exponentially stable with respect to $t \in \mathbb{T}_{t_0}^+$ if and only if there exist constants $\lambda > 0$ $(-\lambda \in \Re^+)$ and $\gamma \geq 1$ such that*

$$||\phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0) \text{ for all } t \in \mathbb{T}_{t_0}^+.$$

Theorem 2.3 *[9] The system of dynamic equations (2.2) is h-stable if there exist a nonincreasing bounded rd-continuous function $h : \mathbb{T}_{t_0}^+ \to \mathbb{R}$, a constant $\gamma \geq 1$ such that*

$$||\phi_A(t, t_0)|| \leq \gamma\|x_0\|h(t)\,(h(t_0))^{-1} \text{ for all } t \in \mathbb{T}_{t_0}^+.$$

The notion of h-stability was introduced by Pinto [22] which is an extension of the notions of exponential stability and uniform stability. For the various definitions of stability, we refer to [12].

3 Main Results

In this study, we consider, a particular class of systems (2.1), i.e the system

$$\begin{aligned} x^{\Delta}(t) &= A(t)x + F(t, x(t)), \\ x(t_0) &= x_0, \end{aligned} \tag{3.1}$$

where $x_0, x \in \mathbb{R}^n$, $F(t, 0) = 0$, $t_0 \in \mathbb{T}$, and $F : \mathbb{T}^+_{t_0} \times \mathbb{R}^n \to \mathbb{R}^n$ is an rd-continuous function. F represents the disturbance of the time-varying linear system :

$$\begin{aligned} x^{\Delta}(t) &= A(t)x, \\ x(t_0) &= x_0, x_0 \neq 0. \end{aligned} \tag{3.2}$$

Theorem 3.1 *[2] Consider the regressive time-varying perturbed system of the form (3.1). Then, every solution can be written in the form*

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))F(s, x(s))\Delta s,\ t \in \mathbb{T}^+_{t_0}. \tag{3.3}$$

Theorem 3.2 *Suppose that the linear system (3.2) is uniformly exponentially stable with positive constants λ and γ and*

$$F(t, x) \leq g(t)w(\|x\|), t \in \mathbb{T}^+_{t_0}, \tag{3.4}$$

where g is a positive and rd-continuous,and $w \in \widehat{H}$ with corresponding multiplier function Φ. Let r be the solution of

$$r^{\Delta}(t) = p(t)w(r(t)),\ r(t_0) = \gamma$$

and assume that there is a bijective function W satisfying

$$(W \circ r)^{\Delta} = p \quad \textit{with} \int_{t_0}^{\infty} p(s)\Delta s < \infty$$

for all $t_0 \in \mathbb{T}$, where $p(t) = \frac{\gamma e_{-\lambda}(t_0,\sigma(t))}{\|x_0\|} g(t) w\left(\frac{\|x_0\|}{\gamma e_{-\lambda}(t_0,t)}\right)$. Then the perturbed system (3.1) is uniformly exponentially stable.

Proof. For any t_0 and $x_0 = x(t_0)$ and from (3.3), the solution of the perturbed system (3.1) is given by :

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))F(s, x(s))\Delta s.$$

By taking the norms of both sides and taking into account the fact that (3.2) is uniformly exponentially stable, we obtain

$$\begin{aligned} \|x(t)\| &\leq \|\Phi_A(t, t_0)\| \|x_0\| + \int_{t_0}^{t} \|\Phi_A(t, \sigma(s))\| \|F(s, (x(s))\| \Delta s \\ &\leq \gamma e_{-\lambda}(t, t_0) \|x_0\| + e_{-\lambda}(t, t_0) \int_{t_0}^{t} \gamma e_{-\lambda}(t_0, \sigma(s)) g(s) w(\|x(s)\|)\Delta s \\ &\leq e_{-\lambda}(t, t_0) \left[\gamma \|x_0\| + \int_{t_0}^{t} \gamma e_{-\lambda}(t_0, \sigma(s)) g(s) w(\|x(s\|)\Delta s\right], \end{aligned}$$

$$\frac{e_{-\lambda}(t_0, t) \|x(t)\|}{\|x_0\|} \leq \gamma + \gamma \int_{t_0}^{t} \frac{e_{-\lambda}(t_0, \sigma(s))}{\|x_0\|} g(s) w\left(\frac{\|x_0\|}{e_{-\lambda}(t_0, s)} \frac{e_{-\lambda}(t_0, s) \|x(s\|}{\|x_0\|}\right) \Delta s.$$

Letting $u(t) = \frac{e_{-\lambda}(t_0,t)\|x(t)\|}{\|x_0\|}$, then the above inequality becomes

$$u(t) \leq \gamma + \int_{t_0}^{t} \frac{\gamma e_{-\lambda}(t_0, \sigma(s))}{\|x_0\|} g(s) w\left(\frac{\|x_0\|}{e_{-\lambda}(t_0, s)} u(s)\right) \Delta s.$$

Since $w \in \widehat{H}$, then we have

$$u(t) \leq \gamma + \int_{t_0}^{t} \frac{\gamma e_{-\lambda}(t_0, \sigma(s))}{\|x_0\|} g(s) \phi\left(\frac{\|x_0\|}{e_{-\lambda}(t_0, s)}\right) w(u(s)) \Delta s,$$

then the above inequality can be expressed as

$$u(t) \leq \gamma + \int_{t_0}^{t} p(s) w(u(s)) \Delta s,$$

where

$$p(t) = \frac{\gamma e_{-\lambda}(t_0, \sigma(t))}{\|x_0\|} g(t) \phi\left(\frac{\|x_0\|}{e_{-\lambda}(t_0, t)}\right).$$

Applying Lemma 2.2, we get

$$\begin{aligned} u(t) &\leq W^{-1}\left[W(\gamma) + \int_{t_0}^{t} p(s) \Delta s\right] \\ &\leq W^{-1}\left[W(\gamma) + \int_{t_0}^{\infty} p(s) \Delta s\right] \end{aligned}$$

for all $t \in \mathbb{T}_{t_0}^{+}$.

Then, we have

$$\|x(t)\| \leq d e_{-\lambda}(t, t_0) \|x_0\|.$$

where

$$d = W^{-1}\left[W(\gamma) + \int_{t_0}^{\infty} p(s) \Delta s\right].$$

It is easy to prove that $d \geq 1$.

Then the perturbed system (3.1) is uniformly exponentially stable.

Remark 3.1 In $[9, Theorem\ 2.10]$, the authors proved the h-stability of the system (3.1).

Theorem 3.3 *Assume that there exist d, $k \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ that satisfy the following conditions:*

($\mathfrak{B}_1$) $\|F(t,x)\| \leq \eta\left(d(t)\|x\| + k(t)\right)$ *and* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *is a differentiable increasing function on* $]0, +\infty[$ *with continuous nonincreasing first derivative* η'*on* $]0, \infty[$,

($\mathfrak{B}_2$) *Suppose that the linear system (3.2) is uniformly exponentially stable with growth constants* λ *and* γ,

($\mathfrak{B}_3$) $\int_{t_0}^{+\infty} \frac{\eta'(k(s))d(s)}{1-\lambda\mu(s)} \Delta s \leq \widetilde{d} < +\infty$, $\int_{t_0}^{+\infty} \eta(k(s)) e_{-\lambda}(t_0, \sigma(s)) \Delta s \leq \widetilde{k} < +\infty$.

Then the perturbed system (3.1) is uniformly exponentially stable.

Proof. Let $t_0 \in \mathbb{T}, x_0 \in \mathbb{R}^n$ and $t \in \mathbb{T}^+_{t_0}$. For any t_0 and $x_0 = x(t_0)$ and from (3.3), the solution of the perturbed system (3.1) is given by :

$$x(t) = \Phi_A(t,t_0)x_0 + \int_{t_0}^{t} \Phi_A(t,\sigma(s))F(s,x(s))\Delta s.$$

Taking into account the fact that (3.2) is uniformly exponentially stable, we obtain

$$\|x(t)\| \le \gamma e_{-\lambda}(t,t_0)\,\|x_0\| + \gamma e_{-\lambda}(t,t_0)\int_{t_0}^{t} e_{-\lambda}(t_0,\sigma(s))\eta\left(d(s)\,\|x\| + k(s)\right)\Delta s. \quad (3.5)$$

Applying the mean value Theorem for the function η, then for every $x_1 \ge y_1 > 0$, there exists $c \in\,]y_1, x_1[$ such taht

$$\eta(x_1) - \eta(y_1) = \eta'(c)(x_1 - y_1) \le \eta'(y_1)(x_1 - y_1)$$

which gives:

$$\eta\left(d(s)\,\|x\| + k(s)\right) \le \eta'(k(s)) \times d(s)\,\|x\| + \eta(k(s)). \quad (3.6)$$

From (3.5) and (3.6), we get

$$\|x(t)\| \le \gamma e_{-\lambda}(t,t_0)\,\|x_0\| + \gamma e_{-\lambda}(t,t_0)\int_{t_0}^{t} e_{-\lambda}(t_0,\sigma(s))\left[\eta'(k(s)) \times d(s)\,\|x(s)\| + \eta(k(s)\right]\Delta s.$$

Applying Lemma 2.1 to the above inequality, we obtain

$$\begin{aligned}\|x(t)\| \le \gamma e_{-\lambda}(t,t_0)\Big[&\|x_0\| + (\textstyle\int_{t_0}^{t}(e_{-\lambda}(t_0,\sigma(s))\eta'(k(s))d(s)\gamma e_{-\lambda}(s,t_0)\,\|x_0\| + e_{-\lambda}(t_0,\sigma(s))\eta(k(s))\\ &\times \exp\left(\textstyle\int_{\sigma(s)}^{t} e_{-\lambda}(t_0,\sigma(\tau))\eta'(k(\tau)) \times d(\tau)\gamma e_{-\lambda}(\tau,t_0)\Delta\tau\right)\Delta s\Big].\end{aligned} \quad (3.7)$$

According to the hypothesis $(\mathfrak{B}_3)$ and from (3.7) , we extract the estimate

$$\begin{aligned}\|x(t)\| &\le \gamma\left(1 + \gamma\widetilde{d}\exp\left(\gamma\widetilde{d}\right)\right)e_{-\lambda}(t,t_0)\,\|x_0\| + \gamma\widetilde{k}\exp\left(\gamma\widetilde{d}\right)e_{-\lambda}(t,t_0),\\ &= \gamma e_{-\lambda}(t,t_0)\,\|x_0\|\left[1 + \gamma\widetilde{d}\exp\left(\gamma\widetilde{d}\right) + \frac{\widetilde{k}\exp\left(\gamma\widetilde{d}\right)}{\|x_0\|}\right].\end{aligned}$$

Then the perturbed system (3.1) is uniformly exponentially stable.

Theorem 3.4 *Suppose that (3.2) is uniformly exponentially stable with positive constants λ and γ, and*

$$F(t,x) \le g(t)w(\|x\|), t \in \mathbb{T}^+_{t_0},$$

where g is a positive and rd-continuous function and $w \in \mathfrak{F}$. Let r be the solution of

$$r^{\Delta}(t) = p_1(t)w(r(t)),\; r(t_0) = \gamma,$$

and assume that there is a bijective function W satisfying

$$(W \circ r)^{\Delta} = p_1 \quad \textit{with} \int_{t_0}^{\infty} p_1(s)\Delta s < \infty$$

for all $t_0 \in \mathbb{T}$, where $p_1(t) = \frac{\gamma g(t)}{1-\lambda\mu(t)}$. Then the perturbed system (3.1) is uniformly exponentially stable.

Proof. Let $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^n$ and $t \in \mathbb{T}^+_{t_0}$. For any t_0 and $x_0 = x(t_0)$ and from (3.3), the solution of the perturbed system (3.1) is given by :

$$x(t) = \Phi_A(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi_A(t,\sigma(s))F(s,x(s))\Delta s.$$

Then, by taking into account the fact that $w \in \mathfrak{F}$, we have

$$\begin{aligned}
\|x(t)\| &= \|\Phi_A(t,t_0)\| \|x_0\| + \int_{t_0}^{t} \|\Phi_A(t,\sigma(s))\| \|F(s,(x(s))\| \Delta s \\
&\leq \gamma e_{-\lambda}(t,t_0) \|x_0\| + e_{-\lambda}(t,t_0) \int_{t_0}^{t} \gamma e_{-\lambda}(t_0,\sigma(s))g(s)w(\|x(s)\|)\Delta s \\
&\leq \gamma e_{-\lambda}(t,t_0) \|x_0\| + e_{-\lambda}(t,t_0) \int_{t_0}^{t} \gamma e_{-\lambda}(t_0,\sigma(s))g(s) \|x_0\| \frac{w(\|x(s)\|)}{\|x_0\|}\Delta s \\
&\leq \gamma e_{-\lambda}(t,t_0) \|x_0\| + e_{-\lambda}(t,t_0) \int_{t_0}^{t} \gamma e_{-\lambda}(t_0,\sigma(s))g(s) \|x_0\| e_{-\lambda}(s,t_0) w\left(\frac{\|x(s)\|}{\|x_0\| e_{-\lambda}(s,t_0)}\right)\Delta s \\
&\leq e_{-\lambda}(t,t_0) \|x_0\| \left[\gamma + \gamma \int_{t_0}^{t} e_{-\lambda}(t_0,\sigma(s))g(s)e_{-\lambda}(s,t_0) w\left(\frac{\|x(s)\|}{\|x_0\| e_{-\lambda}(s,t_0)}\right)\Delta s\right].
\end{aligned}$$

Set $u(t) = \frac{\|x(t)\| e_{-\lambda}(t_0,t)}{\|x_0\|}$, then we get

$$u(t) \leq \gamma + \int_{t_0}^{t} \frac{\gamma g(s)}{1-\lambda\mu(s)} w\left(u(s)\right)\Delta s.$$

The last inequality can be reformulated as

$$u(t) \leq \gamma + \int_{t_0}^{t} p_1(s) w\left(u(s)\right)\Delta s,$$

where

$$p_1(t) = \frac{\gamma g(t)}{1-\lambda\mu(t)}.$$

Applying Lemma 2.2 to the above inequality, we obtain

$$u(t) \leq W^{-1}\left[W(\gamma) + \int_{t_0}^{t} p_1(s)\Delta s\right] \leq W^{-1}\left[W(\gamma) + \int_{t_0}^{\infty} p_1(s)\Delta s\right].$$

Then, we have

$$\begin{aligned}
&\|x(t)\| \leq d \|x_0\| e_{-\lambda}(t,t_0), \\
&\qquad d = W^{-1}\left[W(\gamma) + \int_{t_0}^{\infty} p_1(s)\Delta s\right].
\end{aligned}$$

Then the perturbed system (3.1) is uniformly exponentially stable.

4 Numerical examples

Let $\mathbb{T}$ be a mixed continuous-discrete time scale and $t_0 = 0$. The discrete part has non-uniform step size. The graininess function is bounded as follows: $\forall t \in \mathbb{T}_0^+$

$$0 \leq \mu(t) < \mu_{\max} = \frac{1}{2}.$$

Consider the following time-varying system:

$$\begin{aligned} x_1^{\Delta}(t) &= -x_1(t) + \tfrac{1}{2}\ln\left(\tfrac{1}{(t+1)(\sigma(t)+1)}\left|x_1(t)\right| + \tfrac{k(t)|x_2(t)|}{\sqrt{x_1^2(t)+x_2^2(t)+1}} + 1\right), \\ x_2^{\Delta}(t) &= -x_2(t) + \tfrac{\sqrt{3}}{2}\ln\left(\tfrac{1}{(t+1)(\sigma(t)+1)}\left|x_2(t)\right| + \tfrac{k(t)|x_1(t)|}{\sqrt{x_1^2(t)+x_2^2(t)+1}} + 1\right), \\ x(0) &= (x_{1,0}, x_{2,0}), \end{aligned} \tag{4.1}$$

where $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^2$, $k \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ and $k(t) = \frac{t+\sigma(t)+2}{(t+1)^2(\sigma(t)+1)^2}e_{-\lambda}(\sigma(t), 0)$.

System (4.1) can be written as system (3.1):

$$x^{\Delta}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\ln\left(\frac{1}{(t+1)(\sigma(t)+1)}\left|x_1(t)\right| + \frac{k(t)|x_2(t)|}{\sqrt{x_1^2(t)+x_2^2(t)+1}} + 1\right) \\ \frac{\sqrt{3}}{2}\ln\left(\frac{1}{(t+1)(\sigma(t)+1)}\left|x_2(t)\right| + \frac{k(t)|x_1(t)|}{\sqrt{x_1^2(t)+x_2^2(t)+1}} + 1\right) \end{pmatrix} \tag{4.2}$$

where $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$, $\mu \neq 1$, $F(t, 0) = 0$ and

$$\Phi_A(t, 0) = \begin{pmatrix} e_{-1}(t, 0) & 0 \\ 0 & e_{-1}(t, 0) \end{pmatrix},\ t \in \mathbb{T}_0^+, \tag{4.3}$$

and

$$\|\Phi_A(t, t_0)\| = \sqrt{2}e_{-1}(t, 0). \tag{4.4}$$

The perturbation satisfies condition $(\mathfrak{B}_1)$ of Theorem 3.3 with $n(x) = \ln(x+1)$ is a differentiable increasing function on$]0, \infty[$ with continuous nonincreasing first derivative.

$$\|f(t, x)\| \leq \ln\left(\frac{1}{(t+1)(\sigma(t)+1)}\|x\| + k(t) + 1\right) = n(d(t)\|x\| + k(t)), \tag{4.5}$$

It is clear that assumption $(\mathfrak{B}_2)$ of Theorem 3.3 is satisfied with $(\lambda, \gamma) = \left(1, \sqrt{2}\right)$. Moreover, one can verify that hypothesis $(\mathfrak{B}_3)$ of Theorem 3.3 is verified, since

$$\begin{aligned} \int_0^{+\infty} \tfrac{\eta'(k(s))d(s)}{1-\lambda\mu(s)}\Delta s &= \int_0^{+\infty} \tfrac{1}{(1-\mu(s))(1+k(s))}\tfrac{1}{(s+1)(\sigma(s)+1)}\Delta s, \\ &\leq \int_0^{+\infty} \tfrac{1}{(1-\mu(s))}\tfrac{1}{(s+1)(s+\sigma(s))}\Delta s, \\ &\leq 2\int_0^{+\infty} \tfrac{1}{(s+1)(\sigma(s)+1)}\Delta s = 2 = \widetilde{d} < +\infty. \end{aligned} \tag{4.6}$$

And

$$\begin{aligned} \int_0^{+\infty} \eta(k(s))e_{-\lambda}(0, \sigma(s))\Delta s &= \int_0^{+\infty} \ln(k(s)+1)e_{-\lambda}(0, \sigma(s))\Delta s \\ &\leq \int_0^{+\infty} k(s)e_{-\lambda}(0, \sigma(s))\Delta s = \int_0^{+\infty} \frac{s+\sigma(s)+2}{(s+1)^2(\sigma(s)+1)^2}\Delta s \\ &= \int_0^{+\infty} \left(\frac{-1}{(s+1)^2}\right)^{\Delta}\Delta s = 1 = \widetilde{k} < +\infty. \end{aligned}$$

From Theorem 3.3, one can conclude that system (4.1) is uniformly exponentially stable.

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