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On solvability of an inverse boundary value problem for the Boussinesq-Love equation with periodic and integral condition

Nizameddin Sh. Isgenderov⋆ · **Seriye I. Allahverdiyeva**

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Abstract. *An inverse boundary value problem for the Boussinesq-Love equation with periodic and integral condition is investigated. The goal of the paper consists of the determination of the unknown coefficient together with the solution. The problem is considered in a rectangular domain. The definition of the classical solution of the problem is given. First, the given problem is reduced to an equivalent problem in a certain sense. Then, using the Fourier method the equivalent problem is reduced to solving the system of integral equations. Thus, the solution of an auxiliary inverse boundary value problem reduces to a system of three nonlinear integro-differential equations for unknown functions. Concrete Banach space is constructed. Further, in the ball from the constructed Banach space by the contraction mapping principle, the solvability of the system of nonlinear integro-differential equations is proved. This solution is also a unique solution to the equivalent problem. Finally, by equivalence, the theorem of existence and uniqueness of a classical solution to the given problem is proved.*

Keywords. Inverse problems · Boussinesq-Love equation · nonlocal integral condition · classical solution · existence · uniqueness.

Mathematics Subject Classification (2010): Primary 35R30, 35L10, 35L70; Secondary 35A01, 35A02, 35A09.

1 Introduction

There are many cases where the needs of the practice bring about the problems of determining coefficients or the right hand side of differential equations from some knowledge of its solutions. Such problems are called inverse boundary value problems of mathematical physics. Inverse boundary value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control in industry etc., which makes them an active field of contemporary mathematics. Inverse problems for various

⋆ Corresponding author

N.Sh. Isgenderov
Baku State University, Baku, Azerbaijan
E-mail: nizameddin_isgenderov@mail.ru

S.I. Allahverdiyeva
Mingachevir State University, Mingachevir, Azerbaijan
E-mail: seriye_allahverdiyeva@mail.ru

types of PDEs have been studied in many papers. Among them we should mention the papers of A.N. Tikhonov [1], M.M. Lavrentyev [2, 3], V.K. Ivanov [4] and their followers. For a comprehensive overview, the reader should see the monograph by A.M. Denisov [5].

In this paper we prove existence and uniqueness of the solution to an inverse boundary value problem for the Boussinesq-Love equation modeling the longitudinal waves in an elastic bar with the transverse inertia.

2 Problem statement and its reduction to an equivalent problem

Let $T > 0$ be some fixed number and denote by $D_T := \{(x,t) :\ 0 \le x \le 1,\ 0 \le t \le T\}$. Consider a one-dimensional inverse problem of identification of an unknown triple of functions $\{u(x,t)\,,\, a(t),\, b(t)\}$ for the following Boussinesq-Love equation [6]

$$\begin{aligned} &u_{tt}(x,t) - u_{ttxx}(x,t) - \alpha u_{txx}(x,t) - \beta u_{xx}(x,t) \\ &= a(t)u(x,t) + b(t)g(x,t) + f(x,t) \end{aligned} \tag{2.1}$$

with the nonlocal initial conditions

$$u(x,0) = \varphi(x) + \int_0^T p(t)u(x,t)dt, \quad u_t(x,0) = \psi(x), \quad x \in [0,1], \tag{2.2}$$

periodic boundary condition

$$u(0,t) = u(1,t), \ \ t \in [0,T], \tag{2.3}$$

nonlocal integral condition

$$\int_0^1 u(x,t)dx = 0, \ \ t \in [0,T], \tag{2.4}$$

and over determination conditions

$$u(x_i,t) = h_i(t) \ \ \ (i = 1,2;\, x_1 \ne x_2)\,, \ \ t \in [0,T], \tag{2.5}$$

where $x_i \in (0,1)$ $(i = 1,2)$, $\alpha > 0,\ \beta > 0$ the given numbers, $f(x,t)$, $g(x,t)$, $\varphi(x)$, $\psi(x)$, $p(t)$, and $h_i(t)$ $(i = 1,2)$ are given sufficiently smooth functions of $x \in [0,1]$ and $t \in [0,T]$.

We introduce the following set of functions

$$\tilde{C}^{(2,2)}(D_T) = \left\{u(x,t) :\ u \in C^2(D_T), u_{xx}, u_{txx}, u_{ttxx} \in C(D_T)\right\}.$$

Definition 2.1 *The triple* $\{u(x,t)\,,\, a(t),\, b(t)\}$ *is said to be a classical solution to the problem* (2.1)-(2.5)*, if the functions* $u \in \tilde{C}^{(2,2)}(D_T)$*,* $a \in C[0,T]$ *and* $b \in C[0,T]$ *satisfies an equation* (2.1) *in the region* D_T*, the condition* (2.2) *on* $[0,1]$*, and the statements* (2.3)-(2.5) *on the interval* $[0,T]$.

In order to investigate the problem (2.1) - (2.5), first we consider the following auxiliary problem

$$y''(t) = a(t)y(t), \ \ t \in [0,T], \tag{2.6}$$

$$y(0) = \int_0^T p(t)y(t)dt, \ \ y'(0) = 0, \tag{2.7}$$

where $p,\ a \in C[0,T]$ are given functions, and $y = y(t)$ is desired function. Moreover, by the solution of the problem (2.6), (2.7), we mean a function $y(t)$ belonging to $C^2[0,T]$ and satisfying the conditions (2.6), (2.7) in the usual sense.

Lemma 2.1 *[7] Assume that* $p \in C[0,T]$, $a \in C[0,T]$, $\|a\|_{C[0,T]} \leq R = const$, *and the condition*

$$\left(\|p\|_{C[0,T]} + \frac{T}{2}R\right) T < 1$$

hold. Then the problem (2.6)*,* (2.7) *has a unique trivial solution.*

Now along with the inverse boundary-value problem (2.1) - (2.5), we consider the following auxiliary inverse boundary-value problem: It is required to determine a triple $\{u(x,t), a(t), b(t)\}$ of functions $u \in \tilde{C}^{(2,2)}(D_T)$, $a \in C[0,T]$, and $b \in C[0,T]$, from relations (2.1)-(2.3), and

$$u_x(0,t) = u_x(1,t), \quad t \in [0,T]. \tag{2.8}$$

$$\begin{aligned} & h_i''(t) - u_{ttxx}(x_i,t) - \alpha u_{txx}(x_i,t) - \beta u_{xx}(x_i,t) \\ & = a(t)h_i(t) + a(t)g(x_i,t) + f(x_i,t) \ (i=1,2), \ \ t \in [0,T]. \end{aligned} \tag{2.9}$$

The following lemma is valid

Theorem 2.1 *Suppose that* $\varphi(x)$, $\psi(x) \in C^1[0,1]$, $\varphi'(1) = \varphi'(0)$, $\psi'(1) = \psi'(0)$, $p \in C[0,T]$, $p(t) \leq 0$, $t \in [0,T]$, $h_i \in C^2[0,T]\,(i=1,2)$, $f \in C(D_T)$, $\int_0^1 f(x,t)dx = 0$, $t \in [0,T]$, $g \in C(D_T)$, $\int_0^1 g(x,t)dx = 0$, $t \in [0,T]$, $h(t) \equiv h_1(t)g(x_2,t) - h_2(t)g(x_1,t) \neq 0$, $t \in [0,T]$, $\frac{\alpha^2}{4} - \beta > 0$ *and the compatibility conditions*

$$\int_0^1 \varphi(x)dx = 0, \quad \int_0^1 \psi(x)dx = 0, \tag{2.10}$$

$$h_i(0) = \int_0^T p(t)h(t)dt + \varphi(x_i), \quad h'_i(0) = \psi(x_i)\,(i=1,2) \tag{2.11}$$

holds. Then the following assertions are valid:

1 *each classical solution* $\{u(x,t), a(t), b(t)\}$ *of the problem* (2.1)-(2.5) *is a solution of problem* (2.1)-(2.3)*,* (2.8)*,* (2.9)*, as well;*
2 *each solution* $\{u(x,t), a(t), b(t)\}$ *of the problem* (2.1)-(2.3)*,* (2.8)*,* (2.9)*, if*

$$\left(\|p(t)\|_{C[0,T]} + \frac{T}{2}\|a(t)\|_{C[0,T]}\right) T < 1, \tag{2.12}$$

is a classical solution of problem (2.1)-(2.5).

Proof. Let $\{u(x,t), a(t), b(t)\}$ be any classical solution to problem (2.1)-(2.5). By integrating both sides of equation (2.1) with respect to x from 0 to 1, we find

$$\begin{aligned} & \frac{d^2}{dt^2}\int_0^1 u(x,t)dx - (u_{ttx}(1,t) - u_{ttx}(0,t)) - \alpha(u_{tx}(1,t) - u_{tx}(0,t)) - \beta(u_x(1,t) - u_x(0,t)) \\ & = a\int_0^1 u(x,t)dx + b\int_0^1 g(x,t)dx + \int_0^1 f(x,t)dx, \quad t \in [0,T]. \end{aligned} \tag{2.13}$$

Using the fact that $\int_0^1 f(x,t)dx = 0$, $\int_0^1 g(x,t)dx = 0$, $t \in [0,T]$, and the conditions (2.3),(2.4) we find that:

$$u_{ttx}(1,t) - u_{ttx}(0,t) + \alpha\left(u_{tx}(1,t) - u_{tx}(0,t)\right) + \beta\left(u_x(1,t) - u_x(0,t)\right) = 0, t \in [0,T]. \tag{2.14}$$

It's obvious that the general solution of equation (2.14) has the form:

$$u_x(1,t)-u_x(0,t)=c_1e^{\mu_1 t}+c_2e^{\mu_2 t}, \tag{2.15}$$

where c_1, c_2 are the unknown number and

$$\mu_1=-\frac{\alpha}{2}-\sqrt{\frac{\alpha^2}{4}-\beta} \quad , \ \mu_2=-\frac{\alpha}{2}+\sqrt{\frac{\alpha^2}{4}-\beta}$$

By (2.2) and $\varphi'(2.1)=\varphi'(0)$, $\psi'(2.1)=\psi'(0)$ we obtain:

$$\begin{aligned}&u_x(1,0)-u_x(0,0)-\int_0^T p(t)\left(u_x(1,t)-u_x(0,t)\right)dt\\ &=u_x(1,0)-\int_0^T p(t)u_x(1,t)dt--\left(u_x(0,0)-\int_0^T p(t)u_x(0,t)dt\right)\\ &=\varphi'(1)-\varphi'(0)=0,\ u_{tx}(1,0)-u_{tx}(0,0)=\psi'(1)-\psi'(0)=0.\end{aligned} \tag{2.16}$$

Using (2.15) and (2.16) we obtain

$$c_1+c_2-\int_0^T p(t)\left(c_1e^{\mu_1 t}+c_2\,e^{\mu_2 t}\right)dt=0,\ c_1\mu_1+c_2\mu_2\ =0\,.$$

Hence we find:

$$c_2=-\frac{\mu_1}{\mu_2}c_1,\ \ c_1\left(\mu_2-\mu_1\ -\int_0^T p(t)\left(\,(\mu_2-\mu_1)e^{\mu_1 t}-\mu_1(e^{\mu_2 t}-e^{\mu_1 t})\right)dt\right)\ =0.$$

By $p(t)\le 0$, $\mu_1\ <0$, $\mu_2-\mu_1\ =2\sqrt{\frac{\alpha^2}{4}-\beta}\ >0$, from the latter relations we have $c_1=c_2=0$.

Putting the value of $c_1=c_2=0$ in (2.15), we get that the problem (2.14), (2.16) has only the trivial solution, i.e. we conclude that the statement (2.8) is true.

Substituting $x=x_i$ in equation (2.1) we find:

$$\begin{aligned}&u_{tt}(x_i,t)-u_{ttxx}(x_i,t)-\alpha u_{txx}(x_i,t)-\beta u_{xx}(x_i,t)\\ &=a(t)u(x_i,t)+b(t)g(x_i,t)+f(x_i,t)\,(i=1,2),\quad t\in[0,T].\end{aligned} \tag{2.17}$$

Assume now that $h_i\in C^2[0,T]$ $(i=1,2)$. Differentiating (2.5) twice, we get

$$u_{tt}(x_i,t)=h_i''(t)\ \ (i=1,2),\ \ t\in[0,T], \tag{2.18}$$

From (2.17), taking into account (2.5) and (2.18), we conclude that the relation (2.9) is fulfilled.

Now, assume that $\{u(x,t)\,,\ a(t),\ b(t)\}$ is the solution to problem (2.1)-(2.3), (2.8), (2.9). Then from (2.13), taking into account the condition $\int_0^1 f(x,t)dx=0$, $\int_0^1 g(x,t)dx=0$, $t\in[0,T]$ and relations (2.3), (2.8) we have

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx=a\int_0^1 u(x,t)dx,\quad t\in[0,T]. \tag{2.19}$$

Furthermore, from (2.2) and (2.10) it is easy to see that

$$\begin{aligned}&\int_0^1 u(x,0)dx-\int_0^T p(t)\left(\int_0^1 u(x,t)dx\right)dt\\ &=\int_0^1\left(u(x,0)-\int_0^T p(t)u(x,t)dt\right)dx=\int_0^1\varphi(x)dx=0,\end{aligned}$$

$$\int_0^1 u_t(x,0)dx = \int_0^1 \psi(x)dx = 0. \tag{2.20}$$

Since, by Lemma 1, problem (2.19), (2.20) has only a trivial solution. It means that $\int_0^1 u(x,t)dx = 0, \quad t \in [0,T]$, i.e. the condition (2.4) is satisfied.

Next, from (2.9) and (2.17), we obtain

$$\frac{d^2}{dt^2}(u(x_i,t) - h_i(t)) = a(t)(u(x_i,t) - h_i(t)) \ (i = 1,2), \quad 0 \le t \le T. \tag{2.21}$$

By virtue of (2.2) and the compatibility conditions (2.11), we have

$$u(x_i,0) - h_i(0) - \int_0^T p(t)(u(x_i,t) - h_i(t))dt$$

$$= u(x_i,0) - \int_0^T p(t)u(x_i,t)dt - \left(h_i(0) - \int_0^T p(t)h_i(t)dt\right)$$

$$= \varphi(x_i) - \left(h_i(0) - \int_0^T p(t)h_i(t)dt\right) = 0, u_t(x_i,0) - h_i'(0) = 0 \, (i = 1,2). \tag{2.22}$$

Using Lemma 1, and relations (2.21), (2.22), we conclude that condition (2.5) is satisfied. The theorem is proved.

3 Existence and uniqueness of the classical solution to the inverse boundary value problem

It is known [8] that the system

$$1, \ \cos\lambda_1 x, \ \sin\lambda_1 x, \ ..., \ \cos\lambda_k x, \ \sin\lambda_k x, ... \tag{3.1}$$

is a basis in $L_2(0,1)$, where $\lambda_k = 2k\pi \ \ (k = 1,2,...)$.

Then the first component of classical solution $\{u(x,t)\,,\ a(t),\ b(t)\}$ of the problem (2.1)-(2.3), (2.8), (2.9) has the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x, \quad \lambda_k = 2\pi k, \tag{3.2}$$

where

$$u_{10}(t) = \int_0^1 u(x,t)dx,$$

$$u_{1k}(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx, \quad k = 1,2,...,$$

$$u_{2k}(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx, \quad k = 1,2,.....$$

Then, applying the formal scheme of the Fourier method, from (2.1) - (2.2) we have

$$u_{10}''(t) = F_{10}(t;\, u,\, a,\, b), \quad 0 \le t \le T, \tag{3.3}$$

$$(1+\lambda_k^2)\, u_{ik}''(t) + \alpha\lambda_k^2 u_{ik}'(t) + \beta\lambda_k^2 u_{ik}(t) = F_{ik}(t;\, u,\, a,\, b), \ i = 1,2; \quad k = 0,1,2\ldots; \ \ 0 \le t \le T, \tag{3.4}$$

$$u_{10}(0)=\varphi_{10}+\int_0^T p(t)u_{10}(t)dt\ ,\quad u'_{10}(0)=\psi_{10}\ , \tag{3.5}$$

$$u_{ik}(0)=\varphi_{ik}+\int_0^T p(t)u_{ik}(t)dt,\quad u'_{ik}(0)=\psi_{ik},\ \ i=1,2;\ \ k=1,2\ldots. \tag{3.6}$$

where

$$F_{1k}(t;u,a,b)=a(t)u_{1k}(t)+b(t)g_{1k}(t)+f_{1k}(t),\quad k=0,1,...,$$

$$f_{10}(t)=\int_0^1 f(x,t)dx,\quad f_{1k}(t)=2\int_0^1 f(x,t)\cos\lambda_k x dx,\quad k=1,2,...,$$

$$g_{10}(t)=\int_0^1 g(x,t)dx,\quad g_{1k}(t)=2\int_0^1 g(x,t)\cos\lambda_k x dx,\quad k=1,2,...,$$

$$\varphi_{10}=\int_0^1 \varphi(x)dx,\quad \psi_{10}=2\int_0^1 \psi(x)dx,$$

$$\varphi_{1k}=2\int_0^1 \varphi(x)\cos\lambda_k x dx,\quad \psi_{1k}=2\int_0^1 \psi(x)\cos\lambda_k x dx,\quad k=0,1,...,$$

$$F_{2k}(t;u,a,b)=a(t)u_{2k}(t)+b(t)g_{2k}(t)+f_{2k}(t),$$

$$f_{2k}(t)=2\int_0^1 f(x,t)\sin\lambda_k x dx,\quad g_{2k}(t)=2\int_0^1 g(x,t)\sin\lambda_k x dx,\ \ k=1,2,...,$$

$$\varphi_{2k}=2\int_0^1 \varphi(x)\sin\lambda_k x dx,\quad k=1,2,...,\quad \psi_{2k}=2\int_0^1 \psi(x)\sin\lambda_k x dx,\quad k=1,2,....$$

It is obvious that $\lambda_k^2<1+\lambda_k^2<2\lambda_k^2$ $(k=1,2,...)$. Therefore

$$\frac{\alpha^2}{8}-\beta<\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)}-\beta<\frac{\alpha^2}{4}-\beta\quad (k=1,2,...).$$

Now suppose that $\frac{\alpha^2}{8}-\beta>0$. Solving the problem (3.3)–(3.6), we fin

$$u_{10}(t)=\varphi_{10}+\int_0^T p(t)u_{10}(t)dt+t\psi_{10}+\int_0^t (t-\tau)F_{10}(\tau;\ u,\ a,b)d\tau, \tag{3.7}$$

$$u_{ik}(t)=\frac{1}{\gamma_k}\left[\left(\mu_{2k}e^{\mu_{1k}t}-\mu_{1k}e^{\mu_{2k}t}\right)\left(\varphi_{ik}+\int_0^T p(t)u_{ik}(t)dt\right)+\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)\psi_{ik}\right.$$

$$\left.+\frac{1}{1+\lambda_k^2}\int_0^t F_{ik}(\tau;u,a,b)\left(e^{\mu_{2k}(t-\tau)}-e^{\mu_{1k}(t-\tau)}\right)d\tau\right] \tag{3.8}$$

$$(0\le t\le T;\ i=1,2;\ k=1,2,...),$$

where

$$\mu_{1k}=-\frac{\alpha\lambda_k^2}{2(1+\lambda_k^2)}-\lambda_k\sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2}-\frac{\beta}{1+\lambda_k^2}}\quad ,$$

$$\mu_{2k}=-\frac{\alpha\lambda_k^2}{2(1+\lambda_k^2)}+\lambda_k\sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2}-\frac{\beta}{1+\lambda_k^2}}\quad ,$$

$$\gamma_k=\mu_{2k}-\mu_{1k}=2\lambda_k\sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2}-\frac{\beta}{1+\lambda_k^2}}.$$

Differentiating (3.1) twice, we get:

$$u'_{ik}(t)=\frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\left(e^{\mu_{1k}t}-e^{\mu_{2k}t}\right)\left(\varphi_{ik}+\int_0^T P_2(t)u_{ik}(t)dt\right)+\left(\mu_{2k}e^{\mu_{2k}t}-\mu_{1k}e^{\mu_{1k}t}\right)\psi_{ik}\right.$$
$$\left.+\frac{1}{1+\lambda_k^2}\int_0^t F_{ik}(\tau;u,a,b)\left(\mu_{2k}e^{\mu_{2k}(t-\tau)}-\ \mu_{1k}e^{\mu_{1k}(t-\tau)}\right)d\tau\right]\ \ (0\le t\le T;\ i=1,2;\ k=1,2,...),\tag{3.9}$$

$$u''_{ik}(t)=\frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\left(\mu_{1k}e^{\mu_{1k}t}-\mu_{2k}e^{\mu_{2k}t}\right)\left(\varphi_{ik}+\int_0^T P_2(t)u_{ik}(t)dt\right)\right.$$
$$+\left(\mu_{2k}^2e^{\mu_{2k}t}-\mu_{1k}^2e^{\mu_{1k}t}\right)\psi_{ik}+\frac{1}{1+\lambda_k^2}\int_0^t F_{ik}(\tau;u,a,b)\left(\mu_{2k}^2e^{\mu_{2k}(t-\tau)}\right.$$
$$\left.\left.-\ \mu_{1k}^2e^{\mu_{1k}(t-\tau)}\right)d\tau\right]+\frac{1}{1+\lambda_k^2}F_{ik}(\,t;u,a,b)\quad(\,k=1,2,...).\tag{3.10}$$

To determine the first component of the classical solution to the problem (2.1)-(2.3), (2.8), (2.9) we substitute the expressions $u_{1k}(t)\ \ (k=0,\,1,...)$, $u_{2k}(t)\ \ (k=1,\,2,...)$ into (3.2) and obtain

$$u(x,t)=\varphi_{10}+t\left(\psi_{10}+\int_0^T p(t)u_{10}(t)dt\right)+\int_0^t(t-\tau)F_{10}(\tau\,;u,a,b)d\tau$$
$$+\sum_{k=1}^{\infty}\left\{\frac{1}{\gamma_k}\left[\left(\mu_{2k}e^{\mu_{1k}t}-\mu_{1k}e^{\mu_{2k}t}\right)\left(\varphi_{1k}+\int_0^T P_2(t)u_{1k}(t)dt\right)+\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)\psi_{1k}\right.\right.$$
$$\left.\left.+\frac{1}{1+\lambda_k^2}\int_0^t F_{1k}(\tau;u,a,b)\left(e^{\mu_{2k}(t-\tau)}-e^{\mu_{1k}(t-\tau)}\right)d\tau\right]\right\}\cos\lambda_k x$$
$$+\sum_{k=1}^{\infty}\left\{\frac{1}{\gamma_k}\left[\left(\mu_{2k}e^{\mu_{1k}t}-\mu_{1k}e^{\mu_{2k}t}\right)\left(\varphi_{2k}+\int_0^T P_2(t)u_{2k}(t)dt\right)+\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)\psi_{2k}\right.\right.$$
$$\left.\left.+\frac{1}{1+\lambda_k^2}\int_0^t F_{2k}(\tau;u,a,b)\left(e^{\mu_{2k}(t-\tau)}-e^{\mu_{1k}(t-\tau)}\right)d\tau\right]\right\}\sin\lambda_k x.\tag{3.11}$$

It follows from (2.9) and (3.2) that

$$a(t)=[h(t)]^{-1}\left\{g(x_2,t)\ \left(h''_1(t)-f(x_1,t)\right)-g(x_1,t)\left(h''_2(t)-f(x_2,t)\right)\right.$$
$$+\sum_{k=1}^{\infty}(\lambda_k^2u''_{1k}(t)+\alpha\,\lambda_k^2u'_{1k}(t)+\beta\,\lambda_k^2u_{1k}(t))(g(x_2,t)\cos\lambda_kx_1-g(x_1,t)\cos\lambda_kx_2)$$
$$\left.+\sum_{k=1}^{\infty}(\lambda_k^2u''_{2k}(t)+\alpha\,\lambda_k^2u'_{2k}(t)+\beta\,\lambda_k^2u_{2k}(t))(g(x_2,t)\sin\lambda_kx_1-g(x_1,\,t)\sin\lambda_kx_2)\right\},\tag{3.12}$$

$$b(t) = [h(t)]^{-1} \left\{ h_1(t) \ \left(h_2''(t) - f(x_2,t)\right) - h_2(t) \left(h_1''(t) - f(x_1,t)\right) \right.$$

$$+ \sum_{k=1}^{\infty} (\lambda_k^2 u_{1k}''(t) + \alpha\, \lambda_k^2 u_{1k}'(t) + \beta\, \lambda_k^2 u_{1k}(t))(h_1(t)\cos\lambda_k x_2 - h_2(t)\cos\lambda_k x_1)$$

$$\left. + \sum_{k=1}^{\infty} (\lambda_k^2 u_{2k}''(t) + \alpha\, \lambda_k^2 u_{2k}'(t) + \beta\, \lambda_k^2 u_{2k}(t))(h_1(t)\sin\lambda_k x_2 - h_2(t)\sin\lambda_k x_1) \right\}. \tag{3.13}$$

By (3.4) and (3.10) we have:

$$\lambda_k^2\, u_{ik}''(t) + \alpha\lambda_k^2 u_{ik}'(t) + \beta\lambda_k^2 u_{ik}(t) = F_{ik}(\,t; u, a, b) - \ u_{ik}''(t)$$

$$= -\frac{1}{\gamma_k} \left[\mu_{1k}\mu_{2k} \ \left(\mu_{1k} e^{\mu_{1k} t} - \mu_{2k} e^{\mu_{2k} t}\right) \left(\varphi_{ik} + \int_0^T p(t) u_{ik}(t) dt \right) + \left(\mu_{2k}^2 e^{\mu_{2k} t} - \mu_{1k}^2 e^{\mu_{1k} t}\right) \psi_{ik} \right.$$

$$\left. + \frac{1}{1+\lambda_k^2} \int_0^t F_{ik}(\tau; u, a, b) \left(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)} - \ \mu_{1k}^2 e^{\mu_{1k}(t-\tau)} \right) d\tau \right]$$

$$+ \frac{\lambda_k^2}{1+\lambda_k^2} F_{ik}(\,t; u, a, b) \qquad (0 \le t \le T; \ i = 1, 2; \ k = 1, 2, ...). \tag{3.14}$$

We substitute expression (3.14) into (3.12), (3.13) and have

$$a(t) = [h(t)]^{-1} \left\{ g(x_2,t) \ \left(h_1''(t) - f(x_1,t)\right) - g(x_1,t) \left(h_2''(t) - f(x_2,t)\right) \right.$$

$$- \sum_{k=1}^{\infty} \left[\frac{1}{\gamma_k} \left[\mu_{1k}\mu_{2k} \ \left(\mu_{1k} e^{\mu_{1k} t} - \mu_{2k} e^{\mu_{2k} t}\right) (\varphi_{1k} + \int_0^T p(t) u_{1k}(t) dt) \right. \right.$$

$$+ \left(\mu_{2k}^2 e^{\mu_{2k} t} - \mu_{1k}^2 e^{\mu_{1k} t}\right) \psi_{1k}$$

$$\left. \left. + \frac{1}{1+\lambda_k^2} \int_0^t F_{1k}(\tau; u, a, b) \left(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)} - \ \mu_{1k}^2 e^{\mu_{1k}(t-\tau)} \right) d\tau \right] - \frac{\lambda_k^2}{1+\lambda_k^2} F_{1k}(\,t; u, a, b) \right]$$

$$\times (g(x_2,t)\cos\lambda_k x_1 - g(x_1,t)\cos\lambda_k x_2)$$

$$- \sum_{k=1}^{\infty} \left[\frac{1}{\gamma_k} \left[\mu_{1k}\mu_{2k} \ \left(\mu_{1k} e^{\mu_{1k} t} - \mu_{2k} e^{\mu_{2k} t}\right) \left(\varphi_{2k} + \int_0^T p(t) u_{2k}(t) dt \right) \right. \right.$$

$$+ \left(\mu_{2k}^2 e^{\mu_{2k} t} - \mu_{1k}^2 e^{\mu_{1k} t}\right) \psi_{2k}$$

$$\left. \left. + \frac{1}{1+\lambda_k^2} \int_0^t F_{2k}(\tau; u, a, b) \left(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)} - \ \mu_{1k}^2 e^{\mu_{1k}(t-\tau)} \right) d\tau \right] - \frac{\lambda_k^2}{1+\lambda_k^2} F_{2k}(\,t; u, a, b) \right]$$

$$\left. \times (g(x_2,t)\sin\lambda_k x_1 - g(x_1,\, t)\sin\lambda_k x_2) \right\}, \tag{3.15}$$

$$b(t) = [h(t)]^{-1} \left\{ h_1(t) \ \left(h_2''(t) - f(x_2,t)\right) - h_2(t) \left(h_1''(t) - f(x_1,t)\right) \right.$$

$$- \sum_{k=1}^{\infty} \left[\frac{1}{\gamma_k} \left[\mu_{1k}\mu_{2k} \ \left(\mu_{1k} e^{\mu_{1k} t} - \mu_{2k} e^{\mu_{2k} t}\right) \left(\varphi_{1k} + \int_0^T p(t) u_{1k}(t) dt \right) + \left(\mu_{2k}^2 e^{\mu_{2k} t} - \mu_{1k}^2 e^{\mu_{1k} t}\right) \psi_{1k} \right. \right.$$

$$+\frac{1}{1+\lambda_k^2}\int_0^t F_{1k}(\tau;u,a,b)\left(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)}- \ \mu_{1k}^2 e^{\mu_{1k}(t-\tau)}\right)d\tau\Big] - \frac{\lambda_k^2}{1+\lambda_k^2}F_{1k}(\,t;u,a,b)\Big]$$

$$\times(h_1(t)\cos\lambda_k x_2 - h_2(t)\cos\lambda_k x_1)$$

$$-\sum_{k=1}^{\infty}\left[\frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\ \left(\mu_{1k}e^{\mu_{1k}t}-\mu_{2k}e^{\mu_{2k}t}\right)\left(\varphi_{2k}+\int\limits_0^T p(t)u_{2k}(t)dt\right)+\left(\mu_{2k}^2 e^{\mu_{2k}t}-\mu_{1k}^2 e^{\mu_{1k}t}\right)\psi_{2k}\right.\right.$$

$$+\frac{1}{1+\lambda_k^2}\int_0^t F_{2k}(\tau;u,a,b)\left(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)}- \ \mu_{1k}^2 e^{\mu_{1k}(t-\tau)}\right)d\tau\Big] - \frac{\lambda_k^2}{1+\lambda_k^2}F_{2k}(\,t;u,a,b)\Big]$$

$$\times(h_1(t)\sin\lambda_k x_2 - h_2(t)\sin\lambda_k x_1)\Big\}. \qquad (3.16)$$

Thus, the solution of problem (2.1) - (2.3), (2.7), (2.9) was reduced to the solution of system (3.11), (3.15), (38) with respect to unknown functions $u(x,t)$, $a(t)$ and $b(t)$.

Lemma 3.1 *If* $\{u(x,t),a(t),b(t)\}$ *is any solution to problem* (2.1) - (2.3)*,* (2.8)*,* (2.9)*, then the functions*

$$u_{10}(t)=\int_0^1 u(x,t)dx,$$

$$u_{1k}(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx,\quad k=1,2,...,$$

$$u_{2k}(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx,\quad k=1,2,...$$

satisfies the system (3.7), in $C[0,T]$.

It follows from Lemma 3.1 that

Corollary 3.1 *Let system (3.11), (3.15),* (3.16) *have a unique solution. Then problem* (2.1) - (2.3)*,* (2.8)*,* (2.9) *cannot have more than one solution, i.e. if the problem* (2.1) - (2.3)*,* (2.8)*,* (2.9) *has a solution, then it is unique.*

With the purpose to study the problem (2.1) - (2.3), (2.8), (2.9), we consider the following functional spaces.

Denote by $B_{2,T}^3$ [8] a set of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_{1k}(t)\cos\lambda_k x+\sum_{k=1}^{\infty}u_{2k}(t)\sin\lambda_k x,\quad \lambda_k=2\pi k$$

considered in the region D_T, where each of the function $u_{1k}(t)$ $(k=0,\,1,\,2,...)$, $u_{2k}(t)$ $(k=1,\,2,...)$ is continuous over an interval $[0,T]$ and satisfies the following condition:

$$J(u)\equiv\left\|u_{10}\right\|_{C[0,T]}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_{1k}\right\|_{C[0,T]}\right)^2\right\}^{\frac12}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_{2k}\right\|_{C[0,T]}\right)^2\right\}^{\frac12}<+\infty.$$

The norm in this set is defined by

$$\|u\|_{B_{2,T}^3}=J(u)\,.$$

It is known that $B_{2,T}^3$ is Banach .

Obviously, $E_T^3 = B_{2,T}^3 \times C[0,T] \times C[0,T]$ with the norm $\|z\|_{E_T^3} = \|u\|_{B_{2,T}^3} + \|a\|_{C[0,T]} + \|b\|_{C[0,T]}$ is also Banach space.

Now consider the operator

$$\Phi(u,a,b) = \{\Phi_1(u,a,b)\,,\, \Phi_2(u,a,b)\,,\, \Phi_3(u,a,b)\}\,,$$

in the space E_T^3, where

$$\Phi_1(u,a,b) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} \tilde{u}_{2k}(t)\sin\lambda_k x,$$

$$\Phi_2(u,a,b) = \tilde{a}(t),\;\; \Phi_3(u,a,b) = \tilde{b}(t)$$

and the functions $\tilde{u}_{10}(t)$, $\tilde{u}_{ik}(t)$, $i = 1,2;\; k = 1, 2, \ldots,$ $\tilde{a}(t)$and $\tilde{b}(t)$ are equal to the right-hand sides of (3.7), (3.8) ,(3.15), and (3.16), respectively.

It is easy to see that

$$\mu_{ik} < 0, \quad e^{\mu_{ik}t} < 1,\; e^{\mu_{ik}(t-\tau)} < 1\;\;, (i = 1,2;\; 0 \le t \le T;\; 0 \le \tau \le t)$$

$$|\mu_{ik}| \le \lambda_k \left(\frac{\alpha\lambda_k}{2(1+\lambda_k^2)} + \sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \frac{\beta}{1+\lambda_k^2}} \right) \le \frac{\alpha\lambda_k}{1+\lambda_k^2} \le \alpha \;\; (i = 1,2),$$

$$|\mu_{1k}\mu_{2k}| \le \frac{\beta\lambda_k}{1+\lambda_k^2} \le \beta,\, \frac{1}{\gamma_k} = \frac{1}{2\sqrt{\frac{\lambda_k^2}{1+\lambda_k^2}\left(\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \beta\right)}} \le \frac{1}{2\sqrt{\frac{1}{2}\left(\frac{\alpha^2}{8} - \beta\right)}} \equiv \gamma_0\,.$$

Taking into account these relations, by means of simple transformations we find:

$$\|\tilde{u}_{10}\|_{C[0,T]} \le |\varphi_{10}| + T\,\|p\|_{C[0,T]}\,\|u_{10}\|_{C[0,T]} + T\,|\psi_{10}| + T\sqrt{T}\left(\int_0^T |f_{10}(\tau)|^2\,d\tau\right)^{\frac{1}{2}}$$

$$+T\sqrt{T}\,\|b\|_{C[0,T]}\left(\int_0^T |g_{10}(\tau)|^2\,d\tau\right)^{\frac{1}{2}} + T^2\,\|a\|_{C[0,T]}\,\|u_{10}\|_{C[0,T]}\,, \qquad (3.17)$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|\tilde{u}_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{6}\alpha\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}} + \sqrt{6}\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_{ik}|)^2\right)^{\frac{1}{2}}$$

$$+\gamma_0\sqrt{6T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|f_{ik}(\tau)|)^2 d\tau\right)^{\frac{1}{2}} + \gamma_0\sqrt{6T}\,\|b\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|g_{ik}(\tau)|)^2 d\tau\right)^{\frac{1}{2}}$$

$$+\sqrt{6}T\gamma_0\left(\|p\|_{C[0,T]} + \|a\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \qquad (3.18)$$

$$\|\tilde{a}\|_{C[0,T]} \le \left\|h^{-1}\right\|_{C[0,T]}\Big\{\big\|g(x_2,\cdot)\,\left(h_1''(\cdot) - f(x_1,\cdot)\right) - g(x_1,\cdot)\left(h_2''(\cdot) - f(x_2,\cdot)\right)\big\|_{C[0,T]}$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|\,|g(x_1,\cdot)| + |g(x_2,\cdot)|\|_{C[0,T]}\sum_{i=1}^{2}\left[2\alpha\beta\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}}\right.$$

$$+2\alpha^2\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_{ik}|)^2\right)^{\frac{1}{2}}+2\alpha^2\gamma_0 T\,\|p\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\alpha^2\gamma_0\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|f_{ik}(\tau)|)^2d\tau\right)^{\frac{1}{2}}$$

$$+2\alpha^2\gamma_0\sqrt{T}\,\|b\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|g_{ik}(\tau)|)^2d\tau\right)^{\frac{1}{2}}$$

$$+2\alpha^2\gamma_0 T\,\|a\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(\lambda_k\,\|f_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\|b\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\,\|g_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\|a\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}, \tag{3.19}$$

$$\left\|\tilde{b}\right\|_{C[0,T]}\leq\left\|h^{-1}\right\|_{C[0,T]}\Big\{\|\,h_1(\cdot)\left(h_2''(\cdot)-f(x_2,\cdot)\right)-h_2(\cdot)\left(h_1''(\cdot)-f(x_1,\cdot)\right)\|_{C[0,T]}$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|\,|h_1(\cdot)|+|h_2(\cdot)|\|_{C[0,T]}\sum_{i=1}^{2}\Bigg[2\alpha\beta\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}}$$

$$+2\alpha^2\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_{ik}|)^2\right)^{\frac{1}{2}}+2\alpha^2\gamma_0 T\,\|p\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\alpha^2\gamma_0\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|f_{ik}(\tau)|)^2d\tau\right)^{\frac{1}{2}}$$

$$+2\alpha^2\gamma_0\sqrt{T}\,\|b\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|g_{ik}(\tau)|)^2d\tau\right)^{\frac{1}{2}}$$

$$+2\alpha^2\gamma_0 T\,\|a\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(\lambda_k\,\|f_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\|b\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\,\|g_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\|a\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}, \tag{3.20}$$

Suppose that the data for problem (2.1)-(2.3), (2.8), (2.9) satisfy the assumptions:

1 $\varphi\in C^2[0,1]$, $\varphi'''\in L_2(0,1)$, $\varphi(0)=\varphi(1)$, $\varphi'(0)=\varphi'(1)$, $\varphi''(0)=\varphi''(1)$;
2 $\psi\in C^2[0,1]$, $\psi'''\in L_2(0,1)$, $\psi(0)=\psi(1)$, $\psi'(0)=\psi'(1)$, $\psi''(0)=\psi''(1)$;
3 $f, f_x, f_{xx}\in C(D_T)$, $f_{xxx}\in L_2(D_T)$, $f(0,t)=f(1,t)$,

$$f_x(0,t)=f_x(1,t),\ f_{xx}(0,t)=f_{xx}(1,t),\ 0\leq t\leq T;$$

4 $g(x,t), g_x(x,t), g_{xx} \in C(D_T)\,, g_{xxx} \in L_2(D_T),\ g(0,t) = g(1,t)\,,$

$$g_x(0,t) = g_x(1,t)\,, g_{xx}(0,t) = g_{xx}(1,t)\,,\ 0 \le t \le T;$$

5 $p \in C[0,T]\,, h_i \in C^2[0,T]\,(i = 1,2),\quad h(t) \equiv h_1(t)g(x_2,t) - h_2(t)g(x_1,t) \neq 0,\quad 0 \le t \le T;$

6 $\alpha > 0,\ \beta > 0,\ \frac{\alpha^2}{8} - \beta > 0.$

Then, from (3.17)–(3.20), we get

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \le A_1(T) + B_1(T)\,\|a\|_{C[0,T]}\,\|u\|_{B^3_{2,T}} + C_1(T)\,\|u\|_{B^3_{2,T}} + D_1(T)\,\|b\|_{C[0,T]}\ , \tag{3.21}$$

$$\|\tilde{a}\|_{C[0,T]} \le A_2(T) + B_2(T)\,\|a\|_{C[0,T]}\,\|u\|_{B^3_{2,T}} + C_2(T)\,\|u\|_{B^3_{2,T}} + D_2(T)\,\|b\|_{C[0,T]}\ , \tag{3.22}$$

$$\left\|\tilde{b}\right\|_{C[0,T]} \le A_3(T) + B_3(T)\,\|a\|_{C[0,T]}\,\|u\|_{B^3_{2,T}} + C_3(T)\,\|u\|_{B^3_{2,T}} + D_3(T)\,\|b\|_{C[0,T]}\ , \tag{3.23}$$

where

$$A_1(T) = \|\varphi\|_{L_2(0,1)} + T\,\|\psi\|_{L_2(0,1)} + T\sqrt{T}\,\|f\|_{L_2(D_T)}$$
$$+2\sqrt{6}\alpha\gamma_0\,\|\varphi'''\|_{L_2(0,1)} + 2\sqrt{6}\gamma_0\,\|\psi'''\|_{L_2(0,1)} + 2\gamma_0\sqrt{6T}\,\|f_{xxx}\|_{L_2(D_T)}\,,$$
$$B_1(T) = T^2 + \gamma_0\sqrt{6}T, C_1(T) = T(T + \sqrt{6}\gamma_0)\,\|p\|_{C[0,T]}\,,$$
$$D_1(t) = T\sqrt{T}\,\|g\|_{L_2(D_T)} + 2\gamma_0\sqrt{6T}\,\|g_{xxx}\|_{L_2(D_T)}\,,$$

$$A_2(T) = \left\|h^{-1}\right\|_{C[0,T]}\Big\{\big\|\,g(x_1,\cdot)\left(h_1''(\cdot) - f(x_1,\cdot)\right) - g(x_1,\cdot)\left(h_2''(\cdot) - f(x_2,\cdot)\right)\big\|_{C[0,T]}$$
$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|\,|g(x_1,\cdot)| + |g(x_2,\cdot)|\,\|_{C[0,T]}\Big[4\alpha\beta\gamma_0\,\|\varphi'''\|_{L_2(0,1)}$$
$$+4\alpha^2\gamma_0\,\|\psi'''\|_{L_2(0,1)} + 4\alpha^2\gamma_0\sqrt{T}\,\|f_{xxx}\|_{L_2(D_T)} + 2\,\Big\|\|f_x\|_{C[0,T]}\Big\|_{L_2(0,1)}\Big]\Big\},$$

$$B_2(T) = \left\|h^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|\,|g(x_1,\cdot)| + |g(x_2,\cdot)|\,\|_{C[0,T]}\left(2\alpha^2\gamma_0 T + 1\right)\,,$$

$$C_2(T) = \left\|h^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|\,|g(x_1,\cdot)| + |g(x_2,\cdot)|\,\|_{C[0,T]}\,T(T+\sqrt{5}\gamma_0)\,\|p\|_{C[0,T]}\ ,$$

$$D_2(T) = \left\|h^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||g(x_1,\cdot)| + |g(x_2,\cdot)|\|_{C[0,T]}$$
$$\times\left(4\alpha^2\gamma_0\sqrt{T}\,\|g_{xxx}\|_{L_2(D_T)} + 2\,\Big\|\|g_x\|_{C[0,T]}\Big\|_{L_2(0,1)}\Big]\right),$$

$$A_3(T) = \left\| h^{-1} \right\|_{C[0,T]} \left\{ \left\| \, h_1(\cdot) \left(h_2''(\cdot) - f(x_2, \cdot) \right) - h_2(\cdot) \left(h_1''(\cdot) - f(x_1, \cdot) \right) \right\|_{C[0,T]} \right.$$

$$+ \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left\| \, |g(x_1, \cdot)| + |g(x_2, \cdot)| \right\|_{C[0,T]} \left[4\alpha\beta\gamma_0 \left\| \varphi''' \right\|_{L_2(0,1)} \right.$$

$$\left. \left. +4\alpha^2\gamma_0 \left\| \psi''' \right\|_{L_2(0,1)} + 4\alpha^2\gamma_0 \sqrt{T} \left\| f_{xxx} \right\|_{L_2(D_T)} + 2 \left\| \left\| f_x \right\|_{C[0,T]} \right\|_{L_2(0,1)} \right] \right\},$$

$$B_3(T) = \left\| h^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left\| \, |h_1(\cdot)| + |h_2(\cdot)| \right\|_{C[0,T]} \left(2\alpha^2\gamma_0 T + 1 \right),$$

$$C_3(T) = \left\| h^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left\| \, |h_1(\cdot)| + |h_2(\cdot)| \right\|_{C[0,T]} T(T + \sqrt{5}\gamma_0) \left\| p \right\|_{C[0,T]},$$

$$D_3(T) = \left\| h^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left\| \, |h_1(\cdot)| + |h_2(\cdot)| \right\|_{C[0,T]}$$

$$\times \left(4\alpha^2\gamma_0 \sqrt{T} \left\| g_{xxx} \right\|_{L_2(D_T)} + 2 \left\| \left\| g_x \right\|_{C[0,T]} \right\|_{L_2(0,1)} \right] \Bigg) .$$

It follows from the inequalities (3.21)-(3.23) that.

$$\left\| \tilde{u} \right\|_{B_{2,T}^3} + \left\| \tilde{a} \right\|_{C[0,T]} + \left\| \tilde{b} \right\|_{C[0,T]}$$

$$\leq A(T) + B(T) \left\| a \right\|_{C[0,T]} \left\| u \right\|_{B_{2,T}^3} + C(T) \left\| u \right\|_{B_{2,T}^3} + D(T) \left\| b \right\|_{C[0,T]}, \tag{3.24}$$

where

$$A(T) = A_1(T) + A_2(T) + A_3(T), \quad B(T) = B_1(T) + B_2(T) + B_3(T),$$

$$C(T) = C_1(T) + C_2(T) + C_3(T), \quad D(T) = D_1(T) + D_2(T) + D_3(T).$$

So, we can prove the following theorem.

Theorem 3.1 *Assume that statements 1)-6) and the condition*

$$(B(T)(A(T) + 2) + C(T) + D(T))(A(T) + 2) < 1 \tag{3.25}$$

holds, then problem (2.1)-(2.3), (2.8), (2.9) *has a unique solution in the ball* $K = K_R(\|z\|_{E_T^3} \leq R \leq A(T) + 2)$ *of the space* E_T^3.

Proof. In the space E_T^3consider the equation

$$z = \Phi z, \tag{3.26}$$

where $z = \{u, \, a, \, b\}$,, the components $\Phi_i(u, a, \, b)$ $(i = 1, 2, 3)$ of the operator $\Phi(u, a, \, b)$ are determined by the right hand sides of equations (3.11), (3.15) and (3.16). Consider the operator $\Phi(u, a, \, b)$ in the sphere $K = K_R$from E_T^3. Similar to (3.24), we get that for any $z_1, z_2, z_3 \in K_R$ the following estimations are valid:

$$\left\| \Phi z \right\|_{E_T^3} \leq A(T) + B(T) \left\| a \right\|_{C[0,T]} \left\| u \right\|_{B_{2,T}^3} + C(T) \left\| u \right\|_{B_{2,T}^3} + D(T) \left\| b \right\|_{C[0,T]}$$

$$\leq A(T) + B(T)(A(T) + 2)^2 + C(T)(A(T) + 2) + D(T)(A(T) + 2), \tag{3.27}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^3_T} \leq B(T)R(\|a_1 - a_2\|_{C[0,T]} + \|u_1 - u_2\|_{B^3_{2,T}})$$

$$+C(T)\,\|u_1 - u_2\|_{B^3_{2,T}} + D(T)\,\|b_1 - b_2\|_{C[0,T]}\ , \tag{3.28}$$

Then, it follows from (3.25) together with the estimates (3.27) and (3.28) that the operator Φ acts in the ball $K = K_R$and is contractive. Therefore, in the ball $K = K_R$the operator Φ has a unique
fixed point $\{z\} = \{u\,,\ a,\ b\}$, that is a unique solution to the equation (3.26), i.e. a unique solution to the system (3.11),(3.15), (3.16).

Then the function $u(x,t)$ as an element of space $B^3_{2,T}$ is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

Now from (3.9) it is obvious that $u'_{ik}(t)\,(i = 1,2;\ k = 1,2,...)$ is continuous in $[0,T]$ and from the same relation we get:

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u'_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$\leq 2\sqrt{5}\beta\gamma_0\,\|\varphi'''\|_{L_2(0,1)} + 2\sqrt{5}\alpha\,\|\psi'''\|_{L_2(0,1)} + 2\alpha\sqrt{6T}\,\|f_{xxx}\|_{L_2(D_T)}$$

$$+2\alpha\sqrt{6T}\,\|b\|_{C[0,T]}\,\|g_{xxx}\|_{L_2(D_T)}$$

$$+2\alpha\sqrt{6}T\left(\|p\|_{C[0,T]} + \|a\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k\|_{C[0,T]})^2\right)^{\frac{1}{2}}.$$

Hence, it follows that $u_t(x,t), u_{tx}(x,t), u_{txx}(x,t)$ are continuous in D_T .

Next, from (3.4) it follows that $u''_{ik}(t)\,(i = 1,2;\ k = 1,2,...)$ is continuous in $[0,T]$ and consequently we have:

$$\left(\sum_{k=1}^{\infty}(\lambda_k\,\|u''_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq 2\alpha\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u'_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\beta\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}\|_{C[0,T]})^2\right)^{\frac{1}{2}} + 2\left\|\|f_x + bg_x + au_x\|_{C[0,T]}\right\|_{L(0,1)}.$$

From the last relation it is obvious that $u_{tt}(x,t), u_{ttx}(x,t), u_{ttxx}(x,t)$ are continuous in D_T.

It is easy to verify that equation (2.1) and conditions (2.2), (2.3), (2.8), (2.9) satisfy in the usual sense. So, $\{u(x,t), a(t)\}$ is a solution of (2.1)-(2.3), (2.8), (2.9), and by Lemma 2 it is unique in the ball $K = K_R$. The proof is complete.

In summary, from Theorem 2.1 and Theorem 3.1, straightforward implies the unique solvability of the original problem (2.1) - (2.5).

Theorem 3.2 *Suppose that all assumptions of Theorem 3.1, and the conditions*

$$\int_0^1 \varphi(x)dx = 0,\quad \int_0^1 \psi(x)dx = 0, \int_0^1 f(x,t)dx = 0,$$

$$\int_0^1 g(x,t)dx = 0,\ t \in [0,T], p(t) \leq 0,\ t \in [0,T],$$

$$h_i(0) = \int_0^T p(t)h(t)dt + \varphi(x_i), \quad h'_i(0) = \psi(x_i)\,(i = 1, 2),$$

$$\left(\|p\|_{C[0,T]} + \frac{1}{2}(A(T) + 2)\right) T < 1,$$

holds. Then problem (2.1)-(2.5) *has a unique classical solution in the ball* $K = K_R(\|z\|_{E_T^3} \leq A(T) + 2)$ *of the space* E_T^3.

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