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The zeros of modified Bessel functions as functions of their order

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Abstract. *Zeros of the function* $aK_{\nu}(z) + bK_{\nu}'(z)$ *considered as a function of the order are studied, where* $K_{\nu}(z)$ *is the modified Bessel function of the second kind (Macdonald function). It is proved that, for fixed* z, $z > 0$ *and for any real values* a, b, *the function* $aK_{\nu}(z) + bK_{\nu}'(z)$ *has only a countable number of simple purely imaginary zeros* ν_n. *The asymptotics of the zeros* ν_n *as* $n \to +\infty$ *is found.*

Keywords. Bessel functions · zeros of Bessel functions · Schrödinger equation · eigenvalues.

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1 Introduction and formulation of the main results

The zeros of the ordinary and modified Bessel functions, like the functions themselves, have many applications in physics, mechanics, etc. Much of the work on the zeros of Bessel functions has basically been concerned with the functions of their arguments, i.e., when the order is fixed (see [1-4] and references therein). There are very few works devoted to the study of the zeros of Bessel functions considered as functions of an order (see e.g., [5], [6], [9-11]).

Consider the modified Bessel equation

$$z^2u'' + zu' - \left(z^2 + \nu^2\right)u = 0. \tag{1.1}$$

It is known [1-2] that, the equation (1.2) has a solution $K_{\nu}(z)$, which admits the representation

$$K_{\nu}(z) = \int_0^{\infty} e^{-zcht} ch(\nu t)\, dt\,, |\arg z| < \frac{\pi}{2},\ \nu \in C. \tag{1.2}$$

For each fixed $z > 0$ the function $K_{\nu}(z)$ is an entire function of the order ν. In [12], G. Polya studied the problem of the distribution of the zeros of the modified Bessel function

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$K_{\nu}(z)$ for a fixed $z > 0$. Of particular interest is also the problem of studying the zeros of the function $K'_{\nu}(z)$. It should be noted that the zeros of linear combinations of the Hankel function and its derivative, which are closely related to the function $K_{\nu}(z)$, were investigated in [4], [10], [11]. It turns out that a similar problem for the linear combination $aK_{\nu}(z) + bK'_{\nu}(z)$ can be studied using spectral theory.

In this paper, we study the properties of the zeros of the function $aK_{\nu}(z) + bK'_{\nu}(z)$, considered as a function of the order ν, where a and b are real and $z > 0$. The asymptotics of the zeros of the function $aK_{\nu}(z) + bK'_{\nu}(z)$ is found. Moreover, unlike in the works [4], [10], [11] , the oscillatory property of zeros was established.

Let us formulate the main results of this paper.

Theorem 1.1 *For each fixed $z > 0$ and for any real a and b, where $a^2 + b^2 > 0$, the function $aK_{\nu}(z) + bK'_{\nu}(z)$ has a countable number of simple, purely imaginary zeros $\pm i\nu_n$, $\nu_n > 0$, $n = 1, 2, ...$ All zeros are simple and the following asymptotic formula holds*

$$\nu_n \sim \frac{\pi n}{\ln n}, \; n \to +\infty. \tag{1.3}$$

Theorem 1.2 *If $i\nu_n = i\nu_n(z), \nu_n(z) > 0$ is the $n-$th zero of the function $aK_{\nu}(z) + bK'_{\nu}(z)$ to be considered as function of the order ν with fixed argument $z > 0$, then $\nu_n(z)$ is an increasing function of z.*

2 Proof of the Theorem

Let us proceed to the proof of Theorem 1. Consider the equation (1.1) for $z > 0$. If we set $z = 2e^{\frac{x+c}{2}}$, $\nu = 2i\lambda$ and $y(x) = u\left(2e^{\frac{x+c}{2}}\right)$, where c is any finite number, then equation (1.1) takes the form

$$-y'' + e^{x+c}y = \lambda^2 y, \qquad -\infty < x < +\infty. \tag{2.1}$$

Equation (2.1) is a one-dimensional Schrodinger equation with an exponential potential. One of the solutions of this equation is, obviously, the function

$$f(x, \lambda) = K_{2i\lambda}\left(2e^{\frac{x+c}{2}}\right) \tag{2.2}$$

Since the $K_{\nu}(z)$ function is an entire function of the order ν , from (2.2) it follows that for each fixed x, $-\infty < x < +\infty$, solution $f(x, \lambda)$ of equation (2.1) and its derivative serve as entire functions with respect to λ. Using the well-known (see [1]) asymptotic equality

$$K_{\nu}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}\left(1 + O\left(z^{-1}\right)\right), \; z \to \infty,$$

we find that, for each fixed λ the solution $f(x, \lambda)$ belongs to the space $L_2(0, \infty)$.

Consider the boundary value problem

$$-y'' + e^{x+c}y = \lambda^2 y, \quad 0 \le x < +\infty, \tag{2.3}$$

$$\alpha y(0) + \beta y'(0) = 0, \tag{2.4}$$

where α and β are real numbers and are such that $\alpha^2 + \beta^2 > 0$.

Consider also the self-adjoint operator T_c, generated in the Hilbert space $L_2(0, +\infty)$ by the boundary value problem (2.3) - (2.4). Since $e^x \to +\infty$ as $x \to +\infty$, the spectrum of the operator T_c, i.e., of problem (2.3) and (2.4), consists [13], [14] of simple real eigenvalues $\lambda_n^2 = \lambda_n^2(c) > 0, n = 1, 2, ...,$ condensing to $+\infty$.

On the other hand, it follows from relation $f(x,\lambda) \in L_2(0,\infty)$ that λ_n^2 is an eigenvalue of the operator T_c if and only if $\Delta(\pm\lambda_n) = 0$, where

$$\Delta(\lambda) = \alpha f(0,\lambda) + \beta f'(0,\lambda) = \alpha K_{2i\lambda}\left(2e^{\frac{c}{2}}\right) + \beta e^{\frac{c}{2}} K'_{2i\lambda}\left(2e^{\frac{c}{2}}\right). \qquad (2.5)$$

Therefore, the function $\Delta(\lambda)$ has only real zeros $\pm\lambda_n, \lambda_n > 0, n = 1, 2, ...$ Let us show that the zeros of the function $\Delta(\lambda)$ are simple.

Let us agree to denote differentiation with respect to λ and x with a dot and a prime, respectively:

$$f' = \frac{\partial}{\partial x} f,\ \dot{f} = \frac{\partial}{\partial \lambda} f.$$

Note that from (1.2), (2.2), it follows that, for real λ the following equality holds:

$$f(x,\lambda) = \int_0^\infty e^{-zcht}\cos\lambda t dt,$$

where $z = 2e^{\frac{x+c}{2}}$. This equality, in turn, implies the validity of the estimate for $x > 0$

$$|\,F(x,\lambda)\,| \le e^{\frac{x+c}{2}} e^{-e^{\frac{x+c}{2}}} \int_0^\infty te^{-cht} cht\, dt,$$

where $F(x,\lambda)$ means any of the functions $f(x,\lambda)$, $\dot{f}(x,\lambda)$, $f'(x,\lambda)$, $\dot{f}'(x,\lambda)$. The last estimate shows that each of these functions decays like a double exponent as $x \to +\infty$ uniformly with respect to $\lambda \in (-\infty,\infty)$.

Now differentiating the equation

$$-f''(x,\lambda) + e^{x+c} f(x,\lambda) = \lambda^2 f(x,\lambda)$$

with respect to λ, one obtains satisfies the following equation for $\dot{f}(x,\lambda)$

$$-\dot{f}''(x,\lambda) + e^{x+c}\dot{f}(x,\lambda) = \lambda^2 \dot{f}(x,\lambda) + 2\lambda f(x,\lambda).$$

Multiplying the first equation by $\dot{f}(x,\lambda)$, the second by - $f(x,\lambda)$, and then subtracting the first ratio from the second, we get the equality

$$\left\{\dot{f}(x,\lambda), f(x,\lambda)\right\}' = 2\lambda\, f^2(x,\lambda),$$

where $\{u,v\} = uv' - u'v$. Integrating the resulting equality from zero to infinity and assuming λ real, we have

$$\dot{f}'(0,\lambda)\ f(0,\lambda) - \dot{f}(0,\lambda)\ f'(0,\lambda) = 2\lambda \int_0^{+\infty} f^2(x,\lambda)\, dx.$$

Let, for example, $\beta \neq 0$. Multiplying both sides of the last equality by β and using the definition of the function $\Delta(\lambda)$, we obtain

$$\left[\dot{\Delta}(\lambda) - \alpha\dot{f}(0,\lambda)\right] f(0,\lambda) - \dot{f}(0,\lambda)\left[\Delta(\lambda) - \alpha f(0,\lambda)\right] = 2\beta\lambda \int_0^{+\infty} f^2(x,\lambda)\, dx,$$

i.e.

$$\dot{\Delta}(\lambda) f(0,\lambda) - \Delta(\lambda)\dot{f}(0,\lambda) = 2\beta\lambda \int_0^{+\infty} f^2(x,\lambda)\, dx.$$

Setting now $\lambda = \lambda_n$, we get

$$\dot{\Delta}(\lambda_n) f(0, \lambda_n) = 2\beta\lambda_n \int_0^{+\infty} f^2(x, \lambda_n)\, dx.$$

Therefore, $\dot{\Delta}(\lambda_n) \neq 0$,i.e. the zeros of the function $\Delta(\lambda)$ are simple.

Let us now study the asymptotics of the eigenvalues $\lambda_n^2 = \lambda_n^2(c)$. Since the function $q(x) = e^{x+c}$ satisfies all conditions of Theorem 7.3 from the monograph [13] (see also [14]), we have

$$\int_0^{\ln \lambda_n^2 - c} \sqrt{\lambda_n^2 - e^{x+c}} dx \sim \pi n,\ n \to +\infty. \tag{2.6}$$

Next, we notice that

$$\begin{aligned} &\textstyle\int_0^{\ln \lambda_n^2 - c} \sqrt{\lambda_n^2 - e^{x+c}} dx = \int_{e^c}^{\lambda_n^2} t^{-1} \sqrt{\lambda_n^2 - t} dt \\ &= \lambda_n \textstyle\int_{e^c}^{\lambda_n^2} \frac{\lambda_n^2}{t} \sqrt{1 - \frac{t}{\lambda_n^2}} d\frac{t}{\lambda_n^2} = \lambda_n \int_{e^c\lambda_n^{-2}}^{1} u^{-1} \sqrt{1-u} du. \end{aligned} \tag{2.7}$$

Since the function $G(u) = 2\sqrt{1-u} - \ln\left(1 + \sqrt{1-u}\right) + \ln\left(1 - \sqrt{1-u}\right)$ is a primitive function of $g(u) = u^{-1}\sqrt{1-u}$, from formula (2.7), we have

$$\int_0^{\ln \lambda_n^2 - c} \sqrt{\lambda_n^2 - e^{x+c}} dx = 2\lambda_n \ln \lambda_n \left[1 + O\left(\frac{1}{\ln \lambda_n}\right)\right],\ n \to +\infty.$$

Comparing this relationship with (2.6), we obtain

$$\lambda_n \ln \lambda_n = \frac{\pi n}{2}\left[1 + o(1)\right],\ n \to +\infty.$$

We rewrite the last relation in the form

$$\mu_n + \ln \mu_n = \ln \frac{\pi n}{2} + o(1)\ ,\ n \to +\infty, \tag{2.8}$$

where

$$\mu_n = \ln \lambda_n. \tag{2.9}$$

It follows from relation (2.8) that (see [8])

$$\mu_n = \ln \frac{\frac{\pi n}{2}}{\ln \frac{\pi n}{2}} + O\left(\frac{\ln \frac{\pi n}{2}}{\ln \ln \frac{\pi n}{2}}\right)\ ,\ n \to +\infty.$$

Hence, taking into account equality (2.9), we have

$$\lambda_n = \frac{n\pi}{2}\left(\ln \frac{n\pi}{2}\right)^{-1}\left[1 + o(1)\right],\ n \to +\infty.$$

Therefore, for the zeros of the function $\Delta(\lambda)$ we have the asymptotic equality

$$\lambda_n \sim \frac{n\pi}{2\ln n}, n \to +\infty. \tag{2.10}$$

Further, in the boundary condition (2.4) the values α, β can be arbitrary numbers satisfying condition $\alpha^2 + \beta^2 > 0$. Clearly, the numbers $a = \alpha$ and $b = \beta e^{\frac{c}{2}}$ also have this property. Setting then $z = 2e^{\frac{c}{2}}$ and taking into account that $\nu_n = 2\lambda_n$ from (2.5), (2.10), we find that the function $aK_\nu(z) + bK'_\nu(z)$ has countable simple purely imaginary zeros $\pm i\nu_n$, $\nu_n > 0$, $n = 1, 2, ...$ with asymptotics (1.3). This completes the proof of Theorem 1.1.

Now we give the proof of Theorem 1.2. By virtue of Theorem 1.1, it follows from the relation $aK_\nu(z) + bK'_\nu(z) = 0$ that $\lambda^2 = \left(\frac{\nu}{2}\right)^2$ is an eigenvalue of the boundary value problem (2.3) and (2.4). Let $\lambda_n^2(c)$ be the $n-$th eigenvalue of boundary value problem (2.3) and (2.4). If $c_1 < c_2$, then, by the minimax principle (see [7]) we have $\lambda_n^2(c_1) < \lambda_n^2(c_2)$. Therefore, if $i\nu_n(z)$, $\nu_n(z) > 0$ is the $n-$th zero of the function $aK_\nu(z) + bK'_\nu(z)$, then the condition $0 < z_1 < z_2$ implies that $\nu_n(z_1) < \nu_n(z_2)$.

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