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# Nonsingular integral on weighted Orlicz spaces

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**Abstract.** *We show continuity in weighted Orlicz spaces $L_w^{\Phi}(\mathbb{R}_+^n)$ of nonsingular integral operator.*

**Keywords.** Weighted Orlicz spaces · Muckenhoupt weight · nonsingular integral.

**Mathematics Subject Classification (2010):** 42B20, 42B35, 46E30

## 1 Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators etc, from one Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Lebesgue spaces, Orlicz spaces also play an important role

The Orlicz space were first introduced by Orlicz in [13, 14] as generalizations of Lebesgue spaces $L^p(\mathbb{R}^n)$. Since then, the theory of Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis.

Consider the half-space $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times (0, \infty)$. For $x = (x', x_n) \in \mathbb{R}_+^n$, let $\tilde{x} = (x', -x_n)$ be the "reflected point". Let $x \in \mathbb{R}_+^n$. The nonsingular integral operator $\widetilde{T}$ is defined by

$$\widetilde{T}f(x) = \int_{\mathbb{R}_+^n} \frac{|f(y)|}{|\tilde{x} - y|^n}\, dy, \quad \tilde{x} = (x', -x_n). \tag{1.1}$$

The operator $\widetilde{T}$ and its commutator appear in [1–7] in connection with boundary estimates for solutions to elliptic equations.

In [9, 10] we have studied the boundedness of the parabolic non-singular integral operator on Orlicz and generalized Orlicz-Morrey spaces, respectively. Quite recently, we have also studied in [11] the boundedness of the commutator of parabolic non-singular integral

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operator on parabolic generalized Orlicz-Morrey spaces of the third kind $M^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ with $BMO$ functions (see also [12]).

The main purpose of this paper is mainly to study the boundedness of the nonsingular integral operator $\widetilde{T}$ on weighted Orlicz spaces $L^{\Phi}_w(\mathbb{R}^n_+)$ .

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant $C$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.

## 2 Definitions and Preliminary Results

Even though the $A_p$ class is well known, for completeness, we offer the definition of $A_p$ weight functions. Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius $r$ centered at $x$ and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where $v_n$ is the volume of the unit ball in $\mathbb{R}^n$. Let $\mathcal{B} = \{B(x,r) : x \in \mathbb{R}^n,\ r > 0\}$.

**Definition 2.1** *For, $1 < p < \infty$, a locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an $A_p$ weight if*

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}} < \infty.$$

*A locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is said to be an $A_1$ weight if*

$$\frac{1}{|B|}\int_B w(y)dy \leq Cw(x), \qquad a.e.\ \ x \in B$$

*for some constant $C > 0$. We define $A_\infty = \bigcup_{p\geq 1} A_p$.*

For any $w \in A_\infty$ and any Lebesgue measurable set $E$, we write $w(E) = \int_E w(x)dx$.

We recall the definition of Young functions.

**Definition 2.2** *A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function, if $\Phi$ is convex, left-continuous, $\lim\limits_{r\to 0^+} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.*

The convexity and the condition $\Phi(0) = 0$ force any Young function to be increasing. In particular, if there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then it follows that $\Phi(r) = \infty$ for $r \geq s$.

Let $\mathcal{Y}$ be the set of all Young functions $\Phi$ such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty.$$

If $\Phi \in \mathcal{Y}$, then $\Phi$ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function $\Phi$ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) \equiv \inf\{r \geq 0 : \Phi(r) > s\} \qquad (\inf\emptyset = \infty).$$

A Young function $\Phi$ is said to satisfy the $\Delta_2$-condition, denoted by $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq k\Phi(r), \qquad r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function $\Phi$ is said to satisfy the $\nabla_2$-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \qquad r \geq 0$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the $\Delta_2$-condition and it fails the $\nabla_2$-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the $\nabla_2$-condition but it fails the $\Delta_2$-condition.

For a Young function $\Phi$, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) \equiv \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & \text{if } r \in [0,\infty), \\ \infty & \text{if } \quad r = \infty. \end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and it satisfies $\widetilde{\widetilde{\Phi}} = \Phi$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$.

It is also known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r, \qquad r \ge 0. \tag{2.1}$$

We recall an important pair of indices used for Young functions. For any Young function $\Phi$, write

$$h_\Phi(t) = \sup_{s>0} \frac{\Phi(st)}{\Phi(s)}, \quad t > 0.$$

The lower and upper dilation indices of $\Phi$ are defined by

$$i_\Phi = \lim_{t\to 0^+} \frac{\log h_\Phi(t)}{\log t} \text{ and } I_\Phi = \lim_{t\to\infty} \frac{\log h_\Phi(t)}{\log t},$$

respectively.

A Young function $\Phi$ is said to be of upper type $p$ (resp. lower type $p$) for some $p \in [0,\infty)$, if there exists a positive constant $C$ such that, for all $t \in [1,\infty)$ (resp. $t \in [0,1]$) and $s \in [0,\infty)$,

$$\Phi(st) \le Ct^p\Phi(s). \tag{2.2}$$

**Remark 2.1** It is well known that if $\Phi$ is of lower type $p_0$ and upper type $p_1$ with $1 < p_0 \le p_1 < \infty$, then $\widetilde{\Phi}$ is of lower type $p_1'$ and upper type $p_0'$ and $\Phi$ is lower type $p_0$ and upper type $p_1$ with $1 < p_0 \le p_1 < \infty$ if and only if $\Phi \in \Delta_2 \cap \nabla_2$.

It is easy to see that $\Phi$ is of lower type $i_\Phi - \varepsilon$, and of upper type $I_\Phi + \varepsilon$ for every $\varepsilon > 0$, where the constant appearing in (2.2) may depend on $\varepsilon$. We also mention that $i_\Phi$ and $I_\Phi$ may be viewed as the supremum of the lower types of $\Phi$ and the infimum of upper types, respectively.

**Lemma 2.1** *[8, Lemma 1.3.2] Let $\Phi \in \Delta_2$. Then there exist $p > 1$ and $b > 1$ such that*

$$\frac{\Phi(t_2)}{t_2^p} \le \frac{b\Phi(t_1)}{t_1^p}$$

*for $0 < t_1 < t_2$.*

**Lemma 2.2** *[17, Proposition 62.20] Let $\Phi$ be a Young function with canonical representation*

$$\Phi(t) = \int_0^t \varphi(s)ds, \quad t \ge 0.$$

*(1) Assume that $\Phi \in \Delta_2$. More precisely $\Phi(2t) \le A\Phi(t)$ for some $A \ge 2$. If $p > 1 + \log_2 A$, then*

$$\int_t^\infty \frac{\varphi(s)}{s^p}ds \lesssim \frac{\Phi(t)}{t^p}, \quad t > 0.$$

*(2) Assume that $\Phi \in \nabla_2$. Then*

$$\int_0^t \frac{\varphi(s)}{s} ds \lesssim \frac{\Phi(t)}{t}, \ \ t > 0.$$

**Definition 2.3** *For a Young function $\Phi$ and $w \in A_\infty$, the set*

$$L_w^\Phi(\mathbb{R}^n) \equiv \left\{ f - \textit{measurable} : \int_{\mathbb{R}^n} \Phi(k|f(x)|) w(x) dx < \infty \, \textit{for some } k > 0 \right\}$$

*is called the weighted Orlicz space. The local weighted Orlicz space $L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^n)$ is defined as the set of all functions $f$ such that $f\chi_B \in L_w^\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.*

Note that $L_w^\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_w^\Phi(\mathbb{R}^n)} \equiv \|f\|_{L_w^\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) w(x) dx \leq 1 \right\}$$

and

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_w^\Phi}}\Big) w(x) dx \leq 1.$$

The following analogue of the Hölder inequality is known.

$$\left| \int_{\mathbb{R}^n} f(x) g(x) w(x) dx \right| \leq 2 \|f\|_{L_w^\Phi} \|g\|_{L_w^{\widetilde{\Phi}}}. \tag{2.3}$$

For the proof of (2.1) and (2.3), see, for example [16].

For a weight $w$, a measurable function $f$ and $t > 0$, let

$$m(w, \ f, \ t) = w(\{x \in \mathbb{R}^n : |f(x)| > t\}).$$

**Definition 2.4** *The weak weighted Orlicz space*

$$WL_w^\Phi(\mathbb{R}^n) = \{ f - \textit{measurable} : \|f\|_{WL_w^\Phi} < \infty \}$$

*is defined by the norm*

$$\|f\|_{WL_w^\Phi(\mathbb{R}^n)} \equiv \|f\|_{WL_w^\Phi} = \inf \Big\{ \lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(w, \ \frac{f}{\lambda}, \ t\Big) \ \leq 1 \Big\}.$$

We can prove the following by a direct calculation:

$$\|\chi_B\|_{L_w^\Phi} = \|\chi_B\|_{WL_w^\Phi} = \frac{1}{\Phi^{-1}\left(w(B)^{-1}\right)}, \quad B \in \mathcal{B}, \tag{2.4}$$

where $\chi_B$ denotes the characteristic function of the $B$.

## 3 Nonsingular integral operators in the weighted Orlicz space $L_w^\Phi(\mathbb{R}_+^n)$

The following theorem is valid (see, for example, [8,15]).

**Theorem 3.1** *Let $\widetilde{T}$ be a nonsingular integral operator, defined by* (1.1)*, $f \in L_w^p(\mathbb{R}_+^n)$, $1 \le p < \infty$ and $w \in A_p$. Then there exists a constant $C_p$ independent of $f$, such that*

$$\|\widetilde{T}f\|_{L_w^p(\mathbb{R}_+^n)} \le C_p \|f\|_{L_w^p(\mathbb{R}_+^n)}, \ \ 1 < p < \infty$$

*and*

$$\|\widetilde{T}f\|_{WL_w^1(\mathbb{R}_+^n)} \le C_1 \|f\|_{L_w^1(\mathbb{R}_+^n)}.$$

**Theorem 3.2** *Let $\Phi$ be a Young function, $w \in A_{i_\Phi}$ and $\widetilde{T}$ be a nonsingular integral operator, defined by* (1.1)*. If $\Phi \in \Delta_2$, then the operator $\widetilde{T}$ is bounded from $L_w^\Phi(\mathbb{R}_+^n)$ to $WL_w^\Phi(\mathbb{R}_+^n)$ and if $\Phi \in \Delta_2 \cap \nabla_2$, then the operator $\widetilde{T}$ is bounded on $L_w^\Phi(\mathbb{R}_+^n)$.*

**Proof.** At first proved that for $\Phi \in \Delta_2$ the nonsingular integral operator $\widetilde{T}$ is bounded from $L_w^\Phi(\mathbb{R}_+^n)$ to $WL_w^\Phi(\mathbb{R}_+^n)$.

We take $f \in L_w^\Phi(\mathbb{R}_+^n)$ satisfying $\|f\|_{L_w^\Phi} = 1$. Fix $\lambda > 0$ and define $f_1 = \chi_{\{|f|>\lambda\}} \cdot f$ and $f_2 = \chi_{\{|f|\le\lambda\}} \cdot f$. Then $f = f_1 + f_2$. We have

$$\begin{aligned} w\big(\{x \in \mathbb{R}_+^n : \big|\widetilde{T}f(x)\big| > \lambda\}\big) &\le w\big(\big\{x \in \mathbb{R}_+^n : \big|\widetilde{T}f_1(x)\big| > \frac{\lambda}{2}\big\}\big) \\ &\quad + w\big(\big\{x \in \mathbb{R}_+^n : \big|\widetilde{T}f_2(x)\big| > \frac{\lambda}{2}\big\}\big) \end{aligned}$$

and

$$\begin{aligned} &\Phi(\lambda) w\big(\{x \in \mathbb{R}_+^n : \big|\widetilde{T}f(x)| > \lambda\}\big) \\ &\le \Phi(\lambda) w\big(\big\{x \in \mathbb{R}_+^n : \big|\widetilde{T}f_1(x)\big| > \frac{\lambda}{2}\big\}\big) + \Phi(\lambda) w\big(\big\{x \in \mathbb{R}_+^n : \big|\widetilde{T}f_2(x)\big| > \frac{\lambda}{2}\big\}\big). \end{aligned}$$

We know that from the weighted weak (1,1) boundedness and weighted $L^p$, $p \in (1,\infty)$ boundedness of $\widetilde{T}$

$$w\big(\{x \in \mathbb{R}_+^n : \big|\widetilde{T}(\chi_{\{|f|>\lambda\}} \cdot f)(x)\big| > \lambda\}\big) \lesssim \frac{1}{\lambda} \int_{\{x \in \mathbb{R}_+^n : |f(x)|>\lambda\}} |f(x)| w(x) dx$$

and

$$w\big(\{x \in \mathbb{R}_+^n : \big|\widetilde{T}(\chi_{\{|f|\le\lambda\}} \cdot f)(x)\big| > \lambda\}\big) \lesssim \frac{1}{\lambda^p} \int_{\{x \in \mathbb{R}_+^n : |f(x)|\le\lambda\}} |f(x)|^p w(x) dx.$$

Since $f_1 \in WL_w^1(\mathbb{R}_+^n)$ and $\frac{\Phi(\lambda)}{\lambda}$ increasing we have

$$\begin{aligned} \Phi(\lambda) w\big(\big\{x \in \mathbb{R}_+^n : |\widetilde{T}f_1(x)| > \frac{\lambda}{2}\big\}\big) &\lesssim \frac{\Phi(\lambda)}{\lambda} \int_{\mathbb{R}_+^n} |f_1(x)| w(x) dx \\ &= \frac{\Phi(\lambda)}{\lambda} \int_{\{x \in \mathbb{R}_+^n : |f(x)|>\lambda\}} |f(x)| w(x) dx \\ &\lesssim \int_{\mathbb{R}_+^n} |f(x)| \frac{\Phi(|f(x)|)}{|f(x)|} w(x) dx \\ &= \int_{\mathbb{R}_+^n} \Phi(|f(x)|) w(x) dx. \end{aligned}$$

By Lemma 2.1 and $f_2 \in L^p_w(\mathbb{R}^n_+)$ we have

$$\varPhi(\lambda)\, w\big(\{x \in \mathbb{R}^n_+ : |\widetilde{T} f_2(x)| > \frac{\lambda}{2}\}\big) \lesssim \frac{\varPhi(\lambda)}{\lambda^p} \int_{\mathbb{R}^n_+} |f_2(x)|^p w(x)dx$$
$$= \frac{\varPhi(\lambda)}{\lambda^p} \int_{\{x\in\mathbb{R}^n_+ : |f(x)|\le\lambda\}} |f(x)|^p w(x)dx$$
$$\lesssim \int_{\mathbb{R}^n_+} |f(x)|^p \frac{\varPhi(|f(x)|)}{|f(x)|^p} w(x)dx = \int_{\mathbb{R}^n_+} \varPhi(|f(x)|) w(x)dx.$$

Thus we get

$$w\big(\{x \in \mathbb{R}^n_+ : \big|\widetilde{T} f(x)\big| > \lambda\}\big) \le \frac{C}{\varPhi(\lambda)} \int_{\mathbb{R}^n_+} \varPhi(|f(x)|) w(x)dx$$
$$\le \frac{1}{\varPhi\left(\frac{\lambda}{C\|f\|_{L^\varPhi_w}}\right)}.$$

Since $\|\cdot\|_{L^\varPhi_w}$ norm is homogeneous this inequality is true for every $f \in L^\varPhi_w(\mathbb{R}^n_+)$.

Now proved that for $\varPhi \in \varDelta_2 \bigcap \nabla_2$ the nonsingular integral operator $\widetilde{T}$ is bounded in $L^\varPhi_w(\mathbb{R}^n_+)$.

Using the distribution functions we have that

$$\int_{\mathbb{R}^n_+} \varPhi\left(\frac{|\widetilde{T} f(x)|}{\varLambda}\right) w(x)dx = \frac{1}{\varLambda} \int_0^\infty \varphi\left(\frac{\lambda}{\varLambda}\right) w\big(\{x \in \mathbb{R}^n_+ : |\widetilde{T} f(x)| > \lambda\}\big) d\lambda$$
$$= \frac{2}{\varLambda} \int_0^\infty \varphi\left(\frac{2\lambda}{\varLambda}\right) w\big(\{x \in \mathbb{R}^n_+ : |\widetilde{T} f(x)| > 2\lambda\}\big) d\lambda.$$

The following inequality is valid.

$$w\big(\{x \in \mathbb{R}^n_+ : \big|\widetilde{T} f(x)\big| > 2\lambda\}\big) \le w\big(\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}(\chi_{\{|f|>\lambda\}} \cdot f)\big| > \lambda\}\big)$$
$$+\, w\big(\{x \in \mathbb{R}^n_+ : |\widetilde{T}(\chi_{\{|f|\le\lambda\}} \cdot f)(x)| > \lambda\}\big).$$

Let $p > 1$ be sufficiently large. By the weighted weak $(1, 1)$ boundedness and weighted $L^p$-boundedness of $\widetilde{T}$ (see Theorem 3.1) gives us

$$w\big(\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}(\chi_{\{|f|>\lambda\}} \cdot f)(x)\big| > \lambda\}\big) \lesssim \frac{1}{\lambda} \int_{\{x\in\mathbb{R}^n_+ : |f(x)|>\lambda\}} |f(x)| w(x)dx$$

and

$$w\big(\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}(\chi_{\{|f|\le\lambda\}} \cdot f)(x)\big) > \lambda\}\big| \lesssim \frac{1}{\lambda^p} \int_{\{x\in\mathbb{R}^n_+ : |f(x)|\le\lambda\}} |f(x)|^p w(x)dx.$$

The same calculation as we used for the maximal operator works for the first term to obtain

$$\frac{1}{\varLambda} \int_0^\infty \varphi\left(\frac{2\lambda}{\varLambda}\right) w\big(\{x \in \mathbb{R}^n_+ : |\widetilde{T}(\chi_{\{|f|>\lambda\}} \cdot f)(x)| > \lambda\}\big) d\lambda$$
$$\le \int_{\mathbb{R}^n_+} \varPhi\left(\frac{c|f(x)|}{\varLambda}\right) w(x)dx. \tag{3.1}$$

As for the second term a similar computation still works but we use the fact that $\Phi \in \Delta_2$.

$$\begin{aligned}
&\frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right) w\big(\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)\big| > \lambda\}\big)d\lambda \\
&\lesssim \frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\left(\int_{\{x\in\mathbb{R}^n_+ : |f(x)|\le\lambda\}} |f(x)|^p w(x)dx\right)\frac{d\lambda}{\lambda^p} \\
&\lesssim \frac{1}{\Lambda}\int_{\mathbb{R}^n_+} |f(x)|^p \left(\int_{|f(x)|}^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\frac{d\lambda}{\lambda^p}\right) w(x)dx.
\end{aligned}$$

Using Lemma 2.2 (1), we have

$$\begin{aligned}
&\frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right) w\big(\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)\big| > \lambda\}\big)d\lambda \\
&\lesssim \int_{\mathbb{R}^n_+} \Phi\left(\frac{2|f(x)|}{\Lambda}\right) w(x)dx \le \int_{\mathbb{R}^n_+} \Phi\left(\frac{c|f(x)|}{\Lambda}\right) w(x)dx. \qquad (3.2)
\end{aligned}$$

Thus, putting together (3.1) and (3.2), we obtain

$$\int_{\mathbb{R}^n_+} \Phi\left(\frac{|\widetilde{T}f(x)|}{\Lambda}\right) w(x)dx \le \int_{\mathbb{R}^n_+} \Phi\left(\frac{c_0|f(x)|}{\Lambda}\right) w(x)dx.$$

Again we shall label the constant we want to distinguish from other less important constants. As before, if we set $\Lambda = c_2\|f\|_{L^\Phi_w(\mathbb{R}^n_+)}$, then we obtain

$$\int_{\mathbb{R}^n_+} \Phi\left(\frac{|\widetilde{T}f(x)|}{\Lambda}\right) w(x)dx \le 1.$$

Hence the operator norm of $\widetilde{T}$ is less than $c_2$:

$$\|\widetilde{T}f\|_{L^p_w(\mathbb{R}^n_+)} \le \Lambda = c_2\|f\|_{L^\Phi_w(\mathbb{R}^n_+)}.$$

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