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On a boundary value problem for Benney–Luke type differential equation with nonlinear function of redefinition and integral conditions

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Abstract. *In three-dimensional domain a Benney–Luke type partial differential equation of the even order with integral form conditions, spectral parameter and small positive parameters in mixed derivatives is considered. The solution of this partial differential equation is studied in the class of regular functions. The Fourier method of separation of variables (Fourier series method) and the method of successive approximation in combination with the method of compressing mapping are used. Using the method of Fourier series, we obtain countable system of ordinary differential equations. So, the nonlocal boundary value problem is integrated as an ordinary differential equation. When we define the arbitrary integration constants there are possible five cases with respect to the spectral parameter. The problem is reduced to solving countable system of linear algebraic equations. Using the given additional condition, we obtained the nonlinear countable system of functional equation with respect to redefinition function. Using the Cauchy-Schwarz inequality and the Bessel inequality, we proved the absolute and uniform convergence of the obtained Fourier series.*

Keywords. Benney–Luke type differential equation, regular solutions, Fourier series method, integral conditions, inverse problem, nonlinear functional equation.

Mathematics Subject Classification (2010): 35A02, 35M10, 35S05

1 Formulation of the problem

The theory of direct boundary and inverse boundary value problems is currently one of the most important sections of the theory of differential equations. Studies of many problems of gas dynamics, theory of elasticity, theory of plates and shells are described by high-order partial differential equations. Partial differential equations of Boussinesq type and Benney–Luke type have a lot of applications in different branches of sciences (see, for example, [5], [6,17]). Therefore, a large number of works are devoted to the study of inverse problems for differential and integro-differential equations (see, for example, [2,7,11,13–16,18,23, 24]). In cases where the boundary of the flow domain of a physical process is unavailable

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for measurements, nonlocal conditions in integral form can serve as additional information sufficient for unique solvability of the problem [8]. Therefore, in recent years, research on the study of direct and inverse nonlocal boundary value problems for differential and integro-differential equations with integral conditions has been intensified (see, for example, [1,3,4,9,10,12,19–23]).

In this paper, we study the regular solvability of a nonlocal inverse boundary value problem for a Benney–Luke type differential equation with spectral parameter and small positive parameters. In studying one-valued solvability and constructing solutions, the presence of spectral parameter plays an important role.

In three-dimensional domain $\Omega = \{(t, x, y)\,|\, 0 < t < T,\ 0 < x, y < l\}$ a partial differential equation of the following form is considered

$$D\,[U] = \alpha\,(t\left[\beta\,(x,\,y) - f\left(x,\,y,\int_0^l\int_0^l \Theta\;(\xi,\,\eta,\,\beta(\xi,\,\eta))\;d\,\xi\,d\,\eta\right)\right], \tag{1.1}$$

where T and l are given positive real numbers, ω is positive spectral parameter,

$$D\,[U] = \left[\frac{\partial^2}{\partial\,t^2} - \frac{\partial^2}{\partial\,t^2}\left(\varepsilon_1\frac{\partial^{2k}}{\partial\,x^{2k}} - \varepsilon_2\frac{\partial^{4k}}{\partial\,x^{4k}} + \varepsilon_1\frac{\partial^{2k}}{\partial\,y^{2k}} - \varepsilon_2\frac{\partial^{4k}}{\partial\,y^{4k}}\right)\right.$$
$$\left.-\omega^2\left(\frac{\partial^{2k}}{\partial\,x^{2k}} - \frac{\partial^{4k}}{\partial\,x^{4k}} + \frac{\partial^{2k}}{\partial\,y^{2k}} - \frac{\partial^{4k}}{\partial\,y^{4k}}\right)\right]U\,(t,\,x,\,y),$$

$\varepsilon_1,\ \varepsilon_2$ are positive small parameters, k is given positive integer, $\alpha\,(t\,) \in C\,(\Omega_T),\ f\,(x,\,y,\,\beta) \in C^{4k}_{x,\,y}(\Omega_l \times \Omega_l \times \mathbb{R}),\ \Theta\,(x,\,y,\,\beta) \in C\,(\Omega_l \times \Omega_l \times \mathbb{R}),\ \Omega_T \equiv [0;\,T],\ \Omega_l \equiv [0;\,l]$, $\beta\,(x,\,y) \in C\,(\Omega_l \times \Omega_l)$ is redefinition function. We assume that for given functions are true the following boundary conditions

$$\beta\,(0,\,y) = \beta\,(l,\,y) = \beta\,(x,\,0) = \beta\,(x,\,l) = 0,$$
$$f\,(0,\,y,\,\cdot) = f\,(l,\,y,\,\cdot) = f\,(x,\,0,\,\cdot) = f\,(x,\,l,\,\cdot) = 0.$$

Problem Statement. We find a pair of functions $\{U\,(t,\,x,\,y);\,\beta\,(x,\,y)\}$, first of which satisfies differential equation (1.1), following nonlocal conditions on the integral form

$$U\,(T,\,x,\,y) + \int_0^T U\,(t,\,x,\,y)\,d\,t = \varphi_1(x,\,y), \quad 0 \le x,\,y \le l, \tag{1.2}$$

$$U_t\,(T,\,x,\,y) + \int_0^T U_t\,(t,\,x,\,y)\,t\,d\,t = \varphi_2\,(x,\,y), \quad 0 \le x,\,y \le l, \tag{1.3}$$

zero boundary value conditions for $0 \le t \le T$

$$U\,(t,\,0,\,y) = U\,(t,\,l,\,y) = U\,(t,\,x,\,0) = U\,(t,\,x,\,l)$$
$$= \frac{\partial^2}{\partial\,x^2}U\,(t,\,0,\,y) = \frac{\partial^2}{\partial\,x^2}U\,(t,\,l,\,y) = \frac{\partial^2}{\partial\,x^2}U\,(t,\,x,\,0) = \frac{\partial^2}{\partial\,x^2}U\,(t,\,x,\,l)$$
$$= \frac{\partial^2}{\partial y^2}U\,(t,\,0,\,y) = \frac{\partial^2}{\partial y^2}U\,(t,\,l,\,y) = \frac{\partial^2}{\partial y^2}U\,(t,\,x,\,0) = \frac{\partial^2}{\partial y^2}U\,(t,\,x,\,l) = \ldots$$
$$= \frac{\partial^{4k-2}}{\partial x^{4k-2}}U\,(t,0,y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}}U\,(t,l,y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}}U\,(t,x,0) = \frac{\partial^{4k-2}}{\partial x^{4k-2}}U(t,x,l)$$
$$= \frac{\partial^{4k-2}}{\partial y^{4k-2}}U(t,\,0,\,y) = \frac{\partial^{4k-2}}{\partial\,y^{4k-2}}U(t,\,l,\,y)$$

$$= \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,\, x,\, 0) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,\, x,\, l) = 0, \tag{1.4}$$

class of functions

$$U\,(t,\, x,\, y) \in C\,(\overline{\Omega}) \cap C^{2,\,4k,\,4k}_{t,\,x,\,y}(\Omega) \cap C^{2+4k+0}_{t,\,x,\,y}(\Omega) \cap C^{2+0+4k}_{t,\,x,\,y}(\Omega) \tag{1.5}$$

and additional condition

$$U\,(t_0,\, x,\, y) = \psi\,(x,\, y), \quad 0 < t_0 < T, \quad 0 \le x,\, y \le l, \tag{1.6}$$

where $\varphi_i(x,\, y),\ \psi\,(x,\, y)$ are given smooth functions and

$$\varphi_i(0,\, y) = \varphi_i(l,\, y) = \varphi_i(x,\, 0) = \varphi_i\,(x,\, l) = 0,$$
$$\psi\,(0,\, y) = \psi\,(l,\, y) = \psi\,(x,\, 0) = \psi\,(x,\, l) = 0,$$

$C^{2+4k+0}_{t,\,x,\,y}(\Omega)$ is the class of continuous functions $\frac{\partial^{2+4k}\, U\,(t,x,y)}{\partial\, t^2 \partial\, x^{4k}}$ on Ω, while $C^{2+0+4k}_{t,\,x,\,y}(\Omega)$ is the class of continuous functions $\frac{\partial^{2+4k}\, U\,(t,x,y)}{\partial\, t^2 \partial\, y^{4k}}$ on Ω, $\overline{\Omega} = \{(t,\, x,\, y)\,|\, 0 \le t \le T,\ 0 \le x,\ y \le l\}$, by $\frac{\partial^{4k-2}}{\partial y^{4k-2}} U\,(t,\, x,\, l)$ we understand $\left.\frac{\partial^{4k-2}}{\partial y^{4k-2}} U\,(t,\, x,\, y)\right|_{y=l}$.

2 Expansion of the solution of the problem in a Fourier series for regular values of spectral parameter

Nontrivial solutions of the direct problem (1.1)–(1.5) are sought as a Fourier series

$$U\,(t,\, x,\, y) = \sum_{n,\, m=1}^{\infty} u_{\,n\,,\,m}\,(t)\, \vartheta_{\,n,\,m}\,(x,\, y), \tag{2.1}$$

where

$$u_{\,n,\,m}\,(t) = \int_0^l \int_0^l U\,(t,\, x,\, y)\, \vartheta_{\,n,\,m}\,(x,\, y)\ d\,x\, d\,y, \tag{2.2}$$
$$\vartheta_{\,n,\,m}\,(x,\, y) = \frac{2}{l}\, \sin\, \frac{\pi\, n}{l}\, x\, \sin\, \frac{\pi\, m}{l}\, y, n,\, m = 1,\, 2, \ldots\,.$$

We also suppose that the following functions are expand to Fourier series

$$\beta(x,\, y) = \sum_{n,\, m=1}^{\infty} \beta_{\,n,\,m}\, \vartheta_{\,n,\,m}\,(x,\, y), \quad f(x,\, y,\, \cdot) = \sum_{n,\, m=1}^{\infty} f_{\,n,\,m}\,(\cdot)\, \vartheta_{\,n,\,m}\,(x,\, y), \tag{2.3}$$

where

$$\begin{aligned} \beta_{\,n,\,m} &= \textstyle\int_0^l \int_0^l \beta\,(x,\, y)\, \vartheta_{\,n,\,m}\,(x,\, y)\, d\,x\, d\,y, \\ f_{\,n,\,m}\,(\cdot) &= \textstyle\int_0^l \int_0^l f\,(x,\, y,\, \cdot)\, \vartheta_{\,n,\,m}\,(x,\, y)\, d\,x\, d\,y. \end{aligned} \tag{2.4}$$

Substituting Fourier series (2.1) and (2.3) into partial differential equation (1.1), we obtain a countable system of ordinary differential equations of second order

$$u''_{\,n,\,m}\,(t) + \lambda^{2k}_{n,\,m}\, \omega^{\,2}\, u_{\,n,\,m}\,(t) = \frac{\alpha\,(t)}{1 + \mu^{\,2k}_{n,\,m}\left(\varepsilon_1 + \varepsilon_2 \mu^{\,2k}_{n,\,m}\right)}\, \left(\beta_{\,n,\,m} - f_{\,n,\,m}\,(\cdot)\right), \tag{2.5}$$

where

$$\lambda^{2k}_{n,\,m} = \frac{\mu^{\,2k}_{n,\,m}\left(1 + \mu^{\,2k}_{n,\,m}\right)}{1 + \mu^{\,2k}_{n,\,m}\left(\varepsilon_1 + \varepsilon_2 \mu^{\,2k}_{n,\,m}\right)}, \ \ \mu^{\,k}_{n,\,m} = \left(\frac{\pi}{l}\right)^k\, \sqrt{n^{\,2k} + m^{\,2k}}.$$

The second order countable system of differential equations (2.5) is solved by the variation method of arbitrary constants

$$u_{n,m}(t) = A_{1n,m}\cos\left(\lambda_{n,m}^k \omega\, t\right) + A_{2n,m}\sin\left(\lambda_{n,m}^k \omega\, t\right) + \gamma_{n,m}(t), \tag{2.6}$$

where $\gamma_{n,m}(t) = \frac{1}{\lambda_{n,m}^k \omega}\left[\beta_{n,m} - f_{n,m}(\cdot)\right] h_{n,m}(t)$, $A_{1,n,m}$ and $A_{2,n,m}$ are arbitrary constants,

$$h_{n,m}(t) = \frac{1}{1+\mu_{n,m}^{2k}\left(\varepsilon_1 + \varepsilon_2 \mu_{n,m}^{2k}\right)} \int_0^t \sin\left(\lambda_{n,m}^k \omega\,(t-s)\right)\,\alpha(s)\,ds.$$

Using Fourier coefficients (2.2), the integral conditions (1.2) and (1.3) are written in the following form

$$u_{n,m}(T) + \int_0^T u_{n,m}(t)\,dt = \int_0^l \int_0^l \left[(U(T,x,y) + \int_0^T U(t,x,y)\,dt\right] \vartheta_{n,m}(x,y)\,dx\,dy$$

$$= \int_0^l \int_0^l \varphi_1(x,y)\,\vartheta_{n,m}(x,y)\,dx\,dy = \varphi_{1n,m}, \tag{2.7}$$

$$u'_{n,m}(T) + \int_0^T u'_{n,m}(t)\,t\,dt$$

$$= \int_0^l \int_0^l \left[U_t(T,x,y) + \int_0^T U_t(t,x,y)\,t\,dt\right] \vartheta_{n,m}(x,y)\,dx\,dy$$

$$= \int_0^l \int_0^l \varphi_2(x,y)\,\vartheta_{n,m}(x,y)\,dx\,dy = \varphi_{2n,m}. \tag{2.8}$$

To find the unknown coefficients $A_{1n,m}$ and $A_{2n,m}$ in (2.6), we use conditions (2.7) and (2.8) and obtain the system

$$\begin{cases} A_{1n,m}\sigma_{1n,m}(\omega) + A_{2n,m}\sigma_{2n,m}(\omega) = \varphi_{01n,m}, \\ A_{1n,m}\sigma_{3n,m}(\omega) + A_{2n,m}\sigma_{4n,m}(\omega) = \varphi_{02n,m}, \end{cases} \tag{2.9}$$

where

$$\sigma_{1n,m}(\omega) = \frac{\lambda_{n,m}^k \omega\,\cos\left(2\lambda_{n,m}^k \omega T\right) + \sin\left(2\lambda_{n,m}^k \omega T\right)}{\lambda_{n,m}^k \omega},$$

$$\sigma_{2n,m}(\omega) = \frac{-\cos\left(2\lambda_{n,m}^k \omega T\right) + \lambda_{n,m}^k \omega\,\sin\left(2\lambda_{n,m}^k \omega T\right) + 1}{\lambda_{n,m}^k \omega},$$

$$\sigma_{3n,m}(\omega) = \frac{-\lambda_{n,m}^k \omega T\,\cos\left(2\lambda_{n,m}^k \omega T\right) - \lambda_{n,m}^k \omega T + \left[1+\left(\lambda_{n,m}^k \omega\right)^2\right]\sin\left(2\lambda_{n,m}^k \omega T\right)}{\left(\lambda_{n,m}^k \omega\right)^2},$$

$$\sigma_{4n,m}(\omega) = \frac{\left[1+\left(\lambda_{n,m}^k \omega\right)^2\right]\cos\left(2\lambda_{n,m}^k \omega T\right) + \lambda_{n,m}^k \omega T\,\sin\left(2\lambda_{n,m}^k \omega T\right) - 1}{\left(\lambda_{n,m}^k \omega\right)^2},$$

$$\varphi_{01n,m} = \varphi_{1n,m} - \left[\gamma_{n,m}(T) + \int_0^T \gamma_{n,m}(t)\,dt\right],$$

$$\varphi_{02n,m} = \varphi_{2n,m} - \left[\gamma'_{n,m}(T) + \int_0^T \gamma'_{n,m}(t)\,t\,dt\right].$$

To uniquely determine $A_{1\,n,\,m}$ and $A_{2\,n,\,m}$ from system (2.9), we calculate the values of the spectral parameter ω presented in the coefficients $\sigma_{\,i\,n,\,m}\,(\omega),\ \ i=\overline{1,\,4}$. The coefficients $\sigma_{\,i\,n,\,m}\,(\omega),\ \ i=\overline{1,\,4}$ can go to zero for some values of the parameter ω from the positive semi-axis $(0;\,\infty)$. The set of all values of the spectral parameter ω, consisting of positive solutions of trigonometric equations $\sigma_{\,i\,n,\,m}\,(\omega)=0$, we denote by $\Lambda_i,\ \ i=\overline{1,\,4}$. We take into account, that $\Lambda_i\cap\Lambda_j=\emptyset,\ \ i,\,j=\overline{1\,,\,4},\ \ i\neq j$. Also we introduce the denotation $\Lambda_5=(0;\,\infty)\setminus\left(\bigcup_{j=1}^{4}\Lambda_j\right)$. It is possible there five cases: **1)** $\sigma_{\,1\,n,\,m}\,(\omega)=0$; **2)** $\sigma_{\,2\,n,\,m}\,(\omega)=0$; **3)** $\sigma_{\,3\,n,\,m}\,(\omega)=0$; **4)** $\sigma_{\,4\,n,\,m}\,(\omega)=0$; **5)** $\sigma_{\,j\,n,\,m}\,(\omega)\neq 0\,,\ \ j=\overline{1\,,\,4}$.

Solve the system of algebraic equations (2.9). Then from presentation (2.6) we derived that

$$u_{\,n,\,m}\,(t)=\varphi_{\,1\,n,\,m}\,B_{j\,n,\,m}\,(t)+\varphi_{\,2\,n,\,m}\,C_{j\,n,\,m}\,(t)$$
$$+\frac{1}{\lambda_{n,\,m}^{k}\,\omega}\,(\beta_{\,n,\,m}-f_{\,n,\,m}\,(\cdot))\;E_{j\,n,\,m}\,(t),\quad \omega\in\Lambda_j,\,j=\overline{1,\,5},\tag{2.10}$$

where Fourier coefficients $\beta_{\,n,\,m}$ and $f_{\,n,\,m}\,(\cdot)$ are defined by the presentations (2.4),

$$E_{\,j\,n,\,m}\,(t)=h_{\,n,\,m}\,(t)-B_{\,j\,n,\,m}\,(t)\left[\int_0^T h_{\,n,\,m}\,(t)\,d\,t+h_{\,n,\,m}\,(T)\right]$$
$$-C_{\,j\,n,\,m}\,(t)\left[\int_0^T h'_{\,n,\,m}\,(t)\,t\,d\,t+h'_{\,n,\,m}\,(T)\right],$$

$$B_{\,1\,n,\,m}\,(t)=\frac{\sin\left(\lambda_{\,n\,,\,m}^{k}\omega\,t\right)}{\sigma_{\,2\,n,\,m}\,(\omega)}-\frac{\sigma_{\,4\,n,\,m}\,(\omega)}{\sigma_{\,2\,n,\,m}\,(\omega)}\frac{\cos\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)}{\sigma_{\,3\,n,\,m}\,(\omega)},\quad C_{\,1\,n,\,m}\,(t)=\frac{\cos\left(\lambda_{\,n\,,\,m}^{k}\omega\,t\right)}{\sigma_{\,3\,n,\,m}\,(\omega)},$$

$$B_{\,2\,n,\,m}\,(t)=\frac{\cos\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)}{\sigma_{\,1\,n,\,m}\,(\omega)}-\frac{\sigma_{\,3\,n,\,m}\,(\omega)}{\sigma_{\,1\,n,\,m}\,(\omega)}\frac{\sin\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)}{\sigma_{\,4\,n,\,m}\,(\omega)},\quad C_{\,2\,n,\,m}\,(t)=\frac{\sin\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)}{\sigma_{\,4\,n,\,m}\,(\omega)},$$

$$B_{\,3\,n,\,m}\,(t)=\frac{\cos\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)}{\sigma_{\,1\,n,\,m}\,(\omega)},\quad C_{\,3\,n,\,m}\,(t)=\frac{\sin\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)}{\sigma_{\,4\,n,\,m}\,(\omega)}-\frac{\sigma_{\,2\,n,\,m}\,(\omega)}{\sigma_{\,1\,n,\,m}\,(\omega)}\frac{\cos\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)}{\sigma_{\,4\,n,\,m}\,(\omega)},$$

$$B_{\,4\,n,\,m}\,(t)=\frac{\sin\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)}{\sigma_{\,2,\,n,\,m}\,(\omega)},\quad C_{\,4\,n,\,m}\,(t)=\frac{\cos\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)}{\sigma_{\,3\,n,\,m}\,(\omega)}-\frac{\sigma_{\,1\,n,\,m}\,(\omega)}{\sigma_{\,2\,n,\,m}\,(\omega)}\frac{\sin\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)}{\sigma_{\,3\,n,\,m}\,(\omega)},$$

$$B_{\,5\,n,\,m}\,(t)=\frac{1}{\sigma_{\,5\,n,\,m}\,(\omega)}\,\left[\sigma_{\,4\,n,\,m}\,(\omega)\,\cos\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)-\sigma_{\,3\,n,\,m}\,(\omega)\,\sin\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)\right],$$

$$C_{\,5\,n,\,m}\,(t)=\frac{1}{\sigma_{\,5\,n,\,m}\,(\omega)}\,\left[-\sigma_{\,2\,n,\,m}\,(\omega)\,\cos\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)+\sigma_{\,1\,n,\,m}\,(\omega)\,\sin\left(\lambda_{\,n,\,m}^{k}\omega\,t\right)\right],$$

$$\sigma_{5\,n,\,m}\,(\omega)=\sigma_{\,1\,n,\,m}\,(\omega)\,\sigma_{\,4\,n,\,m}\,(\omega)-\sigma_{\,2\,n,\,m}\,(\omega)\,\sigma_{\,3\,n,\,m}\,(\omega)\neq 0,\quad \omega\in\Lambda_5.$$

Substituting the presentation of Fourier coefficients (2.10) of main unknown function into Fourier series (2.1), for regular values of parameter $\omega\in\Lambda_j\ \left(j=\overline{1,\,5}\right)$ we obtain

$$U\,(t,\,x,\,y)=\sum_{n,\,m=1}^{\infty}\vartheta_{\,n,\,m}\,(x,\,y)$$
$$\times\left[\varphi_{1\,n,\,m}B_{j\,n,\,m}(t)+\varphi_{2\,n,\,m}C_{j\,n,\,m}(t)+\frac{1}{\lambda_{n,\,m}^{k}\omega}\,(\beta_{n,\,m}-f_{n,\,m}(\cdot))\;E_{j\,n,\,m}(t)\right].\tag{2.11}$$

Fourier series (2.11) is a formal solution of the direct problem (1.1)–(1.5).

3 Redefinition function

Using the additional condition (1.6) and taking into account (2.3) and (2.4), for regular values of parameter $\omega \in \Lambda_j$ $\left(j=\overline{1,\,5}\right)$ we obtain from Fourier series (2.11) following nonlinear countable system for Fourier coefficients of redefinition function

$$\beta_{n,\,m} = \Im_j\left(\beta_{n,\,m}\right) \equiv \psi_{n,\,m}\,\tau_{1\,j\,n,\,m} + \varphi_{\,1\,n,\,m}\,\tau_{2\,j\,n,\,m} + \varphi_{\,2\,n,\,m}\,\tau_{3\,j\,n,\,m}$$

$$+\int_0^l\int_0^l f\Big(x,\,y,\int_0^l\int_0^l \Theta\Big(\xi,\eta,\sum_{n,\,m=1}^{\infty}\beta_{n,\,m}\vartheta_{n,\,m}(\xi,\eta)\Big)d\xi d\eta\Big)\vartheta_{n,\,m}(x,y)dxdy, \quad (3.1)$$

where $E_{j\,n,\,m}\left(t_0\right)\neq 0, \quad j=\overline{1,\,5}$,

$$\tau_{1\,j\,n,\,m} = \frac{\lambda_{n,\,m}^{\,k}\,\omega}{E_{j\,n,\,m}\left(t_0\right)}, \;\; \tau_{2\,j\,n,\,m} = -\tau_{1\,j\,n,\,m}B_{j\,n,\,m}\left(t_0\right), \;\;\; \tau_{3\,j\,n,\,m} = -\tau_{1\,j\,n,\,m}C_{j\,n,\,m}\left(t_0\right),$$

$$\psi_{n,\,m} = \int_0^l\int_0^l \psi\left(x,\,y\right)\vartheta_{\,n,\,m}\left(x,\,y\right)d\,x\,d\,y. \quad (3.2)$$

The unique solvability of countable system (3.1). We use the concepts of the following well-known Banach spaces. Hilbert coordinate space ℓ_2 of number sequences $\{\,\varphi_{\,n,\,m}\}_{n,\,m=1}^{\infty}$ with norm

$$\|\,\varphi\,\|_{\,\ell_{\,2}} = \sqrt{\sum_{n,\,m=1}^{\infty}|\,\varphi_{\,n,\,m}\,|^{\,2}} < \infty.$$

The space $L_2\left(\Omega_l^2\right)$ of square-summable functions on the domain $\Omega_l^2 = \Omega_l\times\Omega_l$ with norm

$$\|\,\vartheta\left(x,\,y\right)\|_{\,L_2\left(\Omega_l^2\right)} = \sqrt{\int_0^l\int_0^l|\,\vartheta\left(x,\,y\right)|^2\,d\,x\,d\,y} < \infty.$$

Conditions of smoothness. Let for functions

$$\varphi_i(x,\,y),\;\psi(x,\,y)\in C^{\,4k}\left(\Omega_l^2\right),\; f\,\left(x,\,y,\,\cdot\right)\in C_{x,\,y}^{\,4k}\left(\Omega_l^2\times\mathbb{R}\right),\; i=1,\,2$$

in the domain Ω_l^2 there exist piecewise continuous $4k+1$ order derivatives.

Then by integrating in parts the functions (2.4) and (3.2) $4k+1$ times over every variable $x,\,y$, we obtain following relations [23]

$$|\,\varphi_{i\,n,\,m}\,| = \left(\frac{l}{\pi}\right)^{8k+2}\frac{\left|\varphi_{i\,n,\,m}^{(8k+2)}\right|}{n^{\,4k+1}m^{\,4k+1}}, \;\; |\,\psi_{n,\,m}\,| = \left(\frac{l}{\pi}\right)^{8k+2}\frac{\left|\psi_{n,\,m}^{(8k+2)}\right|}{n^{\,4k+1}m^{\,4k+1}}, \quad (3.3)$$

$$\sum_{n,\,m=1}^{\infty}\left[\varphi_{i\,n,\,m}^{(8k+2)}\right]^{\,2} \leq \left(\frac{2}{l}\right)^{2}\int_0^l\int_0^l\left[\frac{\partial^{\,8k+2}\varphi_i(x,\,y)}{\partial\,x^{4k+1}\,\partial\,y^{4k+1}}\right]^{\,2}d\,x\,d\,y,\;\; i=1,\,2, \quad (3.4)$$

$$\sum_{n,\,m=1}^{\infty}\left[\psi_{\,n,\,m}^{\,(8k+2)}\right]^{\,2} \leq \left(\frac{2}{l}\right)^{2}\int_0^l\int_0^l\left[\frac{\partial^{\,8k+2}\psi(x,\,y)}{\partial\,x^{4k+1}\,\partial\,y^{4k+1}}\right]^{\,2}d\,x\,d\,y, \quad (3.5)$$

where

$$\varphi_{i\,n,\,m}^{(8k+2)} = \int_0^l\int_0^l\frac{\partial^{\,8k+2}\varphi_i(x,\,y)}{\partial\,x^{4k+1}\,\partial\,y^{4k+1}}\,\vartheta_{n,\,m}(x,\,y)\,d\,x\,d\,y,\;\; i=1,\,2,$$

$$\psi_{n,m}^{(8k+2)} = \int_0^l \int_0^l \frac{\partial^{8k+2}\psi(x,\, y)}{\partial\, x^{4k+1}\, \partial\, y^{4k+1}}\, \vartheta_{n,\,m}(x,\, y)\, d\, x\, d\, y.$$

We obtain also that

$$|\, f_{n,\,m}(\cdot)\,| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|\, f_{n,\,m}^{(8k+2)}(x,\, y,\, \cdot)\,\right|}{n^{\,4k+1} m^{\,4k+1}}, \tag{3.6}$$

$$\sum_{n,\,m=1}^{\infty} \left[f_{\,n,\,m}^{\,(8k+2)}(\cdot)\right]^{\,2} \le \left(\frac{2}{l}\right)^{2} \int_0^l \int_0^l \left[\frac{\partial^{\,8k+2} f(x,\, y,\, \cdot)}{\partial\, x^{4k+1} \partial\, y^{4k+1}}\right]^{\,2} d\, x\, d\, y, \tag{3.7}$$

where

$$f_{n,\,m}^{(8k+2)}(\cdot) = \int_0^l \int_0^l \frac{\partial^{8k+2} f(x,\, y,\, \cdot)}{\partial\, x^{4k+1}\, \partial\, y^{4k+1}}\, \vartheta_{n,\,m}(x,\, y)\, d\, x\, d\, y.$$

For regular values of parameter $\omega \in \Lambda_j$ $\left(j = \overline{1,\, 5}\right)$ we prove that there holds

Theorem 3.1 *Suppose that the conditions of smoothness and following conditions are fulfilled:*

1) $\chi_1 = \max\limits_{n,\,m} \{|\,\tau_{1\,j\,n,\,m}\,|\,;\, |\,\tau_{2\,j\,n,\,m}\,|\,;\, |\,\tau_{3\,j\,n,\,m}\,|\} < \infty$*;*

2) $\chi_2 = \|\, f\,(x,\, y,\, \cdot)\,\|_{\,L_2\left(\Omega_l^2\right)} < \infty$*;*

3) $|\, f\,(x,\, y,\, \gamma_1) - f\,(x,\, y,\, \gamma_2)\,| \le M_0(x,\, y)\,|\,\gamma_1 - \gamma_2\,|$*;*

4) $|\Theta(\xi, \eta, \beta_1) - \Theta(\xi, \eta, \beta_2)| \le \Theta_0(\xi, \eta)\,|\beta_1 - \beta_2|\,,\ 0 < \|\Theta_0(\xi, \eta)\|_{L_2\left(\Omega_l^2\right)} < \infty$*;*

5) $\rho < 1$*, where*

$$\rho = \gamma_3 \left(\frac{2}{l}\right)^{2} \|\,\Theta_0\,(x,\, y)\,\|_{\,L_2\left(\Omega_l^2\right)}\,, \quad \gamma_3 = \gamma_2 \left(\frac{2}{l}\right)^{2} \left\|\, \frac{\partial^{\,8k+2} M_0(x,\, y)}{\partial\, x^{4k+1} \partial\, y^{4k+1}} \,\right\|_{\,L_2\left(\Omega_l^2\right)},$$

$$\gamma_2 = C_1 \left(\frac{2}{l}\right)^{2} \left(\frac{l}{\pi}\right)^{8k+2}, \quad C_1 = \sqrt{\sum_{n,\,m=1}^{\infty} \frac{1}{n^{\,8k+2}\, m^{\,8k+2}}} < \infty.$$

Then the countable system (3.1) *is uniquely solvable in the space* ℓ_2 *for regular spectral values from the numerical set* $\omega \in \Lambda_j$ *for each* $j = \overline{1,\, 5}$ *and all possible* n *and* m*.*

Proof. We use the method of compressing mappings in the Hilbert coordinate space ℓ_2. Successive approximations are defined as follows:

$$\begin{cases} \beta_{\,n,\,m}^{\,0} = \psi_{\,n,\,m}\,\tau_{1\,j\,n,\,m} + \varphi_{\,1\,n,\,m}\,\tau_{2\,j\,n,\,m} + \varphi_{\,2\,n,\,m}\,\tau_{3\,j\,n,\,m}, \\ \beta_{\,n,\,m}^{\,i+1} = \Im_j\left(\beta_{\,n,\,m}^{\,i}\right), \quad i = 0,\, 1,\, 2,\, \ldots, \quad j = \overline{1,\, 5}. \end{cases} \tag{3.8}$$

We use formulas (3.3)–(3.7). According to the first condition of theorem and formula (3.3), we have

$$\left|\, \beta_{\,n,\,m}^{\,0}\,\right| \le |\,\psi_{\,n,\,m}\,| \cdot |\tau_{1\,j\,n,\,m}\,| + |\,\varphi_{\,1\,n,\,m}\,| \cdot |\,\tau_{2\,j\,n,\,m}\,| + |\,\varphi_{\,2\,n,\,m}\,| \cdot |\,\tau_{3\,j\,n,\,m}\,|$$

$$\le \chi_1 \left(\frac{l}{\pi}\right)^{8k+2} \left[\frac{\left|\,\psi_{\,n,\,m}^{(8k+2)}\,\right|}{n^{\,4k+1}\, m^{\,4k+1}} + \frac{\left|\,\varphi_{\,1\,n,\,m}^{(8k+2)}\,\right|}{n^{\,4k+1} m^{\,4k+1}} + \frac{\left|\,\varphi_{\,1\,n,\,m}^{(8k+2)}\,\right|}{n^{\,4k+1} m^{\,4k+1}} \right].$$

Hence by the aid of the Cauchy–Schwartz inequality and the Bessel inequalities (3.4), (3.5) for the zero approximation of the coefficients of redefinition function we obtain from successive approximations (3.8), that

$$
\begin{aligned}
\| \beta^0 \|_{\ell_2} &\leq \chi_1 \Big(\frac{l}{\pi}\Big)^{8k+2} \sum_{n,\,m=1}^{\infty} \left[\frac{\left| \psi_{n,\,m}^{(8k+2)} \right|}{n^{4k+1}\, m^{4k+1}} + \frac{\left| \varphi_{1\,n,\,m}^{(8k+2)} \right|}{n^{4k+1} m^{4k+1}} + \frac{\left| \varphi_{1\,n,\,m}^{(8k+2)} \right|}{n^{4k+1} m^{4k+1}} \right] \\
&\leq \chi_1 \Big(\frac{l}{\pi}\Big)^{8k+2} \left[\sum_{n,m=1}^{\infty} \frac{\left| \psi_{n,m}^{(8k+2)} \right|}{n^{4k+1} m^{4k+1}} + \sum_{n,m=1}^{\infty} \frac{\left| \varphi_{1\,n,m}^{(8k+2)} \right|}{n^{4k+1} m^{4k+1}} + \sum_{n,m=1}^{\infty} \frac{\left| \varphi_{2\,n,m}^{(8k+2)} \right|}{n^{4k+1} m^{4k+1}} \right] \\
&\leq \chi_1 \Big(\frac{l}{\pi}\Big)^{8k+2} \sqrt{\sum_{n,m=1}^{\infty} \frac{1}{n^{8k+2} m^{8k+2}}} \left[\left\| \psi^{(8k+2)} \right\|_{\ell_2} + \left\| \varphi_1^{(8k+2)} \right\|_{\ell_2} + \left\| \varphi_2^{(8k+2)} \right\|_{\ell_2} \right] \\
&\leq \gamma_1 \left[\left\| \frac{\partial^{8k+2} \psi(x,\, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} + \left\| \frac{\partial^{8k+2} \varphi_1(x,\, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} \right. \\
&\qquad \left. + \left\| \frac{\partial^{8k+2} \varphi_2(x,\, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} \right] < \infty, \qquad (3.9)
\end{aligned}
$$

where

$$
\gamma_1 = \chi_1 C_1 \Big(\frac{2}{l}\Big)^2 \Big(\frac{l}{\pi}\Big)^{8k+2}, \quad C_1 = \sqrt{\sum_{n,\,m=1}^{\infty} \frac{1}{n^{8k+2}\, m^{8k+2}}} < \infty.
$$

According to the first and second conditions of theorem and formulas (3.3), (3.6), using the Cauchy–Schwartz inequality and Bessel inequalities (3.4), (3.5), (3.7) for the first difference of approximation (3.8) we obtain

$$
\| \beta^1 - \beta^0 \|_{\ell_2} \leq \sum_{n,\,m=1}^{\infty} \left| \int_0^l \int_0^l f\left(x, y, \int_0^l \int_0^l \Theta\left(\xi, \eta, \Delta^0\right) d\xi d\eta\right) \vartheta_{n,\,m}(x,y) dx dy \right|
$$

$$
\leq \left(\frac{l}{\pi}\right)^{8k+2} \sum_{n,\,m=1}^{\infty} \frac{\left| f_{n,\,m}^{(8k+2)}(x,\, y,\, \cdot) \right|}{n^{4k+1} m^{4k+1}} \leq \gamma_2 \left\| \frac{\partial^{8k+2} f(x,\, y,\, \cdot)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} < \infty, \qquad (3.10)
$$

where

$$
\Delta^0 = \sum_{n,\,m=1}^{\infty} \beta_{n,\,m}^0 \vartheta_{n,\,m}(\xi,\, \eta), \;\; \gamma_2 = C_1 \Big(\frac{2}{l}\Big)^2 \Big(\frac{l}{\pi}\Big)^{8k+2}.
$$

Analogously, by the third and fourth conditions of the theorem, using the Cauchy–Schwartz inequality and Bessel inequality for an arbitrary difference of approximation (3.8) we obtain

$$
\| \beta^{i+1} - \beta^i \|_{\ell_2} \leq \gamma_2 \left\| \frac{\partial^{8k+2}}{\partial x^{4k+1} \partial y^{4k+1}} \, \left| f\left(x,\, y,\, V^i\right) - f\left(x,\, y,\, V^{i-1}\right) \right| \right\|_{L_2(\Omega_l^2)}
$$

$$
\leq \gamma_3 \left| \int_0^l \int_0^l \Theta(\xi,\, \eta) \sum_{n,\,m=1}^{\infty} \left| \beta_{n,\,m}^i - \beta_{n,\,m}^{i-1} \right| \vartheta_{n,\,m}(\xi,\, \eta)\, d\xi\, d\eta \right|
$$

$$\leq \gamma_3 \sum_{n,\, m=1}^{\infty} \left| \beta^i_{n,\, m} - \beta^{i-1}_{n,\, m} \right| \left| \int_0^l \int_0^l \Theta(\xi,\, \eta)\, \vartheta_{\, n,\, m}\,(\xi,\, \eta)\, d\,\xi\, d\,\eta \right|$$

$$\leq \rho \, \left\| \, \beta^{\, i} - \beta^{\, i-1} \, \right\|_{\ell_2}, \tag{3.11}$$

where

$$\rho = \gamma_3 \Big(\frac{2}{l} \Big)^2 \, \| \, \Theta_0\,(x,\, y) \, \|_{L_2(\Omega^2_l)}, \quad \gamma_3 = \gamma_2 \Big(\frac{2}{l} \Big)^2 \left\| \frac{\partial^{8k+2} M_0(x,\, y)}{\partial\, x^{4k+1} \partial\, y^{4k+1}} \right\|_{L_2(\Omega^2_l)}.$$

By the fifth condition of the theorem, $\rho < 1$. Therefore, it follows from estimate (3.11) that the operator on the right-hand side of (3.1) is contracting. From the estimates (3.9)–(3.11) implies that there exists a unique fixed point, which is a solution of countable system (3.1) in the space ℓ_2. The Theorem 3.1 is proved.

Convergence Fourier series of redefinition function. Substituting representations (3.1) into the Fourier series (2.3), we obtain

$$\beta\,(x,\, y) = \sum_{n,\, m=1}^{\infty} \vartheta_{\, n,\, m}\,(x,\, y) \Big[\psi_{\, n,\, m}\, \tau_{1\, j\, n,\, m} + \varphi_{\, 1\, n,\, m}\, \tau_{2\, j\, n,\, m} + \varphi_{\, 2\, n,\, m}\, \tau_{3\, j\, n,\, m}$$

$$+ \int_0^l \int_0^l f\left(x,\, y,\, \int_0^l \int_0^l \Theta\Big(\xi, \eta,\, \sum_{n,m=1}^{\infty} \beta_{n,m} \vartheta_{n,m}(\xi,\eta)\Big) d\xi d\eta \right) \vartheta_{n,m}(x,y) dx dy \Big]. \tag{3.12}$$

Theorem 3.2 *Assume that the conditions of theorem 3.1 are fulfilled. Then for regular values of spectral parameter* $\omega \in \Lambda_j \; \big(j = \overline{1,\, 5}\big)$ *the series* (3.12) *converge absolutely and uniformly.*

Proof. We use formulas (3.3)–(3.7) and estimates (3.9), (3.10). Using the Cauchy–Schwartz inequality and Bessel inequalities for series (3.12), we obtain the following estimate

$$|\, \beta\,(x,\, y)\,| \leq \sum_{n,\, m=1}^{\infty} |\, \vartheta_{\, n,\, m}\,(x,\, y)\,| \, [|\, \psi_{\, n,\, m}\, \tau_{1\, j\, n,\, m} + \varphi_{\, 1\, n,\, m}\, \tau_{2\, j\, n,\, m} + \varphi_{\, 2\, n,\, m}\, \tau_{3\, j\, n,\, m}\,|$$

$$+ \left| \int_0^l \int_0^l f\left(x, y,\, \int_0^l \int_0^l \Theta\Big(\xi, \eta,\, \sum_{n,\, m=1}^{\infty} \beta_{n,\, m} \vartheta_{n,\, m}(\xi,\eta)\Big) d\xi d\eta \right) \vartheta_{n,\, m}(x,y) dx dy \right| \Bigg]$$

$$\leq \gamma_1 \, \frac{2}{l} \left[\left\| \frac{\partial^{8k+2} \psi(x,\, y)}{\partial\, x^{4k+1} \partial\, y^{4k+1}} \right\|_{L_2(\Omega^2_l)} + \left\| \frac{\partial^{8k+2} \varphi_1\,(x,\, y)}{\partial\, x^{4k+1} \partial\, y^{4k+1}} \right\|_{L_2(\Omega^2_l)} \right.$$

$$\left. + \left\| \frac{\partial^{8k+2} \varphi_2\,(x,\, y)}{\partial\, x^{4k+1} \partial\, y^{4k+1}} \right\|_{L_2(\Omega^2_l)} \right] + \gamma_2 \frac{2}{l} \left\| \frac{\partial^{8k+2} f(x,\, y,\, \cdot)}{\partial\, x^{4k+1} \partial\, y^{4k+1}} \right\|_{L_2(\Omega^2_l)} < \infty. \tag{3.13}$$

From estimate (3.13) implies the absolutely and uniformly convergence of Fourier series (3.12). The Theorem 3.2 is proved.

4 Main unknown function

We determined the redefinition functions as a Fourier series (3.12). So, redefinition function is known. Using representations (3.1), Fourier series (2.11), the main unknown function we can present as

$$U(t, x, y) = \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x, y)$$
$$\times \left[\varphi_{1\,n,m} P_{j\,n,m}(t) + \varphi_{2\,n,m} Q_{j\,n,m}(t) + \psi_{n,m} S_{j\,n,m}(t)\right], \tag{4.1}$$

where

$$P_{j\,n,m}(t) = B_{j\,n,m}(t) + \frac{\tau_{2\,j\,n,m}}{\lambda_{n,m}^{k}\,\omega} E_{j\,n,m}(t), Q_{j\,n,m}(t) = C_{j\,n,m}(t) + \frac{\tau_{3\,j\,n,m}}{\lambda_{n,m}^{k}\,\omega} E_{j\,n,m}(t),$$

$$S_{j\,n,m}(t) = \frac{\tau_{3\,j\,n,m}}{\lambda_{n,m}^{k}\,\omega} E_{j\,n,m}(t), \quad j = \overline{1, 5}.$$

To establish the uniqueness of the function $U(t, x, y)$ we suppose that there are two functions U_1 and U_2 satisfying the given conditions (1.1)–(1.6). Then their difference $U = U_1 - U_2$ is a solution of differential equation (1.1), satisfying conditions (1.2)–(1.6) with functions

$$\varphi_i(x, y) \equiv 0 \ (i = 1, 2), \quad \psi(x, y) \equiv 0.$$

By virtue of relations (2.4) and (3.2) we have $\varphi_{i\,n,m} = \psi_{n,m} = 0 \ (i = 1, 2)$. Hence, we obtain from formulas (2.2) and (4.1) in the domain Ω, that there holds the following zero identity

$$\int_0^l \int_0^l U(t, x, y)\, \vartheta_{n,m}(x, y)\, dx\, dy \equiv 0.$$

Hence, by virtue of the completeness of the systems of eigenfunctions

$$\left\{ \sqrt{\frac{2}{l}} \sin \frac{\pi n}{l} x \right\}, \ \left\{ \sqrt{\frac{2}{l}} \sin \frac{\pi m}{l} y \right\}$$

in $L_2\left(\Omega_l^2\right)$, we deduce that $U(t, x, y) \equiv 0$ for all $x \in \Omega_l^2 \equiv [0, l]^2$ and $t \in \Omega_T \equiv [0; T]$.

Therefore, for regular values of spectral parameter ω the function $U(t, x, y)$ is unique solution of differential equation (1.1) with conditions (1.2)–(1.6), if this function exists in the domain Ω.

Theorem 4.1 *Let the conditions of the theorem 3.1 be fulfilled. Then for regular values of spectral parameter $\omega \in \Lambda_j$ $\left(j = \overline{1, 5}\right)$ the series (4.1) converge. At the same time, their term by term differentiation is possible.*

Proof. By virtue of conditions of the theorem 3.1, the functions $P_{j\,n,m}(t)$, $Q_{j\,n,m}(t)$ and $S_{j\,n,m}(t)$ $\left(j = \overline{1, 5}\right)$ uniformly bounded on the segment Ω_T. So, for any positive integers n, m there exists finite constant C_2, that there takes place the following estimate

$$\max\left\{ \max_{n,m} \max_{j=\overline{1,5}} |P_{jn,m}(t)|\,;\ \max_{n,m} \max_{j=\overline{1,5}} |Q_{jn,m}(t)|\,;\ \max_{n,m} \max_{j=\overline{1,5}} |S_{jn,m}(t)| \right\} \le C_2. \tag{4.2}$$

Using estimate (4.2), analogously to the estimate (3.13), for series (4.1) we obtain

$$|U(t,x,y)| \le \sum_{n,m=1}^{\infty} |\vartheta_{n,m}(x,y)|\, |\varphi_{1\,n,m} P_{j\,n,m}(t) + \varphi_{2\,n,m} Q_{j\,n,m}(t) + \psi_{n,m} S_{j\,n,m}(t)|$$

$$\leq \gamma_4 \left[\left\| \frac{\partial^{8k+2}\varphi_1(x,y)}{\partial x^{4k+1}\partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} + \left\| \frac{\partial^{8k+2}\varphi_2(x,y)}{\partial x^{4k+1}\partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} + \left\| \frac{\partial^{8k+2}\psi(x,y)}{\partial x^{4k+1}\partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} \right] < \infty,$$

where $\gamma_4 = C_1 C_2 \left(\frac{2}{l}\right)^3 \left(\frac{l}{\pi}\right)^{8k+2}$.

Similarly, it is proved that the function $U(t, x, y)$ belongs to the class of functions (1.5). Due to the limitation of the volume of the article, we will not present this proof here. A similar proof you can see in the work [23]. Theorem 4.1 is proved.

Remark 4.1 By virtue of limitation of the volume of this article, we will not include to this paper the results on stability of the function $U(t, x, y)$ on redefinition function $\beta(x, y)$, on given data functions $\varphi_i(x, y)$ $(i = 1, 2)$, $\psi(x, y)$ and on parameters ω, ε_1, ε_2. Moreover, we did not consider this inverse problem (1.1)–(1.6) for the case of irregular values of spectral parameter ω.

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