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On spectral properties of multivalued production mappings in economic dynamics models

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Abstract. Spectral theory of superlinear mappings was created by A.M. Rubinov, who introduced the concepts of eigenvalue and eigenset. He also established the finiteness and discreteness of the spectrum of superlinear mapping, proved the existence of eigencompact and, under some additional conditions, provided a full description of the spectrum. But, in general case, a description of the spectrum of superlinear mapping has not been obtained. This work finishes the construction of above spectral theory and provides a full description of the spectrum of arbitrary normal superlinear mapping.

Keywords. Superlinear mappings, Eigenvalue, Invariant subspace

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1 Introduction

Multivalued mappings are largely used in the study of economic models by mathematical methods. For more information about multivalued mappings we refer the readers to [3, 4, 9, 13,14]. A monograph by V.L. Makarov and A.M. Rubinov [11] is a fundamental work in the field of applications of multivalued mappings in economic studies. In that work, under some additional conditions, the authors created the spectral theory of superlinear mappings, introduced the concepts of eigenvalue and eigenset, established the finiteness and discreteness of the spectrum of superlinear mapping, proved the existence of eigencompact and provided a description of spectrum. But, in general case, a description of the spectrum of superlinear mapping has not been obtained. Some results have been obtained later in [1-9, 13-15]. Our work finishes the construction of above spectral theory and provides a full description of the spectrum of arbitrary superlinear mapping.

Let $\mathbb{R}^n$ be an n -dimensional real vector space, $\mathbb{R}^n_+$ be a cone from $\mathbb{R}^n$ with non-negative components, x^i be an i-th component of the vector $x \in \mathbb{R}^n$, J be an index set $\{1,\ 2,\ \ldots,\ n\}$, R^I be a subspace of the space $\mathbb{R}^n$ with the indices of its unit vectors taken from the index set $I \subset J$, R^I_+ be a cone of vectors of the subspace R^I with non-negative components, and $\pi\left(\Omega\right)$ be a set of all nonempty subspaces of the space Ω.

Definition [4, 11]. *A multivalued mapping* $a : \mathbb{R}^n_+ \to \pi\left(\mathbb{R}^n_+\right)$ *is called superlinear if:*

1. $a(\lambda,\ x) = \lambda\, a(x)\,, \quad \lambda > 0,\ x \in \mathbb{R}^n_+,$
2. $a\left(x_1\right) +\ a\left(x_2\right) = a\left(x_1 + x_2\right), \quad x_1,\ x_2 \in \mathbb{R}^n_+,$
3. $a(0) = \{0\},$

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4. *the graph of* a *is closed*,
5. $a\left(\mathbb{R}^n_+\right) \cap int\,\mathbb{R}^n_+ \neq \emptyset$.

Definition [11]. *A multivalued mapping* $a : \mathbb{R}^n_+ \to \pi\left(\mathbb{R}^n_+\right)$ *is called normal if for every* $x \in \mathbb{R}^n$ *the set* $a(x)$ *is normal, i.e.* $\left(a(x) - \mathbb{R}^n_+\right) \cap \mathbb{R}^n_+ = a(x)$.

2 Main part

In what follows, we only consider the normal superlinear mappings.

Definition 2.1. *Positive number* λ *is called an eigenvalue of the superlinear mapping* $a : \mathbb{R}^n_+ \to \pi\left(\mathbb{R}^n_+\right)$ *if there is a nonempty convex subset* Ω *of the cone* $\mathbb{R}^n_+$ *different from the face of this cone such that*

$$\overline{a\left(\Omega\right)} = \lambda\,\overline{\Omega}. \tag{2.1}$$

A set Ω which satisfies (2.1) is called an eigenset of the mapping a.

The totality of all eigensets of the mapping a corresponding to the eigenvalue λ will be denoted by $\pi\alpha\left(\lambda\right)$. Obviously, any eigenset is normal. Besides, due to the superadditivity and closedness of superlinear mapping (properties 2 and 4), the closure $\overline{\Omega}$ of the set $\Omega \subset \pi\alpha\left(\lambda\right)$ also belongs to $\pi\alpha\left(\lambda\right)$.

For every set $Q \subset \mathbb{R}^n_+$, define the index sets

$$I\left(Q\right) = \left\{i \in J \;\mid\; \exists\, y \in \overline{Q} \;:\; \; y^i > 0\right\},$$

$$I_\infty\left(Q\right) = \left\{i \in J \;\mid\; \exists\, y \in \overline{Q} \;:\; \{(\mu y)\,\mu \geq 0\} \subset \overline{Q}, \;\; y^i > 0\right\}.$$

Definition [10, 12, 15]. *A subspace* R^I *is called invariant with respect to the mapping* a *if*

$$a\left(R^I_+\right) = R^I_+. \tag{2.2}$$

Lemma 2.1. *Let* $a \;:\; \mathbb{R}^n_+ \to \pi\left(\mathbb{R}^n_+\right)$ *be a superlinear mapping,* λ *be its eigenvalue,* $\Omega \subset \pi\alpha\left(\lambda\right)$. *Then the subspaces* R^I *are invariant with respect to* a *for* $I = I\left(\Omega\right)$ *and* $I = I_\infty\left(\Omega\right)$.

Proof. Let's prove this lemma for $I = I\left(\Omega\right)$. The closure of the conical shell $\overline{C\left(\Omega\right)}$ of the set Ω coincides with the cone R^I_+. As the superlinear mapping is positively homogeneous (property 1), we have

$$a\left(R^I_+\right) = a\left(\overline{C\left(\Omega\right)}\right) = C\left(\overline{a\left(\Omega\right)}\right) = C\left(\overline{\lambda\Omega}\right) = C\left(\overline{\Omega}\right) = R^I_+.$$

So, for $I = I\left(\Omega\right)$ the subspace R^I is invariant.

Now let's prove the lemma for $I = I_\infty\left(\Omega\right)$. Let x_0 be an arbitrary point on the cone R^I_+. As R^I_+ is a cone, we have $\left\{\left(\mu\, x_0\right)_\mu \geq 0\right\} \subset R^I_+$. It is not difficult to show that $R^I_+ \subset \Omega$ for $I = I_\infty\left(\Omega\right)$.

This follows from the properties of superlinear mapping and the definition of the set $I_\infty\left(\Omega\right)$ [2, 3, 9,13]. Then $\left\{\left(\mu\, x_0\right)_\mu \geq 0\right\} \subset R^I_+ \subset \Omega$, and therefore $\mu\, a\left(x_0\right) = a\left(\mu\, x_0\right) \subset \overline{a\left(\Omega\right)} = \lambda\overline{\Omega}$ for $\mu \geq 0$.

Hence we obtain $I\left(a\left(x_0\right)\right) \subset I_\infty\left(\Omega\right)$. Consequently, $a\left(x_0\right) \subset R^I_+$ for $I = I_\infty\left(\Omega\right)$. As the inclusion is proved for arbitrary $x_0 \in R^I_+$, we have

$$a\left(R^I_+\right) \subset R^I_+. \tag{2.3}$$

Let's prove the inverse inclusion. As noted above, $R_+^I \subset \Omega$. Therefore,

$$R_+^I \subset \lambda\overline{\Omega} = \overline{a\,(\Omega)}.$$

Normality of the set $a\,(\Omega)$ implies $R_+^I \subset a\,(\Omega)$. Let $y_0 \in ri\; R_+^I$. For $k = 1,\ 2,\ \ldots$ we have

$$ky_0 \in ri\; R_+^I \subset R_+^I \subset a\,(\Omega)\,.$$

Fix $x_k \in a^{-1}\,(ky_0) \bigcap \Omega$ for every k and denote by $\tilde{x}_k$ the projection of the point $\frac{x_k}{k}$ on the subspace R^I. For $i \notin I_\infty\,(\Omega)$ we have

$$\lim_{k\to\infty} \frac{x_k^i}{k} = 0.$$

Then

$$\lim_{k\to\infty} \left\| \tilde{x}_k - \frac{x_k}{k} \right\| = 0.$$

As $y_0 \in a\left(\frac{x_k}{k}\right)$ for every k and the superlinear mapping is closed and superadditive (properties 4 and 2), the distance between $a\,(\tilde{x}_k)$ and $y_0 \in ri\; R_+^I$ tends to zero as $k \to \infty$. For sufficiently great k we have

$$a\,(x_k) \cap ri\; R_+^I = \emptyset.$$

As $x_k \in R_+^I$ and the mapping a is normal and positively homogeneous, we have $R_+^I \subset a\left(R_+^I\right)$. The subspace R_+^I is invariant for $I = I_\infty\,(\Omega)$. Lemma is proved.

Definition [10, 11, 12]. *By restriction of superlinear mapping a to the subspace R_+^I we mean a multivalued mapping*

$$a_I : R_+^I \to \pi\left(R_+^I\right),$$

defined by the following equality:

$$a_I(x) = a(x) \cap R_+^I \quad \text{for} \quad x \in R_+^I.$$

The graph Z of the mapping a and the graph Z_I of the restriction a_I are related by the equality

$$Z_I = Z \cap \left(R_+^I \times R_+^I\right).$$

If R^I is an invariant subspace, then $a_I(x) = a(x)$ for $x \in R_+^I$. It is not difficult to show that a_I is superlinear in invariant subspace R^I. In this case, the condition

$$a\left(\mathbb{R}_+^n\right) \cap int\; \mathbb{R}_+^n \neq \emptyset$$

is satisfied instead of

$$a_I\left(\mathbb{R}_+^n\right) \cap ri\; R_+^I \neq \emptyset.$$

Let I_1, I_2 be the subsets of indices from J with $I_2 \subset I_1$ and $I_2 \neq I_1$. Define the multivalued mapping

$$b_{I_1,\,I_2} : R_+^{I_2} \to \pi\left(R_+^{I_2}\right),$$

by letting

$$b_{I_1,\, I_2}(x) = \left\{ P_{r_{R^{I_2}}} \left(\cup a_{I_1} \left(\tilde{x} \right) \right) \mid \tilde{x} \in R_+^{I_1} \text{ and } P_{\mathrm{r}_{R^{I_2}}} \tilde{x} = x \right\},$$

where $P_{\mathrm{r}_{R^I}}$ is a projection operator on the subspace R^I.

Lemma 2.2. *If the subspaces R^{I_1}, R^{I_3} are invariant with respect to the superlinear mapping $a : \mathbb{R}_+^n \to \pi\left(\mathbb{R}_+^n\right)$ with $I_2 \cap I_3 = \emptyset$, $I_2 \cup I_3 = I_1$, then the mapping $b_{I_1,\, I_2}$ is superlinear in the subspace R^{I_2}.*

Proof. The graph Z_{I_1} of the restriction a_{I_1} and the graph Z_b of the mapping $b_{I_1,\, I_2}$ are related by the equality

$$Z_b = \overline{P_{\mathrm{r}_{R_+^{I_2} \times R_+^{I_2}}} Z}\ ,$$

from which it follows that Z_b is a convex closed cone lying in $R^{I_2} \times R^{I_2}$, with $P_r\ Z_b \cap ri\ R_+^{I_2} \neq \emptyset$.

As the subspace R^{I_3} is invariant, we have $(0,\ y) \in Z_b$ only if $y = 0$. Then, as shown in [11], the mapping $b_{I_1,\, I_2}$ is superlinear in the subspace R^{I_2}. Lemma is proved.

Definition [11]. *Neumann growth rate of superlinear mapping $a : \mathbb{R}_+^n \to \pi\left(\mathbb{R}_+^n\right)$ is defined as*

$$\alpha(a) = \sup\left\{ \alpha(x,\ y) \mid x \in \mathbb{R}_+^n, \quad y \in a(x) \right\},$$

where

$$\alpha\left(x,\ y\right) = \sup\left\{ \alpha \mid y \geq \alpha x \right\}.$$

Obviously, for every invariant subspace R^I we have $\alpha\left(a\right) \geq \alpha\left(a_I\right)$.

Theorem 2.1. *Let $a : \mathbb{R}_+^n \to \pi\left(\mathbb{R}_+^n\right)$ be a superlinear mapping, λ be its eigenvalue, $\Omega \subset \pi a(\lambda)$. Then $\lambda = \alpha\left(b_{I_1,\, I_2}\right)$ for $I_1 = I\left(\Omega\right)$, $I_2 = I\left(\Omega\right) \backslash I_\infty\left(\Omega\right)$.*

Proof. By Lemma 1, the subspace R^{I_1} is invariant with respect to a. Then λ is an eigenvalue of the restriction a_{I_1}, $\Omega ? \pi\left(a\left(\lambda\right)\right)$, because

$$\overline{a_{I_1}\left(\Omega\right)} = \overline{a\left(\Omega\right)} = \lambda\, \overline{\Omega}.$$

If $I_3 = I_\infty\left(\Omega\right) = \emptyset$, then the closure $\overline{\Omega}$ is an eigencompact of the restriction a_I, with

$$ri\ \overline{\Omega} \cap ri\ R_+^{I_1} \neq \emptyset.$$

In this case, as shown in [11], $\alpha\left(a_{I_1}\right) = \lambda$. It is not difficult to see that $a_{I_1} = b_{I_1,\, I_1}$. Consequently, $\alpha\left(b_{I_1,\, I_1}\right) = \alpha\left(a_{I_1}\right) = \lambda$.

Let $I_3 = I_\infty\left(\Omega\right) = \emptyset$. Consider the set

$$\Psi = \overline{P_{\mathrm{r}_{R^{I_2}}}\left(\Omega\right)}\ .$$

Obviously, Ψ is convex and closed. Besides, it is bounded, because $I_2 \cap I_3 = \emptyset$. Further, as $\Omega \cap ri\ R_+^{I_1} \neq \emptyset$, we have

$$\Omega \cap ri\ R_+^{I_2} \neq \emptyset.$$

The following equalities hold:

$$b_{I_1,\, I_2}\left(\Omega\right) = \left\{ P_{\mathrm{r}_{R^{I_2}}} a_{I_1}\left(\tilde{x}\right) \mid \tilde{x} \in \Omega \right\} = \overline{P_{\mathrm{r}_{R_+^{I_2}}} a\left(\Omega\right)} = \overline{P_{\mathrm{r}_{R_+^{I_2}}}\left(\lambda \Omega\right)} = \lambda P_{\mathrm{r}_{R_+^{I_2}}} a\left(\Omega\right) = \lambda \Psi.$$

Thus, Ψ is an eigencompact set of the mapping $b_{I_1,\, I_2}$, with ri $\Psi \neq \Phi$, corresponding to the eigenvalue λ. In this case, as shown in [6], $\alpha\left(b_{I_1,\, I_2}\right) = \lambda$. Theorem is proved.

Definition [11]. *We will say that the generalized equilibrium state σ of superlinear mapping $a : \mathbb{R}^n_+ \to \pi\left(\mathbb{R}^n_+\right)$ is given, if there are the positive number α, $\overline{x} \in \mathbb{R}^n_+$, $\overline{y} \in a\left(\overline{x}\right)$ and the functional $\overline{p} \in \left(\mathbb{R}^n_+\right)^*$ (hereafter, $*$ denotes the conjugacy) such that*

$$\alpha\overline{x} \leq \overline{y},$$

$$\overline{p}\left(y\right) \leq \alpha\, \overline{p}(x)$$

for every $x \in \mathbb{R}^n_+$, $y \in a(x)$.

The number α in this definition is called a generalized growth rate of superlinear mapping. If, in addition, $\overline{p}\left(\overline{y}\right) > 0$, then the triplet $\sigma = \left(\alpha,\ \left(\overline{x},\ \overline{y}\right),\ \overline{p}\right)$ is called an equilibrium state, and the number α is called a growth rate of superlinear mapping. If the equilibrium state σ exists for $\alpha = \alpha\left(a\right)$, then it is called a Neumann equilibrium state.

Lemma 2.3. *Let $a : \mathbb{R}^n_+ \to \pi\left(\mathbb{R}^n_+\right)$ be a superlinear mapping. There exists an invariant subspace R^{I_1} such that the restriction a_{I_1} has a unique generalized growth rate $\alpha\left(a_{I_1}\right) = \alpha(a)$.*

Proof. As shown in [6], there exists a subset I_1 of J such that for every $\varepsilon > 0$ there are $x \in R^{I_1}_+$, $y \in a(x)$, which satisfy the conditions

$$\alpha(a) - \varepsilon \leq \min_{i \in J} \frac{y^i}{x^i}\ ; \quad y^i = 0 \text{ for } i \notin I_1; \quad y^i > 0 \text{ for } i \in I_1.$$

The lemma implies $R^{I_1}_+ \subset a\left(R^{I_1}_+\right)$. On the other hand, it has been shown in [14] that there exists a functional $\overline{p} \in \left(\mathbb{R}^n_+\right)^*$ with $I_2 = J \backslash I_1$ such that for every sufficiently small $\varepsilon > 0$ the following relations hold:

$\overline{p}\left(y\right) \leq \left(\alpha - \varepsilon\right)\ \overline{p}(x)$ for $x \in \mathbb{R}^n_+$, $y \in a(x)$,

$\overline{p}^i > 0$ for $i \in I_2$,

$\overline{p}^i = 0$ for $i \notin I_2$.

Hence, $a\left(R^{I_1}_+\right) \subset R^{I_1}_+$. Combining opposite inclusions, we obtain $a\left(R^{I_1}_+\right) = R^{I_1}_+$, which proves the invariance of the subspace R^{I_1}.

Choose $\varepsilon_k > 0$ in such a way that $\lim_{k \to \infty} \varepsilon_k = 0$. By Lemma 6.1 of [11], we find $x_k \in R^{I_1}_+$, $y_k \in a\left(x_k\right)$ such that $\left(\alpha(a) - \varepsilon_k\right)\ x_k \leq y_k$. The following inequality holds:

$$\alpha\left(a_{I_1}\right) = \sup\left\{\alpha\left(x,\ y\right)\ |\ x \in R^{I_1}_+, \quad y \in a_{I_1}(x)\right\} \geq \sup\left\{\alpha\left(x_k,\ y_k\right)\right\} = \alpha(a).$$

As R^{I_1} is an invariant subspace, we have $\alpha\left(a_{I_1}\right) \leq \alpha(a)$. So, $\alpha\left(a\right) = \alpha\left(a_{I_1}\right)$. Lemma is proved.

Lemma 2.4. *For every superlinear mapping $a : \mathbb{R}^n_+ \to \pi\left(\mathbb{R}^n_+\right)$ there is an invariant subspace R^{J_N} such that the restriction a_{J_N} has a growth rate $\alpha\left(a\right)$.*

Proof. The proof consists of constructing the sequence of embedded invariant subspaces $R^{J_1} \supset \cdots \supset R^{J_N}$ such that every restriction a_{J_i} has a Neumann growth rate $\alpha\left(a\right)$, and the last restriction a_{J_N} has a growth rate $\alpha(a)$.

Let $J_1 = J$. If the mapping a has a growth rate $\alpha\left(a\right)$, then $N = 1$ and the construction is finished. If otherwise, by Lemma 4 we find a subset $I_0 \subset J_1$ such that R^{I_0} is invariant, and the restriction a_{I_0} of the mapping a_{I_1} has a unique generalized growth rate $\alpha\left(a_{J_1}\right) = \alpha(a)$.

Let $\sigma = \left(\alpha\left(a_{I_0}\right),\ \left(\overline{x_1},\ \overline{y_1}\right),\ \overline{p_1}\right)$ be a generalized equilibrium state. If $\overline{p_1}\left(\overline{y_1}\right) > 0$, then $\alpha\left(a\right) = \alpha\left(a_{I_0}\right)$ is a growth rate of the restriction a_{I_0}, and the proof is complete.

If $\overline{p_1}(\overline{y_1}) = 0$, denote $I_1 = I(\overline{y_1})$. As $\overline{y_1} \geq \alpha(a_{J_1})\ \overline{x_1}$ and $\overline{y_1} \subset a_{J_1}(\overline{x_1})$, we have $I(\overline{y_1}) \subset I(a_{J_2}(\overline{x_1}))$, consequently, $R_+^{I_1} \subset a_{J_1}\left(R_+^{I_1}\right)$. If $a_{J_1}\left(R_+^{I_1}\right) = R_+^{I_1}$, then the subspace R^{I_1} is invariant. Assuming $J_2 = I_1$, we finish the first step of construction. If $a_{J_1}\left(R_+^{I_1}\right) \neq R_+^{I_1}$, denote $I_2 = I\left(a\left(R_+^{I_1}\right)\right)$. As $R_+^{I_1} \subset R_+^{I_2}$, we have $R_+^{I_2} \subset a_{J_1}\left(R_+^{I_1}\right) \subset a_{J_1}\left(R_+^{I_2}\right)$. If the subspace $R_+^{I_2}$ is invariant with respect to a_{J_1}, we assume $J_2 = I_2$ and finish the first step of construction, if not, we construct $I_3 = I\left(a_{J_1}\left(R_+^{I_2}\right)\right)$, and so on. This process is finite, because at every step i we have

$$I_{i+1} \subset I_i,\ \ I_{i+1} \neq I_i,\ \ I_i \subset I_0 \backslash I(\overline{p_1}).$$

At the last L-th step we obtain the subspace R^{I_2}, which is invariant with respect to a_{J_1}, and, consequently, is invariant also with respect to a. Besides, $\overline{x_1},\ \overline{y_1} \subset R^{I_1} \subset \cdots \subset R^{I_2}$, therefore,

$$\alpha(a_{J_2}) = \alpha(a_{I_2}) \geq \alpha(\overline{x_1},\ \overline{y_1}) = \alpha(a).$$

Consequently, $\alpha(a) = \alpha(a_{J_2})$. If $\alpha(a)$ is a growth rate of the restriction a_{J_2}, then $N = 2$ and the construction is finished. If otherwise, we construct R^{J_3} and so on. The process of construction will be finished at some N-th step when the invariant subspace R^{J_N} is found such that a_{J_N} has a growth rate $\alpha(a)$, or an index set J_N consisting of one element is obtained. In this case it is easy to see that the generalized growth rate $\alpha(a)$ of the restriction a_{J_N} will be equal to the growth rate of this restriction. Lemma is proved.

As shown in [11], every growth rate of superlinear mapping is its eigenvalue. If $\sigma = (\alpha,\ (\overline{x},\ \overline{y}),\ \overline{p})$ is an equilibrium state, then

$$\Omega = \bigcup_{t=1}^{\infty} \alpha^{-t} a^t(\overline{x})$$

is an eigenset from $\pi\ a(\alpha)$. But, in general, the set Ω is unbounded. Let's prove that if there exists a Neumann equilibrium state, then the Neumann growth rate is an eigenvalue with a corresponding eigencompact.

Recall that, by Lemma 1, the subspace R^I is invariant with respect to a for $I = I_\infty(\Omega)$.

Lemma 2.5. *Let the superlinear mapping a have a Neumann equilibrium $(\alpha(a), (\overline{x}, \alpha(a)\,\overline{x}), p)$, and the eigenset $\Omega = \bigcup_{t=1}^{\infty} \alpha^{-t} a^t(\overline{x})$ be unbounded. Then the restriction a_I to the invariant subspace R^I for $I = I_\infty(\Omega)$ has a Neumann growth rate $\alpha(a)$.*

Proof. For definiteness assume that $\alpha(a) = 1$. Due to the homogeneity of superlinear mapping, this assumption does not limit generality of the proof. Then $\Omega = \bigcup_{t=1}^{\infty} a^t(\overline{x})$. It is easy to see that for $I = I_\infty(\Omega)$ we have $R_+^I \subset \Omega$.

Fix $y \in ri\ R_+^I \subset \Omega$. As R_+^I is a cone, we have $k^2 y \in ri\ R_+^I \subset \Omega$ for $k = 1,\ 2,\ \ldots$. For every k there exists the finite T_k -step trajectory

$$x_k = \left\{x_{t,\,k} \mid x_{0,\,k} = \overline{x},\ x_{T_k,\,k} = k^2 y,\ x_{t+1,\,k} \in a(x_{t,\,k})\ \ for\ \ t = 0,\ \ldots,\ T_k - 1\right\}.$$

Let

$$Z_k = \frac{1}{k} \sum_{t=0}^{T_k - 1} x_{t,\,k}\,, \qquad \tilde{Z}_k = \frac{1}{k} \sum_{t=1}^{T_k} x_{t,\,k}.$$

The following relation is true:

$$\tilde{Z}_k = \sum_{t=1}^{T_k} \frac{1}{k}\, x_{t,\, k} \in \sum_{t=0}^{T_k-1} \frac{1}{k}\, a\left(x_{t,\, k}\right) \subset a\, \left(\sum_{t=0}^{T_k-1} \frac{1}{k}\, x_{t,\, k}\right) = a\left(Z_k\right).$$

As Ω is convex and $x_{t,\, k} \in \Omega$, we have $Z_k,\ \tilde{Z}_k \in \Omega$, with $\left\|\tilde{Z}_k\right\| \geq \frac{1}{k}\ \|x_{T_k,\, k}\| = \|ky\|$. Consequently, $\left\|\tilde{Z}_k\right\| \to \infty$ as $k \to \infty$. Superlinear mapping is bounded, $\tilde{Z}_k \in a\left(Z_k\right)$, therefore $\|Z_k\| \to \infty$ as $k \to \infty$. Then

$$\lim_{k\to\infty} \frac{Z_k^i}{\left\|\tilde{Z}_k\right\|} = \lim_{k\to\infty} \frac{\tilde{Z}_k^i}{\left\|\tilde{Z}_k\right\|} = 0 \quad for \quad i \in I = I_\infty\left(\Omega\right).$$

It follows that every thickening point $\left(Z,\ \tilde{Z}\right)$ of the bounded sequence $\left(\frac{Z_k}{\|\tilde{Z}_k\|},\ \frac{\tilde{Z}_k}{\|\tilde{Z}_k\|}\right)$ lies in $R_+^I \times R_+^I$. Further, $\tilde{Z}_k - Z_k \geq ky - \frac{\overline{x}}{k}$. Then $\overline{Z} > Z$. The latter means $\alpha\left(a_I\right) \geq 1$. As $1 = \alpha\left(a\right) \geq \alpha\left(a_I\right)$, we have $\alpha\left(a_I\right) = \alpha(a) = 1$. Lemma is proved.

Lemma 2.6. *Neumann growth rate of superlinear mapping is an eigenvalue of this mapping. There exists an eigencompact which corresponds to this eigenvalue.*

Proof. Let's construct a finite sequence of restrictions $a_{J_1},\ \ldots,\ a_{J_N}$ to unequal embedded invariant subspaces $R^{J_1} \supset \cdots \supset R^{J_N}$. Let $J_1 = J$ and $a_{J_1} = a$. By Lemma 4, there exists an invariant subspace $R^{I_1} \subset R^{J_1}$ such that the restriction a_{I_1} has a unique generalized growth rate $\alpha\left(a_{I_1}\right)$. The restriction a_{I_1} is superlinear. By Lemma 5, there exists a subspace R^{I_2} invariant with respect to a_{I_1}, and the restriction a_{I_2} has a Neumann equilibrium state $\sigma_1 = \left(\alpha\left(a_{I_2}\right),\ \left(x_1,\ y_1\right),\ p_1\right)$ with $\alpha\left(a_{I_2}\right) = \alpha\left(a\right)$. As

$$a_{J_1}\left(R_+^{I_2}\right) = a_{I_1}\left(R_+^{I_2}\right) = a_2\left(R_+^{I_2}\right) = R_+^{I_2},$$

the subspace R^{I_2} is invariant also with respect to a_{J_1}.

As shown in [13], the growth rate $\alpha\left(a_{I_2}\right)$ is an eigenvalue of the mapping a_{I_2}, and the set

$$\Omega_1 = \bigcup_{t=1}^{\infty} \alpha^{-t}\left(a_{I_2}\right) \cdot \left(a_{I_2}^t\left(x_1\right)\right) \subset \pi\ \left(a_{I_2}\ \left(\alpha\left(a_{I_2}\right)\right)\right).$$

is its eigenset.

If $I_\infty\left(\Omega_1\right) = \emptyset$, then the closure $\overline{\Omega}_1$ is an eigencompact from $\pi\ \left(a_{I_2}\ \left(\alpha\left(a_{I_2}\right)\right)\right)$. Taking into account the invariance of R^{J_1}, R^{I_1}, R^{I_2}, and the equalities $\alpha(a) = \alpha\left(a_{J_1}\right) = \alpha\left(a_{I_1}\right) = \alpha\left(a_{I_2}\right)$, we obtain

$$a\left(\overline{\Omega}_1\right) = a_{J_1}\left(\overline{\Omega}_1\right) = a_{I_1}\left(\overline{\Omega}_1\right) = a_{I_2}\left(\overline{\Omega}_1\right) = \alpha\left(a_{I_2}\right)\ \overline{\Omega}_1 = \alpha(a)\ \overline{\Omega}_1.$$

So we have shown that $\alpha(a)$ is an eigenvalue of the mapping a, and the compact $\overline{\Omega} \subset \pi\ \left(a\ \left(\alpha(a)\right)\right)$ is its eigenset.

If $I_\infty\left(\Omega_1\right) \neq \emptyset$, we set $J_2 =\ I_\infty\left(\Omega_1\right)$. By Lemma 1, the subspace R^{I_2} is invariant with respect to a_{I_2}, and, consequently, also with respect to a. The restriction a_{J_2} of the mapping a to this subspace is superlinear. Besides, $\alpha\left(a_{J_2}\right) = \alpha\left(a_{I_2}\right) = \alpha\left(a\right)$. Let's treat a_{I_2} in the same way as we did with a_{J_1}. Then either we will find an eigencompact of the mapping a from $\pi\left(a\ \left(\alpha\left(a\right)\right)\right)$ or we will construct an invariant subspace $R^{J_3} \subset R^{J_2}$ and so on. As every eigenset is different from the face of the cone $\mathbb{R}_+^n$, we will always have $I_\infty\left(\Omega_i\right) \subset J_i$

and $I_\infty(\Omega_i) \neq J_i$. It follows that $N < n$, and this process is finite. At the last step we obtain an eigencompact $\overline{\Omega}_N$ of the mapping a which corresponds to the eigenvalue $\alpha(a)$. Lemma is proved.

It was proved in [11] that every growth rate is an eigenvalue. But it was also noted in [6] that there exist eigenvalues which differ from growth rates and the example was given where the eigenvalue exceeds the Neumann growth rate. The theorem below describes all eigenvalues of superlinear mapping.

Theorem 2.2. *Let $a : \mathbb{R}^n_+ \to \pi\left(\mathbb{R}^n_+\right)$ be a superlinear mapping, R^{I_1}, R^{I_3} be the subspaces invariant with respect to a, with $I_2 \cup I_3 = I_1$, $I_2 \cap I_3 = \emptyset$. Then $\alpha(b_{I_1,\ I_2})$ is an eigenvalue of the mapping a. There exists the corresponding eigenset Ω such that $I(\Omega) = I_1$, $I_\infty(\Omega) = I_3$. If $I_3 = \emptyset$, then Ω is an eigencompact.*

Proof. By Lemma 2, the mapping $b_{I_1,\,I_2}$ is superlinear. Then, by Lemma 6, the Neumann growth rate $\alpha(b_{I_1,\ I_2})$ is an eigenvalue of the mapping $b_{I_1,\,I_2}$ and there exists an eigencompact $Q \subset R^{I_2}_+$ which corresponds to $\alpha(b_{I_1,\ I_2})$.

If $I_3 = \emptyset$, then $I_1 = I_2$. Hence,

$$\alpha(b_{I_1,\ I_2})\,Q = b_{I_1,\,I_2}(Q) = \left\{\overline{P_{\mathrm{r}_{R^{I_2}\times R^{I_2}}} \cup a_{I_1}(x)}\ ,\ x \in R^{I_1}_+ \mid P_{\mathrm{r}_{R^{I_2}}} x \in R^{I_1}_+,\quad x \in Q\right\}$$

$$= \left\{\overline{\cup a_{I_1}(x)}\ \mid\ \ x \in Q\right\} = a_{I_1}(Q)\,.$$

Thus, $\alpha(b_{I_1,\ I_2})$ is an eigenvalue of the mapping a_{I_1}, and Q is a corresponding eigencompact. The subspace R^{I_1} is invariant with respect to a. Consequently,

$$a(Q) = a_{I_1}(Q) = \alpha(b_{I_1,\ I_2})\ Q.$$

So we have shown that $Q \subset \pi(a\ \ (\alpha(b_{I_1,\ I_2})))$ and $\alpha(b_{I_1,\ I_2})$ is an eigenvalue of the mapping a.

Now let's prove the theorem for $I_3 \neq \emptyset$. Using the set Q, we construct $\Omega \subset R^{I_1}_+$. Assume

$$\Omega = \left\{x \in R^{I_1}_+\ \mid P_{\mathrm{r}_{R^{I_2}_+}} x \in Q\right\}.$$

By the conditions of theorem, the subspaces R^{I_1}, R^{I_3} are invariant with respect to a and $I_3 \subset I_1$. Consequently,

$$a_{I_1}\left(R^{I_3}_+\right) = a\left(R^{I_3}_+\right) = R^{I_3}_+.$$

Therefore, R^{I_3} is invariant with respect to a_{I_1}. Taking into account $R^{I_3} \subset \Omega$, we obtain

$$R^{I_3}_+ = a_{I_1}\left(R^{I_3}_+\right) \subset a_{I_1}(\Omega)\,.$$

Then

$$\overline{a_{I_1}(\Omega)} = \left\{y \in R^{I_1}_+ \mid P_{\mathrm{r}_{R^{I_2}_+}} y \in P_{\mathrm{r}_{R^{I_2}_+}}\ a_{I_1}(\Omega)\right\}.$$

As $Q = P_{\mathrm{r}_{R^{I_2}_+}}\ Q$, we have

$$b_{I_1,\ I_2}(Q) = P_{\mathrm{r}_{R^{I_2}\times R^{I_2}}}(\Omega)$$

and therefore

$$\overline{a_{I_1}(\Omega)} = \left\{ y \in R_+^{I_1} \mid P_{\mathrm{r}_{R_+^{I_2}}} y \in b_{I_1,\ I_2}(\mathrm{Q}) \right\} = \left\{ y \in R_+^{I_1} \mid P_{\mathrm{r}_{R_+^{I_2}}} y \in \alpha\left(b_{I_1,\ I_2}\right) Q \right\}$$

$$= \alpha\left(b_{I_1,\ I_2}\right) \left\{ y \in R_+^{I_1} \mid P_{\mathrm{r}_{R_+^{I_2}}} y \in Q \right\} = \alpha\left(b_{I_1,\ I_2}\right) \overline{\Omega}.$$

The invariance of the subspace R^{I_1} implies

$$\overline{a(\Omega)} = \overline{a_{I_1}(\Omega)} = \alpha\left(b_{I_1,\ I_2}\right) \overline{\Omega}.$$

The set Ω does not coincide with the face of $\mathbb{R}_+^n$. Consequently, $\alpha\left(b_{I_1,\ I_2}\right)$ is an eigenvalue of the mapping a, and the set $\Omega \subset \pi\left(\alpha\left(b_{I_1,\ I_2}\right)\right)$ is its eigenset, with $I_1 = I(\Omega)$, $I_3 = I_\infty(\Omega)$. Theorem is proved.

References

1. Altum, I., Minak, G.: *Multivalued F-contractions on complete metric spaces*, J. Nonlinear Convex Anal. 16, 659-669 (2015).
2. Aubin, J.P.: Viability Theory, *Boston, Basel, Berlin; Birkhausued* (1991).
3. Baidosov, V.A.: *Mappings conjuqate to multivalued mappings of topological spaces, and their application tu dinamical games*, Math. USSR-Sb, 65 (2), 323-331 (1990).
4. Borisovich, Yu.G., Gelman, B.D., Mishkis, A.D., Obukhovskii V.V.: *Multivalued mappings*, J. Soviet Math. 24 (6), 719-791 (1984).
5. Ciric, L.: *Multivalued nonlinear contraction mappings*, Nonlinear Anal. 71, 2716-2223 (2009).
6. Conti, G., Nistri, P.: *A definition of asymptotic spectrum for multivalued maps in Banach spaces*, Atti Sem. Mat. Fis. Univ. Modena, XXVI, 1-14 (1977).
7. Dag, H., Minak, G., Altun, I.: *Some fixed point results for multivalued F-contractions on quasi metric spaces*, RACSAM 111, 177-187 (2017).
8. Feng, Y., Lin, S.: *Fixed point theorems for multivalued contractive mappings and multivalied cariste type mappings*, J. Math. Anal. Appl. 317, 103-112 (2006).
9. Gelman, B.D., Obukhovskii, V.V.: *On fixed points of acychi type multivalued maps.*, J. Math. Sci. 225 (4), 505-574 (2017).
10. Larina, Ya.Yu., Rodina, I.L.: Extension of the concept of invariance and statistically weakly invariant sets of controllable systems, *VINITI RAN, M.* (2017).
11. Makarov, V.L., Rubinov, A.M.: Mahematical theory of economic dynamics and equilibrium, *M. Nauka* (1973).
12. Panasenko, E.A., Tonkov, E.L.: *Invariant and stably invariant sets for differential inclusions*, Tr. Mat. Inst. Steklova 262, 202-221 (2008).
13. Protasov, V.Yn.: *On linear selections of Convex Set-Valued Maps*, Func. Anal. Appl. 45 (1), 46-55 (2011).
14. Rodina, L.I.: *Invariant and weakly invariant sets of controllable systems*, Izv. IMI ULGU. 2 (40), 3-164 (2012).
15. Rodina, I.L., Tonkov, E.L.: *Statistically weakly invariant sets of controllable systems*, Vestnik UGU 1, 67-86 (2011).