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Asymptotic behaviour of solutions of mixed problem for linear thermoelastic systems with microtemperatures and microstretch

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Abstract. *In this paper, we study the mixed problem with dissipative boundary conditions for linear thermoelastic system with microtemperatures. We investigate the correctness of the mixed problem and set the exponential decay of the energy norm of solutions.*

Keywords. Thermoelastic systems with microtemperatures, mixed problem, exponential decay rate.

Mathematics Subject Classification (2010): 35M10, 35M33, 35L20, 74A15

1 Introduction

In this paper, we study decay properties of solutions of the initial boundary-value problem associated with the following thermoelastic system

$$\begin{cases} u_{tt} - \mu^2 \Delta u + b\nabla\theta = 0, \\ \varphi_{tt} - \alpha^2 \Delta\varphi + \omega\nabla w = 0, \\ \theta_t - k\Delta\theta + \beta\nabla u_t + g\nabla w = 0, \\ w_t - \gamma\Delta w + h\nabla\varphi_t + m\nabla\theta = 0, \end{cases} \tag{1.1}$$

where μ, α, k, γ, are some positive constants, b, ω, β, g, h and m are some constants, $u = u(x,t)$, $\varphi = \varphi(x,t)$, $\theta = \theta(x,t)$, $w = w(x,t)$, $x = (x_1, x_2, \ldots, x_n) \in \Pi = [0,1]^n$, $\nabla = div$, $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

In the case $n = 1$, system (1.1) is a one-dimensional model of the theory of thermoelasticity, which is known in the literature as Eringen's theory of micromorphic continua. Eringen [3,4] introduced a class of micromorphic solids, called microstatic solids, simulating porous media filled with gas, non-viscous liquids, or composite materials with chopped elastic fibers. The material points of these materials can stretch and contract regardless of their movement and rotation. To date, there is an extensive literature devoted to the theory of micromorphic continua, in which deformation is described not only by the usual vector displacement field, but also by other vector or tensor fields (see [3-7, 9-12]). Within this theory, u represents the mixing stress, $\varphi-$ microstress, $\theta = T_a - T_0$ the absolute temperature difference, and w microtemperature, respectively.

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2 Formulation of the problem and the main result

Let

$$x_{r,i} = (x_1, ..., x_{i-1}, r, x_{i+1}, ..., x_n),\ \tilde{x}_i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n),$$
$$\Pi_i = \{\tilde{x}_i : \tilde{x}_i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n),\ 0 \le x_k \le 1,\ k = 1, \ldots, n,\ k \ne i\}.$$

For system (1.1), we will investigate the mixed problem with boundary and initial conditions

$$u(x_{0,\,i}, t) = \varphi(x_{0,\,i}, t) = 0, \tag{2.1}$$

$$\theta(x_{0,\,i}, t) = w(x_{0,\,i}, t) = \theta(x_{1,\,i}, t) = w(x_{1,\,i}, t), \tag{2.2}$$

$$\frac{\partial u(x_{1,\,i}, t)}{\partial x_i} + u_t(x_{1,\,i}, t) = 0, \tag{2.3}$$

$$\frac{\partial \varphi(x_{1,\,i}, t)}{\partial x_i} + \varphi_t(x_{1,\,i}, t) = 0, \tag{2.4}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{2.5}$$

$$\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \tag{2.6}$$

$$\theta(x, 0) = \theta_0(x), \quad w(x, 0) = w_0(x), \tag{2.7}$$

where $(\tilde{x}_i, t) \in \Pi_i \times [0, +\infty),\ i = 1,\, 2,\, \ldots, n,\ x \in \Pi.$

Assume that

$$\mu > 0,\ \alpha > 0,\ k > 0,\ \omega > 0, \tag{2.8}$$

and there are constants λ_0, λ_1 and λ_2 such that

$$\begin{cases} \lambda_i > 0,\ i = 0, 1, 2, \\ m = \lambda_0 g, \\ \lambda_0 \beta = \lambda_1 b, \\ \lambda_2 \omega = h. \end{cases} \tag{2.9}$$

We introduce the following notation:

$$L_2 = L_2(\Pi),\ W_2^1 = W_2^1(\Pi),\ W_2^2 = W_2^2(\Pi),$$
$$W_{2,\Gamma_1}^1 = \left\{u : u \in W_2^1(\Pi), u(x_{0,\,i}) = 0,\ \tilde{x}_i \in \Pi_i,\ i = 1, \ldots, n\right\},$$
$$W_{2,\Gamma}^1 = \left\{u : u \in W_2^1(\Pi),\ u(x_{0,\,i}) = u(x_{1,\,i}) = 0,\ \tilde{x}_i \in \Pi_i,\ i = 1, \ldots, n\right\}.$$

In the space $H = W_{2,\Gamma_1}^1 \times L_2 \times W_{2,\Gamma_1}^1 \times L_2 \times L_2 \times L_2$ we introduce the scalar product as follows:

$$\langle w, z\rangle_H = \lambda_1 \mu^2 \int\limits_\Pi \nabla v_1 \nabla z_1 dx + \lambda_1 \int_\Pi v_2 z_2 dx + \lambda_2 \alpha^2 \int_\Pi \nabla v_3 \nabla z_3 dx +$$

$$+ \lambda_2 \int\limits_\Pi v_4 z_4 dx + \lambda_0 \int\limits_\Pi v_5 z_5 dx + \int_\Pi v_6 z_6 dx,$$

where $w = (v_1, \ldots, v_6),\ z = (z_1, \ldots, z_6) \in H.$

Let H_0 be the space defined by

$$H_0 = \{w : w = (v_1, \ldots, v_6),\ v_1, v_3 \in W_2^2 \cap W_{2,\Gamma_1}^1,\ v_2, v_4 \in W_{2,\Gamma_1}^1,$$

$$v_5, v_6 \in W_2^2 \cap W_{2,\Gamma}^1,\ \frac{\partial v_1(x_{1,i})}{\partial x_i} + v_2(x_{1,i}) = 0,\ \frac{\partial v_3(x_{1,i})}{\partial x_i} + v_4(x_{1,i}) = 0,$$

$$\tilde{x}_i \in \Pi_i,\ i = 1, \ldots, n\}.$$

In the space H, we define a linear operator A by setting

$$D(A) = H_0,$$

$$Aw = \big(v_2, \mu^2\, \Delta v_1 - b\nabla v_5,\ \alpha^2 \Delta v_3 - \omega \nabla v_6,\ k\Delta v_5 - \beta \nabla v_2,$$
$$-g\nabla v_6, \gamma \Delta v_6 - h\nabla v_4 - m\nabla v_5).$$

Lemma 2.1 *A is a maximally dissipative operator in H.*

By Lemma 2.1, the operator generates a strongly continuous semigroup $U(t) = e^{tA}$. If $w_0 \in H$, then $w(t) = e^{tA}w_0$ is a weak solution to problem (1.1), (2.1)-(2.4).
Thus, the following theorems holds.

Theorem 2.1 *Let conditions (2.8), (2.9) be satisfied. Then for any $(u_0, u_1, \varphi_0, \varphi_1, \theta_0, w_0) \in H$ problem (1.1), (2.1) - (2.4) has a unique "weak" solution (u, φ, θ, w), where*

$$u, \varphi \in C\left([0,\infty); W_{2,\Gamma_1}^1\right) \cap C^1\left([0,\infty); L_2\right),\ \theta, w \in C\left([0,\infty); L_2\right).$$

Theorem 2.2 *Let conditions (2.8), (2.9) be satisfied. Then for any $(u_0, u_1, \varphi_0, \varphi_1, \theta_0, w_0) \in H_0$ problem (1.1), (2.1)-(2.7) has a unique solution (u, φ, θ, w), where*

$$u, \varphi \in C\left([0,\infty); W_2^2 \cap W_{2,\Gamma_1}^1\right) \cap C^1\left([0,\infty); W_{2,\Gamma_1}^1\right) \cap C^2\left([0,\infty); L_2\right),$$

$$\theta, w \in C\left([0,\infty); W_2^2 \cap W_{2,\Gamma}^1\right) \cap C^1\left([0,\infty); L_2\right).$$

Let us $E(t)$ denote the energy function

$$E(t) = \int_\Pi \Big[|u_t(x,t)|^2 + |\varphi_t(x,t)|^2 + |\nabla u(x,t)|^2$$
$$+ |\nabla \varphi(x,t)|^2 + |\theta(x,t)|^2 + |w(x,t)|^2\Big]\, dx. \tag{2.10}$$

In order to show an exponential decrease of energy, we use the following well-known result.

Lemma 2.2 (see [8]). *Let $t \to y(t) : [0,\infty) \to [0,\infty)$ is a monotonically decreasing function and there is a constant $c > 0$ such that*

$$\int_S^T y(t)dt \le cy(S),\ 0 \le S \le T. \tag{2.11}$$

Then there exist constants $\omega > 0$ and $M \ge 1$ such that

$$y(t) \le Me^{-\omega t}y(0),\ 0 \le t < +\infty. \tag{2.12}$$

Here the following main result is obtained on the behavior of the energy function at infinity.

Theorem 2.3 *Let conditions (2.8), (2.9) be satisfied. Then there are numbers $M \ge 1$ and $\sigma > 0$ such that for any $(u_0, u_1, \varphi_0, \varphi_1, \theta_0, w_0) \in H$ the following inequality holds:*

$$E(t) \le Me^{-\sigma t}E(0), \tag{2.13}$$

where

$$E(0) = \int_\Pi \Big[|u_1|^2 + |\varphi_1|^2 + |\nabla u_0|^2 + |\nabla \varphi_0|^2 + |\theta_0|^2 + |w_0|^2\Big]\, dx. \tag{2.14}$$

3 Proof of Lemma 2.1.

Let $w = (v_1, \ldots, v_6) \in D(A)$. Then we have

$$\langle Aw, w\rangle_H = \lambda_1\mu^2 \int_\Pi \nabla v_2 \cdot \nabla v_1 dx + \lambda_1 \int_\Pi \left[\mu^2 \Delta v_1 - b\nabla v_5\right] v_2 dx$$

$$+ \lambda_2\alpha^2 \int_\Pi \nabla v_4 \cdot \nabla v_3 dx + \lambda_2 \int_\Pi \left[\alpha^2 \Delta v_3 - \omega\nabla v_6\right] v_4 dx$$

$$+ \lambda_0 \int_\Pi \left[k\Delta v_5 - \beta\nabla v_2\right] v_2 dx + \int_\Pi \left[\gamma\Delta v_5 - h\nabla v_4 - m\nabla v_5\right] v_6 dx.$$

Integrating by parts and taking into account (1.1), (2.1)-(2.4), we obtain from this that

$$Re\,\langle Aw, w\rangle_H = -\lambda_0 k \int_\Pi |\nabla v_5|^2 dx - \gamma \int_\Pi |\nabla v_6|^2 dx$$

$$-\lambda_1\mu^2 \int_{\Pi_i} |v_2(x_{1,i})|^2 d\tilde{x}_i - \lambda_2\alpha^2 \int_{\Pi_i} |v_4(x_{1,i})|^2 d\tilde{x}_i \le 0. \tag{3.1}$$

Thus A dissipative operator.

Now we will show that A is an invertible operator.

Consider the equation

$$Aw = h, \tag{3.2}$$

where $h = (h_1, \ldots, h_6) \in H$. Eq. (3.2) is equivalent to the boundary-value problem for the system

$$\begin{cases} v_2(x) = h_1(x), \\ \mu^2\Delta v_1(x) - b\nabla v_5(x) = h_2(x), \\ v_4(x) = h_3(x), \\ \alpha^2\Delta v_3(x) - \omega\nabla v_6(x) = h_4(x), \\ k\Delta v_5(x) - \beta\nabla v_2(x) - g\nabla v_6(x) = h_5(x), \\ \gamma\Delta v_6(x) - h\nabla v_4(x) - m\nabla v_5(x) = h_6(x), \end{cases} \quad x \in \Pi, \tag{3.3}$$

with boundary conditions

$$v_1(x_{0,\,i}) = 0,\; v_2(x_{0,\,i}) = 0,\; v_3(x_{0,i}) = 0,\; v_4(x_{0,\,i}) = 0, \tag{3.4}$$

$$\frac{\partial v_1(x_{1,\,i})}{\partial x_i} + v_2(x_{1,\,i}) = 0, \tag{3.5}$$

$$\frac{\partial v_3(x_{1,\,i})}{\partial x_i} + v_4(x_{1,\,i}) = 0, \tag{3.6}$$

$$v_5(x_{0,\,i}) = v_5(x_{1,\,i}) = 0, \tag{3.7}$$

$$v_6(x_{0,\,i}) = v_6(x_{1,\,i}) = 0, \tag{3.8}$$

where $\tilde{x}_i \in \Pi_i,\; i = 1, \ldots, n$.

Setting $v_2 = h_1$ and $v_4 = h_3$ in the fifth and sixth equations in (3.3) and in the boundary conditions (3.4)-(3.8), we get the boundary-value problem

$$\begin{cases} \mu^2\Delta v_1(x) - b\nabla v_5(x) = h_2(x), \\ \alpha^2\Delta v_3(x) - \omega\nabla v_6(x) = h_4(x), \\ k\Delta v_5(x) - g\nabla v_6(x) = h_5(x) + \beta\nabla h_1(x), \\ \gamma\Delta v_6(x) - m\nabla v_5(x) = h_6(x) + h\nabla h_3(x). \end{cases} \tag{3.9}$$

$$v_1(x_{0,\,i}) = 0,\;\; v_3(x_{0,\,i}) = 0, \tag{3.10}$$

$$\frac{\partial v_1(x_{1,\,i})}{\partial x_i} = -\,h_1(x_{1,\,i}),\;\; \frac{\partial v_3(x_{1,\,i})}{\partial x_i} = -\,h_3(x_{1,\,i}), \tag{3.11}$$

$$v_5(x_{0,\,i}) = \; v_5(x_{1,\,i}) = 0, \tag{3.12}$$

$$v_6(x_{0,\,i}) = \; v_6(x_{1,\,i}) = 0, \tag{3.13}$$

where $\tilde{x}_i \in \Pi_i, i = 1,\,\ldots,n.$

First we solve the system

$$\begin{cases} k\Delta v_5(x) - g\nabla v_6(x) = h_5(x) + \beta\nabla h_1(x), \\ \gamma\Delta v_6(x) - m\nabla v_5(x) = h_6(x) + h\nabla h_3(x), \end{cases} \tag{3.14}$$

with boundary conditions (3.12), (3.13).

In the space $\mathcal{H} = W^1_{2,\Gamma} \times W^1_{2,\Gamma}$ consider the bilinear form

$$B\left[w^1, w^2\right] = \delta k \int_\Pi \nabla v_5^1(x)\nabla v_5^2(x)dx + \delta g \int_\Pi \nabla v_6^1(x)\nabla v_5^2(x)dx$$

$$+\gamma \int_\Pi \nabla v_6^1(x)\nabla v_6^2(x)dx + m \int_\Pi \nabla v_5^1(x)\nabla v_6^2(x)dx,$$

where

$$w^1 = \begin{pmatrix} v_5^1 \\ v_6^1 \end{pmatrix},\; w^2 = \begin{pmatrix} v_5^2 \\ v_6^2 \end{pmatrix},\; \delta = \frac{m}{g}.$$

Using the Hölder and Poincaré inequalities, we obtain

$$\left|B\left[w^1, w^2\right]\right| \le \delta k \left|\int_\Pi \nabla v_5^1(x)\nabla v_5^2(x)dx\right| + \delta g \left|\int_\Pi \nabla v_6^1(x)\nabla v_5^2(x)dx\right|$$

$$+\gamma \left|\int\limits_\Pi \nabla v_6^1(x)\nabla v_6^2(x)dx\right| + m \left|\int_\Pi \nabla v_5^1(x)\nabla v_6^2(x)dx\right|$$

$$\le \frac{1}{2}(\delta k + m) \int_\Pi \left|\nabla v_5^1(x)\right|^2 dx + \frac{\delta}{2}(k + c_0 g) \int_\Pi \left|\nabla v_5^2(x)\right|^2 dx$$

$$+\frac{1}{2}(\delta g + \gamma) \int\limits_\Pi \left|\nabla v_6^1(x)\right|^2 dx + \frac{1}{2}(\gamma + m c_0) \int\limits_\Pi \left|\nabla v_6^2(x)\right|^2 dx$$

$$\le M_0 \left[\left\|v_5^1\right\|^2_{W^1_{2,\Gamma}} + \left\|v_5^2\right\|^2_{W^1_{2,\Gamma}} + \left\|v_6^1\right\|^2_{W^1_{2,\Gamma}} + \left\|v_6^2\right\|^2_{W^1_{2,\Gamma}}\right]$$

$$= M_0 \left[\left\|w^1\right\|^2_{\mathcal{H}} + \left\|w^2\right\|^2_{\mathcal{H}}\right], \tag{3.15}$$

where

$c_0 = \inf\limits_{z \in W^1_{2,\Gamma}} \frac{\int_\Pi |z(x)|^2 dx}{\int_\Pi |\nabla z(x)|^2 dx}$, $M_0 = \max\left\{\frac{1}{2}(\delta k + m), \frac{\delta}{2}(k + c_0 g), \frac{1}{2}(\delta g + \gamma), \frac{1}{2}(\gamma + m c_0)\right\}.$

For $w = \begin{pmatrix} v_5 \\ v_6 \end{pmatrix} \in \mathcal{H}$ we have

$$B\left[w, w\right] = \delta k \int_\Pi \left|\nabla v_5(x)\right|^2 dx + \delta g \int_\Pi \nabla v_5(x) v_6(x)dx$$

$$+\gamma\int\limits_{\Pi}|\nabla v_6(x)|^2\,dx+m\int_{\Pi}\nabla v_6(x)v_5(x)dx$$

$$=\delta k\int_{\Pi}|\nabla v_5(x)|^2\,dx+\gamma\int_{\Pi}|\nabla v_6(x)|^2\,dx\geq m_0\|w\|^2_{\mathcal{H}}. \tag{3.16}$$

It follows from the definition of H and (3.2) that

$$h_5(x)+\beta\nabla h_1(x)\in L_2(\Pi),\ \ h_6(x)+h\nabla h_3(x)\in L_2(\Pi). \tag{3.17}$$

Relations (3.15)-(3.17) show that all conditions of the Lax-Milgram theorem are satisfied. Therefore, there are unique functions satisfying (3.12)-(3.17) in the weak sense.

By (3.14) we get

$$\Delta v_5(x)=\frac{g}{k}\nabla v_6(x)+\frac{1}{k}h_5(x)+\frac{\beta}{k}\nabla h_1(x)\in L_2(\Pi),$$

and consequently, $v_5(x)\in W_2^2\bigcap W^1_{2,\Gamma}$. In a similar way, we obtain that $v_6(x)\in W_2^2\bigcap W^1_{2,\Gamma}$.

Setting $v_5(x)$ and $v_6(x)$ in the first two equations in system (3.9), we obtain a boundary-value problem for systems with respect to the functions $v_1(x)$ and $v_3(x)$ with inhomogeneous boundary conditions.

The resulting problem is solved by the standard method. The proof of this lemma is complete.

4 Proof Theorem 2.3.

We define the following functional

$$E_0(t)=\int_{\Pi}\left[\frac{\lambda_1}{2}|u_t(x,t)|^2+\frac{\lambda_2}{2}|\varphi_t(x,t)|^2+\frac{\mu^2\lambda}{2}|\nabla u(x,t)|^2\right.$$

$$\left.+\frac{\alpha^2\lambda_2}{2}|\nabla\varphi(x,t)|^2+\frac{\lambda_0}{2}|\theta(x,t)|^2+\frac{1}{2}|w(x,t)|^2\right]dx. \tag{4.1}$$

It is easy to see that there are $c_1>0$ and $c_2>0$ such that

$$c_1E(t)\leq E_0(t)\leq c_2E(t). \tag{4.2}$$

Considering (1.1), (2.1)-(2.4) from (4.1) we get

$$\frac{d}{dt}E_0(t)=-k\lambda_0\sum_{i=1}^{n}\int_{\Pi}\theta^2_{x_i}dx-\gamma\sum_{i=1}^{n}\int_{\Pi}w^2_{x_i}dx$$

$$-\lambda_1\mu^2\sum_{i=1}^{n}\int_{\Pi_i}|u_t(x_{1,i},t)|^2\,d\tilde{x}_i-\lambda_2\alpha^2\sum_{i=1}^{n}\int_{\Pi_i}|\varphi_t(x_{1,i},t)|^2\,d\tilde{x}_i\leq 0, \tag{4.3}$$

whence implies that

$$E_0(t)\leq E_0(0),\quad t\geq 0. \tag{4.4}$$

The following statements directly follow from the inequalities (4.3) and (4.4).

Lemma 4.1 *For weak solutions of problem* (1.1)*,* (2.1)-(2.7) *the following estimates hold:*

(i) $\sum\limits_{i=1}^{n}\int_{\Pi}\theta_{x_i}^2dx\le-\frac{1}{k\lambda_0}\frac{d}{dt}E_0(t),\ t\ge0$;

(ii) $\sum\limits_{i=1}^{n}\int_S^T\int_{\Pi}\theta_{x_i}^2dxdt\le\frac{1}{k\lambda_0}E_0(S),\ 0\le S\le T$;

(iii) $\sum\limits_{i=1}^{n}\int\limits_{\Pi}w_{x_i}^2dx\le\frac{1}{\gamma}\frac{d}{dt}E_0(t),\ t\ge0$;

(iv) $\sum\limits_{i=1}^{n}\int_S^T\int\limits_{\Pi}w_{x_i}^2dxdt\le\frac{1}{\gamma}E_0(S),\ \ 0\le S\le T$;

(v) $\sum\limits_{i=1}^{n}\int_{\Pi_i}\left|u_t(x_{1,i},t)\right|^2d\tilde{x}_i\le-\frac{1}{\lambda_1\mu^2}\frac{d}{dt}E_0(t),\ \ t\ge0$;

(vi) $\sum\limits_{i=1}^{n}\int_{\Pi_i}\left|u_{x_i}(x_{1,i},t)\right|^2d\tilde{x}_i\le-\frac{1}{\lambda_1\mu^2}\frac{d}{dt}E_0(t),\ \ t\ge0$;

(vii) $\sum\limits_{i=1}^{n}\int\limits_S^T\int_{\Pi_i}\left|u_t(x_{1,i},t)\right|^2d\tilde{x}_idt\le-\frac{1}{\lambda_1\mu^2}E_0(S),\ 0\le S\le T$;

(viii) $\sum\limits_{i=1}^{n}\int\limits_S^T\int_{\Pi_i}\left|u_{x_i}(x_{1,i},t)\right|^2d\tilde{x}_idt\le\frac{1}{\lambda_1\mu^2}E_0(S),\ 0\le S\le T$;

(ix) $\sum\limits_{i=1}^{n}\int_{\Pi_i}\left|\varphi_t(x_{1,i},t)\right|^2d\tilde{x}_i\le-\frac{1}{\lambda_2\alpha^2}\frac{d}{dt}E_0(t),\ t\ge0$;

(x) $\sum\limits_{i=1}^{n}\int_{\Pi_i}\left|\varphi_{x_i}(x_{1,i},t)\right|^2d\tilde{x}_i\le-\frac{1}{\lambda_2\alpha^2}\frac{d}{dt}E_0(t),\ t\ge0$;

(xi) $\sum\limits_{i=1}^{n}\int\limits_S^T\int_{\Pi_i}\left|\varphi_t(x_{1,i},t)\right|^2d\tilde{x}_idt\le\frac{1}{\lambda_2\alpha^2}E_0(S),\ 0\le S\le T$;

(xii) $\sum\limits_{i=1}^{n}\int_S^T\int_{\Pi_i}\left|\varphi_{x_i}(x_{1,i},t)\right|^2d\tilde{x}_idt\le\frac{1}{\lambda_2\alpha^2}E_0(S),\ 0\le S\le T.$

Multiplying the first equation of system (1.1) by $u_t+\eta_0u+\eta\sum\limits_{i=1}^{n}x_iu_{x_i}$ and integrating the obtained relation over the domain $[S,T]\times\Omega$ we obtain

$$0=\int_S^T\int_{\Pi}\left(u_t(x,t)+\eta_0u(x,t)+\eta\sum_{i=1}^{n}x_iu_{x_i}(x,t)\right)$$

$$\times\left(u_{tt}(x,t)-\mu^2\Delta u(x,t)+b\nabla\theta(x,t)\right)dxdt$$

$$=F(T)-F(S)+\int_S^T\Phi(t)dt+\int_S^T\Psi(t)dt+\int_S^T\Theta(t)dt,\tag{4.5}$$

where

$$F(t)=\frac{1}{2}\int\limits_{\Pi}u_t^2(x,t)dx+\frac{\mu^2}{2}\sum_{i=1}^{n}\int_{\Pi}u_{x_i}^2(x,t)dx+\eta\sum_{i=1}^{n}\int_{\Pi}x_iu_t(x,t)u_{x_i}(x,t)dx$$

$$+\eta_0\int_{\Pi}u_t(x,t)u(x,t)dx+\frac{\eta_0\mu^2}{2}\sum_{i=1}^{n}\int_{\Pi_i}u^2(x_{1,t},t)d\tilde{x}_i,\tag{4.6}$$

$$\Phi(t) = \frac{\eta n}{2}\int_{\Pi} u_t^2(x,t)dx + \frac{\eta}{2}\sum_{i=1}^{n}\int_{\Pi} u_{x_i}^2(x,t)dx - \frac{\eta}{2}\sum_{i=1}^{n}\sum_{\substack{j=1,\\ j\neq i}}^{n}\int_{\Pi} u_{x_j}^2(x,t)dx$$

$$-\eta_0\int_{\Pi} u_t^2(x,t)dx + \eta_0\mu^2\sum_{i=1}^{n}\int_{\Pi} u_{x_i}^2(x,t)dx, \tag{4.7}$$

$$\Psi(t) = -\sum_{i=1}^{n}\int_{\Pi_i} u_t(x_{1,i},t)u_{x_i}(x_{1,i},t)d\tilde{x}_i - \frac{\eta}{2}\sum_{i=1}^{n}\int_{\Pi_i} u_t^2(x_{1,i},t)d\tilde{x}_i$$

$$-\frac{\eta}{2}\sum_{i=1}^{n}\int_{\Pi_i} u_{x_i}^2(x_{1,i},t)d\tilde{x}_i + \frac{\eta\mu^2}{2}\sum_{i=1}^{n}\sum_{\substack{j=1,\\ j\neq i}}^{n}\int_{\Pi_i} u_{x_j}^2(x,t)d\tilde{x}_i$$

$$-\eta\mu^2\sum_{i=1}^{n}\sum_{\substack{j=1,\\ j\neq i}}^{n}\int_{\Pi_j} x_i u_{x_i}(x_{1,j},t)u_{x_j}(x_{1,i},t)d\tilde{x}_i, \tag{4.8}$$

$$\Theta(t) = b\sum_{i=1}^{n}\int_{\Pi} u_t(x,t)\theta_{x_i}(x,t)dx + \eta_0\sum_{i=1}^{n}\int_{\Pi} u(x,t)\theta_{x_i}(x,t)dx$$

$$+\eta\sum_{i=1}^{n}\sum_{\substack{j=1,\\ j\neq i}}^{n}\int_{\Pi} x_i u_{x_i}(x,t)\theta_{x_i}(x,t)dx. \tag{4.9}$$

It follows from (4.5) that

$$\int_S^T \Phi(t)dt = F(S) - F(T) - \int_S^T \Psi(t)dt - \int_S^T \Theta(t)dt. \tag{4.10}$$

The parameters $\eta,\ \eta_0$ can be chosen so that the following inequalities are satisfied

$$\eta_0\mu^2 - \frac{\eta(n-2)}{2} > 0,$$

$$\frac{\eta n}{2} - \frac{\eta_0}{2} > 0,$$

$$1 - \eta - \frac{\eta\mu^2 n}{2} > 0.$$

Then it is easy to see that the following inequalities hold:

$$c_1F_0(t) \le F(t) \le c_2F_0(t), \tag{4.11}$$

$$\Phi(t) \ge c_3F_0(t), \tag{4.12}$$

$$\Psi(t) \ge c_4\sum_{i=1}^{n}\int_{\Pi_i} u_{x_i}^2(x_{1,i},t)d\tilde{x}_i, \tag{4.13}$$

where

$$F_0(t) = \int_{\Pi} u_t^2(x,t)dx + \sum_{i=1}^{n} \int_{\Pi} u_{x_i}^2(x,t)dx, \tag{4.14}$$

$c_1, \ldots, c_4$, are some positive constants.

Using the Hölder's and Young's inequalities, we obtain that for any $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$|\Theta(t)| \le \varepsilon F_0(t) + c(\varepsilon) \sum_{i=1}^{n} \int_{\Pi} |\Theta_{x_i}(x,t)|^2 \, dx. \tag{4.15}$$

In view of (4.10)-(4.14) we get

$$c_3 \int_S^T F_0(t)dt \le c_2 E_0(S) - \int_S^T \Theta(t)dt.$$

Hence, by irelation (4.15), we obtain

$$(c_3 - \varepsilon) \int_S^T F_0(t)dt \le c_2 F_0(S)$$

$$+c(\varepsilon) \sum_{i=1}^{n} \int_{\Pi} |\Theta_{x_i}(x,t)|^2 \, dx \le c_5 F_0(S), \quad 0 \le S \le T. \tag{4.16}$$

Next, we multiply the second equation of system (1.1) by $\varphi_t + \mu_0 \varphi + \mu \sum\limits_{i=1}^{n} x_i \varphi_{x_i}$ and integrate resulting relation over the region $[S,T] \times \Omega$. By following the above arguments we get

$$\int_S^T F_1(t)dt \le c_6 E_0(S), \; 0 \le S \le T < +\infty, \tag{4.17}$$

where $F_1(S) = \int\limits_{\Pi} \varphi_t^2(x,t)dx + \sum\limits_{i=1}^{n} \int_{\Pi} \varphi_{x_i}^2(x,t)dx$.

It follows from Lemma 4.1 and estimates (4.10), (4.17) that

$$\int_S^T E_0(t)dt \le c_7 E_0(S), \;\; 0 \le S \le T < +\infty.$$

By Lemma 2.2, there exist $M \ge 1$ and $\omega > 0$ such that $E_0(t) \le M e^{-\omega t} E_0(0)$, $t \ge 0$. Then by (4.2) from this we get (2.13). The proof of this theorem is complete.

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