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Vertical rescaled berger deformation metric on the tangent bundle

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Abstract. *In this paper, we introduce the vertical rescaled berger deformation metric on the tangent bundle TM over an anti-paraKähler manifold (M^{2m}, φ, g) as a new natural metric with respect to g non-rigid on TM. Firstly, we investigate the Levi-Civita connection of this metric, Secondly we study the curvature tensor and also we characterize the scalar curvature.*

Keywords. Horizontal lift, vertical lift, cotangent bundles, vertical rescaled berger deformation metric , curvature tensor.

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1 Introduction

The tangent bundle equipped with the Sasaki metric has been studied by many authors such as Sasaki, S. [10], Yano, K., Ishihara S. [14], Dombrowski, P. [4], Salimov A., Gezer A. [7]. The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on TM. Musso, E., Tricerri, F. has introced the notion of Cheeger-Gromoll metric [6], this metric has been studied also by many authors (see [5,9,11]). There are other metrics on the tangent bundle studied by [3,15,16].

The main idea in this note consists in the deformation (in the vertical bundle) of the Berger type deformed Sasaki metric on the tangent bundle [2]. Firstly, we introduce the vertical rescaled berger deformation metric on the tangent bundle TM over an anti-paraKähler manifold (M^{2m}, φ, g). Secondly, we investigate the Levi-Civita connection (Theorem 3.1 and Proposition 3.1) and we establish the curvature tensor (Theorem 4.1). Finally we characterize the sectional curvature (Theorem 4.2 and Proposition 4.2) and the scalar curvature (Theorem 4.3 and Proposition 4.3).

2 Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,m}}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and $\Im^1_0(M)$ be the module over $C^\infty(M)$ of C^∞ vector fields on M.

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The Levi Civita connection ∇ defines a direct sum decomposition

$$T_{(x,u)}TM = V_{(x,u)}TM \oplus H_{(x,u)}TM. \tag{2.1}$$

of the tangent bundle to TM at any $(x,u) \in TM$ into vertical subspace

$$V_{(x,u)}TM = Ker(d\pi_{(x,u)}) = \{\xi^i \frac{\partial}{\partial y^i}|_{(x,u)},\ \xi^i \in \mathbb{R}\}, \tag{2.2}$$

and the horizontal subspace

$$H_{(x,u)}TM = \{\xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)},\ \xi^i \in \mathbb{R}\}. \tag{2.3}$$

.

Note that the map $X \to X^H$ is an isomorphism between the vector spaces T_xM and $H_{(x,u)}TM$. Similarly, the map $X \to X^V$ is an isomorphism between the vector spaces T_xM and $V_{(x,u)}TM$. Obviously, each tangent vector $Z \in T_{(x,u)}TM$ can be written in the form $Z = X^H + Y^V$, where $X, Y \in T_xM$ are uniquely determined vectors.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{2.4}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i\{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\}. \tag{2.5}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=\overline{1,m}}$ is a local adapted frame on TTM.

In particular if U be a local vector field constant on each fiber T_xM such that ($U = u$), the vertical lift U^V is called the canonical vertical vector field or Liouville vector field on TM.

Lemma 2.1 *[4, 14] Let (M, g) be a Riemannian manifold. The bracket operation of vertical and horizontal vector fields is given by the formulas*

1 $[X^H, Y^H] = [X, Y]^H - (R(X, Y)u)^V$,
2 $[X^H, Y^V] = (\nabla_X Y)^V$,
3 $[X^V, Y^V] = 0$,

for all vector fields $X, Y \in \Im^1_0(M)$, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M, g).

3 Vertical rescaled berger deformation metric

Let M be a $2m$-dimensional Riemannian manifold with a Riemannian metric g. An almost paracomplex manifold is an almost product manifold (M^{2m}, φ), $\varphi^2 = id$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank.

A Riemannian metric g is said to be an anti-paraHermitian metric if

$$g(\varphi X, \varphi Y) = g(X, Y), \tag{3.1}$$

or equivalently (purity condition), (B-metric)[8]

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{3.2}$$

for all $X, Y \in \Im_0^1(M)$.

If (M^{2m}, φ) is an almost paracomplex manifold with an anti-paraHermitian metric g, then the triple (M^{2m}, φ, g) is said to be an almost anti-paraHermitian manifold (an almost B-manifold)[8]. Moreover, (M^{2m}, φ, g) is said to be anti-paraKähler manifold (B-manifold)[8] if φ is parallel with respect to the Levi-Civita connection ∇ of g i.e. $(\nabla\varphi = 0)$.

As is well known, the anti-paraKähler condition $(\nabla\varphi = 0)$ is equivalent to paraholomorphicity of the anti-paraHermitian metric g, that is, $(\phi_\varphi g) = 0$, where ϕ_φ is the Tachibana operator [13].

It is well known that if (M^{2m}, φ, g) is a anti-paraKähler manifold, the Riemannian curvature tensor is pure [8], and

$$\begin{cases} R(\varphi Y, Z) \; = R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) = R(Y, Z), \end{cases} \tag{3.3}$$

for all $Y, Z \in \Im_0^1(M)$.

Definition 3.1 *Let (M^{2m}, φ, g) be an almost anti-paraHermitian manifold and f be a strictly positive smooth function on M. Define a vertical rescaled berger deformation metric noted $\tilde{g}$ on TM, by*

$$\begin{aligned} &\tilde{g}(X^H, Y^H) = g(X, Y), \\ &\tilde{g}(X^H, Y^V) = 0, \\ &\tilde{g}(X^V, Y^V) = f\big[g(X, Y) + \delta^2 g(X, \varphi u) g(Y, \varphi u)\big], \end{aligned}$$

for all $X, Y \in \Im_0^1(M)$, where δ is some constant [2, 12] and f is called twisting function.

Remark 3.1 1. If $f = 1$ and $\delta = 0$, $\tilde{g}$ is the Sasaki metric[10],
2. If $f = 1$, $\tilde{g}$ is the Berger-type deformed Sasaki metric[2],
3. If $\delta = 0$, $\tilde{g}$ is the vertical rescaled metric [3],
4. $\tilde{g}(X^V, \varphi U^V) = (1 + \delta^2 r^2) f g(X, \varphi u)$ and $r^2 = g(u, u)$, for any $X \in \Im_0^1(M)$.

In the following, we consider $\lambda = 1 + \delta^2 r^2$ and $r^2 = g(u, u) = \|u\|^2$, where $\|.\|$ denote the norm with respect to (M, g).

Lemma 3.1 *[1] Let (M, g) be a Riemannian manifold and $\rho : \mathbb{R} \to \mathbb{R}$ a smooth function. Then we have*

1 $X^H(\rho(r^2)) = 0$,
2 $X^V(\rho(r^2)) = 2\rho'(r^2) g(X, u)$,
3 $X^H g(Y, u) = g(\nabla_X Y, u)$,
4 $X^V g(Y, u) = g(X, Y)$,

for any $X, Y \in \Im_0^1(M)$.

Lemma 3.2 *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, we have the following:*

1 $X^H g(Y, \varphi u) = g(\nabla_X Y, \varphi u)$,
2 $X^V g(Y, \varphi u) = g(Y, \varphi X)$,

for all $X, Y \in \Im_0^1(M)$.

Proof. The results come immediately from Lemma 3.1.

Lemma 3.3 *Let* (M^{2m}, φ, g) *be an anti-paraKähler manifold, we have the following:*

$$\begin{aligned}
&(1)\ X^H\tilde{g}(Y^H, Z^H) = Xg(Y,Z),\\
&(2)\ X^V\tilde{g}(Y^H, Z^H) = 0,\\
&(3)\ X^H\tilde{g}(Y^V, Z^V) = \frac{1}{f}X(f)\tilde{g}(Y^V, Z^V) + \tilde{g}((\nabla_X Y)^V, Z^V) + \tilde{g}(Y^V, (\nabla_X Z)^V),\\
&(4)\ X^V\tilde{g}(Y^V, Z^V) = \delta^2 f\big[g(X,\varphi Y)g(Z,\varphi u) + g(Y,\varphi u)g(X,\varphi Z)\big],
\end{aligned}$$

where $X, Y, Z \in \Im^1_0(M)$.

Proof. The results comes directly from Lemma 3.1 and Lemma 3.2

We shall calculate the Levi-Civita connection $\widetilde{\nabla}$ of TM with vertical rescaled berger deformation metric $\tilde{g}$. This connection is characterized by the Koszul formula:

$$\begin{aligned}
2\tilde{g}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}) &= \widetilde{X}\tilde{g}(\widetilde{Y}, \widetilde{Z}) + \widetilde{Y}\tilde{g}(\widetilde{Z}, \widetilde{X}) - \widetilde{Z}\tilde{g}(\widetilde{X}, \widetilde{Y})\\
&\quad + \tilde{g}(\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]) + \tilde{g}(\widetilde{Y}, [\widetilde{Z}, \widetilde{X}]) - \tilde{g}(\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]).
\end{aligned} \tag{3.4}$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im^1_0(TM)$.

Theorem 3.1 *Let* (M^{2m}, φ, g) *be an anti-paraKähler manifold and* $(TM, \tilde{g})$ *its tangent bundle equipped with the vertical rescaled berger deformation metric , then we have the following formulas.*

$$\begin{aligned}
&1.\ \widetilde{\nabla}_{X^H}Y^H = (\nabla_X Y)^H - \frac{1}{2}(R(X,Y)u)^V,\\
&2.\ \widetilde{\nabla}_{X^H}Y^V = (\nabla_X Y)^V + \frac{1}{2f}X(f)Y^V + \frac{f}{2}(R(u,Y)X)^H,\\
&3.\ \widetilde{\nabla}_{X^V}Y^H = \frac{1}{2f}Y(f)X^V + \frac{f}{2}(R(u,X)Y)^H,\\
&4.\ \widetilde{\nabla}_{X^V}Y^V = \frac{-1}{2f}\tilde{g}(X^V, Y^V)(grad\, f)^H + \frac{\delta^2}{\lambda}g(X,\varphi Y)(\varphi U)^V,
\end{aligned}$$

for all vector fields $X, Y \in \Im^1_0(M)$*, where* ∇ *and* R *denotes respectively the Levi-Civita connection and the curvature tensor of* (M^{2m}, φ, g).

Proof. The proof of Theorem 3.1 follows directly from Kozul formula (3.4), Lemma 2.1, Lemma 3.2 and Lemma 3.3.
(1) Direct calculations give,

$$\begin{aligned}
(i)\ 2\tilde{g}(\widetilde{\nabla}_{X^H}Y^H, Z^H) &= X^H\tilde{g}(Y^H, Z^H) + Y^H\tilde{g}(Z^H, X^H) - Z^H\tilde{g}(X^H, Y^H)\\
&\quad + \tilde{g}(Z^H, [X^H, Y^H]) + \tilde{g}(Y^H, [Z^H, X^H]) - \tilde{g}(X^H, [Y^H, Z^H])\\
&= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z, [X,Y])\\
&\quad + g(Y, [Z,X]) - g(X, [Y,Z])\\
&= 2g(\nabla_X Y, Z)\\
&= 2\tilde{g}((\nabla_X Y)^H, Z^H),
\end{aligned}$$

and

$$\begin{aligned}
(ii)\ 2\tilde{g}(\widetilde{\nabla}_{X^H}Y^H, Z^V) &= X^H\tilde{g}(Y^H, Z^V) + Y^H\tilde{g}(Z^V, X^H) - Z^V\tilde{g}(X^H, Y^H) \\
&\quad + \tilde{g}(Z^V, [X^H, Y^H]) + \tilde{g}(Y^H, [Z^V, X^H]) - \tilde{g}(X^H, [Y^H, Z^V]) \\
&= \tilde{g}(Z^V, [X^H, Y^H]) \\
&= -\tilde{g}((R(X,Y)u)^V, Z^V).
\end{aligned}$$

we have:

$$\widetilde{\nabla}_{X^H}Y^H = (\nabla_X Y)^H - \frac{1}{2}(R(X,Y)u)^V.$$

(2) By straightforward calculations,

$$\begin{aligned}
(iii)\ 2\tilde{g}(\widetilde{\nabla}_{X^H}Y^V, Z^H) &= X^H\tilde{g}(Y^V, Z^H) + Y^V\tilde{g}(Z^H, X^H) - Z^H\tilde{g}(X^H, Y^V) \\
&\quad + \tilde{g}(Z^H, [X^H, Y^V]) + \tilde{g}(Y^V, [Z^H, X^H]) - \tilde{g}(X^H, [Y^V, Z^H]) \\
&= \tilde{g}(Y^V, [Z^H, X^H]) \\
&= -\tilde{g}((R(Z,X)u)^V, Y^V) \\
&= -f[g(R(Z,X)u, Y) + \delta^2 g(Y, \varphi u)g(R(Z,X)u, \varphi u)] \\
&= f\tilde{g}((R(u,Y)X)^H, Z^H).
\end{aligned}$$

Where

$$-g(R(Z,X)u, Y) = g(R(u,Y)X, Z) = \tilde{g}((R(u,Y)X)^H, Z^H),$$

and from (3.3) we have

$$g(R(Z,X)u, \varphi u) = g(\varphi R(Z,X)u, u) = g(R(\varphi Z, X)u, u) = 0.$$

It follows from.

$$\begin{aligned}
(iv)\ 2\tilde{g}(\widetilde{\nabla}_{X^H}Y^V, Z^V) &= X^H\tilde{g}(Y^V, Z^V) + Y^V\tilde{g}(Z^V, X^H) - Z^V\tilde{g}(X^H, Y^V) \\
&\quad + \tilde{g}(Z^V, [X^H, Y^V]) + \tilde{g}(Y^V, [Z^V, X^H]) - \tilde{g}(X^H, [Y^V, Z^V]) \\
&= X^H\tilde{g}(Y^V, Z^V) + \tilde{g}(Z^V, [X^H, Y^V]) + \tilde{g}(Y^V, [Z^V, X^H]) \\
&= \frac{1}{f}X(f)\tilde{g}(Y^V, Z^V) + \tilde{g}((\nabla_X Y)^V, Z^V) + \tilde{g}(Y^V, (\nabla_X Z)^V) \\
&\quad + \tilde{g}(Z^V, (\nabla_X Y)^V) - \tilde{g}(Y^V, (\nabla_X Z)^V) \\
&= 2\tilde{g}((\nabla_X Y)^V, Z^V) + \frac{1}{f}X(f)\tilde{g}(Y^V, Z^V) \\
&= 2\tilde{g}((\nabla_X Y)^V + \frac{1}{2f}X(f)Y^V, Z^V).
\end{aligned}$$

we have:

$$\widetilde{\nabla}_{X^H}Y^V = (\nabla_X Y)^V + \frac{1}{2f}X(f)Y^V + \frac{f}{2}(R(u,Y)X)^H.$$

The other formulas are obtained by a similar calculation.

Lemma 3.4 *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\tilde{g})$ its tangent bundle equipped with the vertical rescaled berger deformation metric, then we have*

1. $\widetilde{\nabla}_{X^H}(\varphi U)^V = \dfrac{1}{2f}X(f)(\varphi U)^V,$
2. $\widetilde{\nabla}_{X^V}(\varphi U)^V = (\varphi X)^V - \dfrac{\lambda}{2}g(X,\varphi u)(grad\, f)^H + \dfrac{\delta^2}{\lambda}g(X,u)(\varphi U)^V,$

for all vector fields $X \in \Im^1_0(M)$.

Definition 3.2 *Let (M,g) be a Riemannian manifold and and $F : TM \to TM$ be a smooth bundle endomorphism of TM. Then the vertical and horizontal vector fields F^V and F^H respectively are defined on TM by*

$$\begin{aligned} F^V :\ & TM \to TTM \\ & (x,u) \mapsto (F_x(u))^V, \\ F^H :\ & TM \to TTM \\ & (x,u) \mapsto (F_x(u))^H, \end{aligned}$$

Locally we have

$$F^V = y^i F^j_i \frac{\partial}{\partial y^j} = y^i (F(\frac{\partial}{\partial x^i}))^V, \tag{3.5}$$

$$F^H = y^i F^j_i \frac{\partial}{\partial x^j} - y^i y^k F^j_i \Gamma^s_{jk} \frac{\partial}{\partial y^s} = y^i (F(\frac{\partial}{\partial x^i}))^H. \tag{3.6}$$

Proposition 3.1 *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\tilde{g})$ its tangent bundle equipped with the vertical rescaled berger deformation metric, then we have the following formulas*

1. $\widetilde{\nabla}_{X^H}F^H = (\nabla_X F)^H - \dfrac{1}{2}(R(X,F(u))u)^V,$
2. $\widetilde{\nabla}_{X^H}F^V = (\nabla_X F)^V + \dfrac{1}{2f}X(f)F^V + \dfrac{1}{2}(R(u,F(u))X)^H,$
3. $\widetilde{\nabla}_{X^V}F^H = (F(X))^H + \dfrac{1}{2f}g(F(u),grad\, f)X^V + \dfrac{1}{2}(R(u,X)F(u))^H,$
4. $\widetilde{\nabla}_{X^V}F^V = (F(X))^V - \dfrac{1}{2f}\tilde{g}(X^V,F^V)(grad\, f)^H + \dfrac{\delta^2}{\lambda}g(\varphi X,F(u))(\varphi U)^V,$

for all vector fields $X \in \Im^1_0(M)$, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M^{2m},φ,g).

Proof. The results comes directly from Theorem 3.1.

4 Curvatures of vertical rescaled berger deformation metric

We shall calculate the Riemannian curvature tensor $\widetilde{R}$ of TM with the vertical rescaled berger deformation metric $\tilde{g}$. This curvature tensor is characterized by the formula

$$\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}, \tag{4.1}$$

for all $\widetilde{X},\widetilde{Y},\widetilde{Z} \in \Im^1_0(TM)$.

Theorem 4.1 *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\tilde{g})$ its tangent bundle equipped with the vertical rescaled berger deformation metric, then we have the following formulas*

$$\begin{aligned}
\widetilde{R}(X^H,Y^H)Z^H &= (R(X,Y)Z)^H+\frac{f}{2}(R(u,R(X,Y)u)Z)^H\\
&+\frac{f}{4}(R(u,R(X,Z)u)Y)^H-\frac{f}{4}(R(u,R(Y,Z)u)X)^H\\
&+\frac{1}{2}((\nabla_Z R)(X,Y)u)^V+\frac{1}{2f}Z(f)(R(X,Y)u)^V\\
&+\frac{1}{4f}Y(f)(R(X,Z)u)^V-\frac{1}{4f}X(f)(R(Y,Z)u)^V, \qquad (4.2)\\
\widetilde{R}(X^H,Y^V)Z^H &= \frac{1}{2}X(f)(R(u,Y)Z)^H+\frac{1}{4}Z(f)(R(u,Y)X)^H\\
&+\frac{f}{2}((\nabla_X R)(u,Y)Z)^H-\frac{1}{4}g(R(X,Z)u,Y)(grad\, f)^H\\
&+\frac{1}{2}(R(X,Z)Y)^V-\frac{f}{4}(R(X,R(u,Y)Z)u)^V\\
&+\Big[\frac{1}{2f}g(\nabla_X grad\, f,Z)-\frac{1}{4f^2}X(f)Z(f)\Big]Y^V\\
&+\frac{\delta^2}{2\lambda}g(R(X,Z)u,\varphi Y)(\varphi U)^V, \qquad (4.3)\\
\widetilde{R}(X^H,Y^H)Z^V &= \frac{f}{2}((\nabla_X R)(u,Z)Y)^H-\frac{f}{2}((\nabla_Y R)(u,Z)X)^H\\
&+\frac{1}{4}X(f)(R(u,Z)Y)^H-\frac{1}{4}Y(f)(R(u,Z)X)^H\\
&-\frac{1}{2}g(R(X,Y)u,Z)(grad\, f)^H+(R(X,Y)Z)^V\\
&-\frac{f}{4}(R(X,R(u,Z)Y)u)^V+\frac{f}{4}(R(Y,R(u,Z)X)u)^V\\
&+\frac{\delta^2}{\lambda}g(R(X,Y)u,\varphi Z)(\varphi U)^V, \qquad (4.4)\\
\widetilde{R}(X^H,Y^V)Z^V &= \frac{1}{4f}X(f)[g(Y,Z)+\delta^2 g(Y,\varphi u)g(Z,\varphi u)](grad\, f)^H\\
&-\frac{1}{2}[g(Y,Z)+\delta^2 g(Y,\varphi u)g(Z,\varphi u)](\nabla_X grad\, f)^H\\
&-\frac{f}{2}(R(Y,Z)X)^H-\frac{f^2}{4}(R(u,Y)R(u,Z)X)^H\\
&+\frac{1}{4}[g(Y,Z)+\delta^2 g(Y,\varphi u)g(Z,\varphi u)](R(X,grad\, f)u)^V\\
&-\frac{1}{4}g(R(u,Z)X,grad\, f)Y^V, \qquad (4.5)\\
\widetilde{R}(X^V,Y^V)Z^H &= f(R(X,Y)Z)^H\\
&+\frac{f^2}{4}[(R(u,X)R(u,Y)Z)^H-(R(u,Y)R(u,X)Z)^H]\\
&+\frac{1}{4}[g(R(u,Y)Z,grad\, f)X^V-g(R(u,X)Z,grad\, f)Y^V], \qquad (4.6)
\end{aligned}$$

$$\begin{aligned}\widetilde{R}(X^V,Y^V)Z^V =& \frac{\delta^2}{2}\big[g(X,\varphi Z)g(Y,\varphi u)-g(Y,\varphi Z)g(X,\varphi u)\big](grad\,f)^H\\ &-\frac{1}{4}\big[\tilde{g}(Y^V,Z^V)(R(u,X)grad\,f)^H-\tilde{g}(X^V,Z^V)(R(u,Y)grad\,f)^H\big]\\ &-\frac{1}{4f^2}\|grad\,f\|^2\big[\tilde{g}(Y^V,Z^V)X^V-\tilde{g}(X^V,Z^V)Y^V\big]\\ &+\frac{\delta^2}{\lambda}\big[g(Y,\varphi Z)(\varphi X)^V-g(X,\varphi Z)(\varphi Y)^V\big]\\ &+\frac{\delta^4}{\lambda^2}\big[g(Y,u)g(X,\varphi Z)-g(X,u)g(Y,\varphi Z)\big](\varphi U)^V, \qquad (4.7)\end{aligned}$$

for all $X,Y,Z\in\Im^1_0(M)$, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M^{2m},φ,g).

Proof. Let $X,Y,Z\in\Im^1_0(M)$, by applying Theorem 3.1 and Lemma 3.4, Proposition 3.1 we have

$1)\widetilde{R}(X^H,Y^H)Z^H=\widetilde{\nabla}_{X^H}\widetilde{\nabla}_{Y^H}Z^H-\widetilde{\nabla}_{Y^H}\widetilde{\nabla}_{X^H}Z^H-\widetilde{\nabla}_{[X^H,Y^H]}Z^H$

$i)$ Let $F:TM\to TM$ be the bundle endomorphism given by $F(u)=R(Y,Z)u$ and $F^V=(R(Y,Z)u)^V$, from direct calculation we get:

$$\begin{aligned}\widetilde{\nabla}_{X^H}\widetilde{\nabla}_{Y^H}Z^H &= \widetilde{\nabla}_{X^H}\big[(\nabla_YZ)^H-\frac{1}{2}F^V\big]\\ &=(\nabla_X\nabla_YZ)^H-\frac{1}{2}(R(X,\nabla_YZ)u)^V-\frac{f}{4}(R(u,R(Y,Z)u)X)^H\\ &\quad-\frac{1}{2}(\nabla_X(R(Y,Z)u)-R(Y,Z)(\nabla_XU))^V-\frac{1}{4f}X(f)(R(Y,Z)u)^V.\end{aligned}$$

$ii)$ With permutation of X by Y in the formula of $\widetilde{\nabla}_{X^H}\widetilde{\nabla}_{Y^H}Z^H$ we get

$$\begin{aligned}\widetilde{\nabla}_{Y^H}\widetilde{\nabla}_{X^H}Z^H &=(\nabla_Y\nabla_XZ)^H-\frac{1}{2}(R(Y,\nabla_XZ)u)^V-\frac{f}{4}(R(u,R(X,Z)u)Y)^H\\ &\quad-\frac{1}{2}(\nabla_Y(R(X,Z)u)-R(X,Z)(\nabla_YU))^V-\frac{1}{4f}Y(f)(R(X,Z)u)^V.\end{aligned}$$

$iii)$ From direct calculation we get

$$\begin{aligned}\widetilde{\nabla}_{[X^H,Y^H]}Z^H &=\widetilde{\nabla}_{[X,Y]^H}Z^H-\widetilde{\nabla}_{(R(X,Y)u)^V}Z^H\\ &=(\nabla_{[X,Y]}Z)^H-\frac{1}{2}(R([X,Y],Z)u)^V-\frac{f}{2}(R(u,R(X,Y)u)Z)^H\\ &\quad-\frac{1}{2f}Z(f)(R(X,Y)u)^V.\end{aligned}$$

Using the second Bianchi identity, we obtain the formula (4.2).
$2)\widetilde{R}(X^H,Y^V)Z^H=\widetilde{\nabla}_{X^H}\widetilde{\nabla}_{Y^V}Z^H-\widetilde{\nabla}_{Y^V}\widetilde{\nabla}_{X^H}Z^H-\widetilde{\nabla}_{[X^H,Y^V]}Z^H$
$i)$ Let $F:TM\to TM$ be the bundle endomorphism given by $F(u)=R(u,Y)Z$ and

$F^H = (R(u,Y)Z)^H$, from direct calculation we get

$$\begin{aligned}
\widetilde{\nabla}_{X^H}\widetilde{\nabla}_{Y^V}Z^H &= \widetilde{\nabla}_{X^H}\big[\frac{1}{2f}Z(f)Y^V + \frac{f}{2}F^H\big] \\
&= \big[\frac{-1}{4f^2}X(f)Z(f) + \frac{1}{2f}X(Z(f))\big]Y^V + \frac{1}{2f}Z(f)(\nabla_X Y)^V \\
&\quad + \frac{1}{4}Z(f)(R(u,Y)X)^H + \frac{1}{2}X(f)(R(u,Y)Z)^H \\
&\quad + \frac{f}{2}(\nabla_X(R(u,Y)Z) - R(\nabla_X U, Y)Z)^H - \frac{f}{4}(R(X,R(u,Y)Z)u)^V.
\end{aligned}$$

$ii)$ Let $F : TM \to TM$ be the bundle endomorphism given by $F(u) = R(X,Z)u$, and $F^V = (R(X,Z)u)^V$, from direct calculation we get

$$\begin{aligned}
\widetilde{\nabla}_{Y^V}\widetilde{\nabla}_{X^H}Z^H &= \widetilde{\nabla}_{Y^V}\big[(\nabla_X Z)^H - \frac{1}{2}F^V\big] \\
&= \frac{1}{2f}(\nabla_X Z)(f)Y^V + \frac{f}{2}(R(u,Y)\nabla_X Z)^H - \frac{1}{2}(R(X,Z)Y)^V \\
&\quad + \frac{1}{4}g(R(X,Z)u,Y)(grad\, f)^H - \frac{\delta^2}{2\lambda}g(R(X,Z)u,\varphi Y)(\varphi U)^V.
\end{aligned}$$

$iii)$ From direct calculation we get:

$$\widetilde{\nabla}_{[X^H,Y^V]}Z^V = \frac{1}{2f}Z(f)(\nabla_X Y)^V + \frac{f}{2}(R(u,\nabla_X Y)Z)^H,$$

which gives the formula (4.3).
3) Applying formula (4.3) and 1^{st} Bianchi identity.

$$\widetilde{R}(X^H,Y^H)Z^V = \widetilde{R}(X^H,Z^V)Y^H - \widetilde{R}(Y^H,Z^V)X^H,$$

we get

$$\begin{aligned}
\widetilde{R}(X^H,Z^V)Y^H &= \frac{1}{2}X(f)(R(u,Z)Y)^H + \frac{1}{4}Y(f)(R(u,Z)X)^H \\
&\quad + \frac{f}{2}((\nabla_X R)(u,Z)Y)^H - \frac{1}{4}g(R(X,Y)u,Z)(grad\, f)^H \\
&\quad + \frac{1}{2}(R(X,Y)Z)^V - \frac{f}{4}(R(X,R(u,Z)Y)u)^V \\
&\quad + \Big[\frac{1}{2f}g(\nabla_X grad\, f, Y) - \frac{1}{4f^2}X(f)Y(f)\Big]Z^V \\
&\quad + \frac{\delta^2}{2\lambda}g(R(X,Y)u,\varphi Z)(\varphi U)^V.
\end{aligned}$$

and

$$
\begin{aligned}
\widetilde{R}(Y^H, Z^V)X^H = & \frac{1}{2}Y(f)(R(u,Z)X)^H + \frac{1}{4}X(f)(R(u,Z)Y)^H \\
& + \frac{f}{2}((\nabla_Y R)(u,Z)X)^H - \frac{1}{4}g(R(Y,X)u,Z)(grad\, f)^H \\
& + \frac{1}{2}(R(Y,X)Z)^V - \frac{f}{4}(R(Y,R(u,Z)X)u)^V \\
& + \Big[\frac{1}{2f}g(\nabla_Y grad\, f, X) - \frac{1}{4f^2}Y(f)X(f)\Big]Z^V \\
& + \frac{\delta^2}{2\lambda}g(R(Y,X)u,\varphi Z)(\varphi U)^V.
\end{aligned}
$$

which gives the formula (4.4).
The other formulas are obtained by a similar calculation.

Now, we consider the sectional curvature $\widetilde{K}$ on $(TM,\tilde{g})$ for P is given by

$$
\widetilde{K}(\widetilde{X},\widetilde{Y}) = \frac{\tilde{g}(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Y},\widetilde{Y})}{\tilde{g}(\widetilde{X},\widetilde{X})\tilde{g}(\widetilde{Y},\widetilde{Y}) - \tilde{g}(\widetilde{X},\widetilde{Y})^2}, \tag{4.8}
$$

where $P = P(\widetilde{X},\widetilde{Y})$ denotes the plane spanned by $\{\widetilde{X},\widetilde{Y}\}$, for all, linearly independent vector fields $\widetilde{X},\widetilde{Y} \in \Im^1_0(TM)$.

Let $\widetilde{K}(X^H,Y^H)$, $\widetilde{K}(X^H,Y^V)$ and $\widetilde{K}(X^V,Y^V)$ denote the sectional curvature of the plane spanned by $\{X^H,Y^H\}$, $\{X^H,Y^V\}$ and $\{X^V,Y^V\}$ on $(TM,\tilde{g})$ respectively, where X,Y are orthonormal vector fields on M.

Proposition 4.1 *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\tilde{g})$ its tangent bundle equipped with the vertical rescaled berger deformation metric. Then we have the following*

$$
\begin{aligned}
& i)\ \tilde{g}(\widetilde{R}(X^H,Y^H)Y^H,X^H) = g(R(X,Y)Y,X) - \frac{3f}{4}\|R(X,Y)u\|^2, \\
& ii)\ \tilde{g}(\widetilde{R}(X^H,Y^V)Y^V,X^H) = \big(1+\delta^2 g(Y,\varphi u)^2\big)\big(\frac{X(f)^2}{4f} - \frac{1}{2}g(\nabla_X grad\, f, X)\big) \\
& \qquad + \frac{f^2}{4}\|R(u,Y)X\|^2, \\
& iii)\ \tilde{g}(\widetilde{R}(X^V,Y^V)Y^V,X^V) = -\frac{\|grad\, f\|^2}{4}\big(1+\delta^2\big(g(X,\varphi u)^2 + g(Y,\varphi u)^2\big)\big) \\
& \qquad + \frac{f\delta^2}{\lambda}\big(g(X,\varphi X)g(Y,\varphi Y) - g(X,\varphi Y)^2\big).
\end{aligned}
$$

Proof. $i)$ From the formula (4.2), we have

$$
\begin{aligned}
\tilde{g}(\widetilde{R}(X^H,Y^H)Y^H,X^H) &= g(R(X,Y)Y,X) + \frac{f}{2}g(R(u,R(X,Y)u)Y,X) \\
& \quad + \frac{f}{4}g(R(u,R(Y,X)u)Y,X) \\
&= g(R(X,Y)Y,X) - \frac{3f}{4}\|R(X,Y)u\|^2.
\end{aligned}
$$

$ii)$ From the formula (4.5), we have

$$\begin{aligned}\tilde{g}(\widetilde{R}(X^H,Y^V)Y^V,X^H) &= -\frac{1}{2}\big(1+\delta^2 g(Y,\varphi u)^2\big)g(\nabla_X grad\, f,X)\\ &\quad+\frac{X(f)^2}{4f}\big(1+\delta^2 g(Y,\varphi u)^2\big)+\frac{f^2}{4}\|R(u,Y)X\|^2\\ &= \big(1+\delta^2 g(Y,\varphi u)^2\big)\big(\frac{X(f)^2}{4f}-\frac{1}{2}g(\nabla_X grad\, f,X)\big)\\ &\quad+\frac{f^2}{4}\|R(u,Y)X\|^2.\end{aligned}$$

$iii)$ The result follows immediately from the formula (4.7)

$$\begin{aligned}\tilde{g}(\widetilde{R}(X^V,Y^V)Y^V,X^V) &= -\frac{\|grad\, f\|^2}{4}\big(1+\delta^2\big(g(X,\varphi u)^2+g(Y,\varphi u)^2\big)\big)\\ &\quad+\frac{f\delta^2}{\lambda}g(X,\varphi X)g(Y,\varphi Y)-\frac{f\delta^2}{\lambda}g(X,\varphi Y)^2\\ &= -\frac{\|grad\, f\|^2}{4}\big(1+\delta^2\big(g(X,\varphi u)^2+g(Y,\varphi u)^2\big)\big)\\ &\quad+\frac{f\delta^2}{\lambda}\big(g(X,\varphi X)g(Y,\varphi Y)-g(X,\varphi Y)^2\big).\end{aligned}$$

Theorem 4.2 *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\tilde{g})$ its tangent bundle equipped with the vertical rescaled berger deformation metric, then the sectional curvature $\widetilde{K}$ satisfy the following equations*

$$\begin{aligned}&(1)\ \widetilde{K}(X^H,Y^H)=K(X,Y)-\frac{3f}{4}\|R(X,Y)u\|^2,\\ &(2)\ \widetilde{K}(X^H,Y^V)=\frac{f\|R(u,Y)X\|^2}{4\big(1+\delta^2 g(Y,\varphi u)^2\big)}+\frac{X(f)^2}{4f^2}-\frac{1}{2f}g(\nabla_X grad\, f,X),\\ &(3)\ \widetilde{K}(X^V,Y^V)=\frac{\delta^2\big(g(X,\varphi X)g(Y,\varphi Y)-g(X,\varphi Y)^2\big)}{f\lambda\big(1+\delta^2\big(g(X,\varphi u)^2+g(Y,\varphi u)^2\big)\big)}-\frac{\|grad\, f\|^2}{4f^2}.\end{aligned}$$

where K denote the sectional curvature of (M^{2m},φ,g).

Proof. Using the Proposition 4.1 and direct calculations, we have

$$\begin{aligned}(1)\ \widetilde{K}(X^H,Y^H) &= \frac{\tilde{g}(\widetilde{R}(X^H,Y^H)Y^H,X^H)}{\tilde{g}(X^H,X^H)\tilde{g}(Y^H,Y^H)-\tilde{g}(X^H,Y^H)^2}\\ &= \tilde{g}(\widetilde{R}(X^H,Y^H)Y^H,X^H)\\ &= K(X,Y)-\frac{3f}{4}\|pR(X,Y)\|^2.\end{aligned}$$

$$\begin{aligned}(2)\ \widetilde{K}(X^H,Y^V) &= \frac{\tilde{g}(\widetilde{R}(X^H,Y^V)Y^V,X^H)}{\tilde{g}(X^H,X^H)\tilde{g}(Y^V,Y^V)-\tilde{g}(X^H,Y^V)^2}\\ &= \frac{f\|R(u,Y)X\|^2}{4\big(1+\delta^2 g(Y,\varphi u)^2\big)}+\frac{X(f)^2}{4f^2}-\frac{1}{2f}g(\nabla_X grad\, f,X).\end{aligned}$$

$$(3)\ \widetilde{K}(X^V,Y^V)=\frac{\tilde{g}(\widetilde{R}(X^V,Y^V)Y^V,X^V)}{\tilde{g}(X^V,X^V)\tilde{g}(Y^V,Y^V)-\tilde{g}(X^V,Y^V)^2}$$
$$=\frac{\delta^2\big(g(X,\varphi X)g(Y,\varphi Y)-g(X,\varphi Y)^2\big)}{f\lambda\big(1+\delta^2\big(g(X,\varphi u)^2+g(Y,\varphi u)^2\big)\big)}-\frac{\|grad\,f\|^2}{4f^2}.$$

Proposition 4.2 *Let (M^{2m},φ,g) be an anti-paraKähler manifold of constant sectional curvature κ and $(TM,\tilde{g})$ its tangent bundle equipped with the vertical rescaled berger deformation metric, then the sectional curvature $\widetilde{K}$ satisfy the following equations*

$$(1)\ \widetilde{K}(X^H,Y^H)=\kappa-\frac{3f\kappa^2}{4}\big(g(X,u)^2+g(Y,u)^2\big),$$
$$(2)\ \widetilde{K}(X^H,Y^V)=\frac{f\kappa^2g(X,u)^2}{4\big(1+\delta^2g(Y,\varphi u)^2\big)}+\frac{X(f)^2}{4f^2}-\frac{1}{2f}g(\nabla_X grad\,f,X),$$
$$(3)\ \widetilde{K}(X^V,Y^V)=\frac{\delta^2\big(g(X,\varphi X)g(Y,\varphi Y)-g(X,\varphi Y)^2\big)}{f\lambda\big(1+\delta^2\big(g(X,\varphi u)^2+g(Y,\varphi u)^2\big)\big)}-\frac{\|grad\,f\|^2}{4f^2}.$$

Proof. M has constant curvature κ then for all $U,V,W\in\Im^1_0(M)$,

$$R(U,V)W=\kappa\big(g(V,W)U-g(U,W)V\big),$$

direct calculations we get

$$\|R(X,Y)u\|^2=\kappa^2\big(g(X,u)^2+g(Y,u)^2\big),$$
$$\|R(u,Y)X\|^2=\kappa^2g(X,u)^2,$$

then the result.

Remark 4.1 Let $(x,u)\in TM$ with $u\neq 0$ and $\{E_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M, such that $E_1=\dfrac{u}{\|u\|}$, then

$$\Big\{F_i=E_i^H,F_{2m+1}=\frac{1}{\sqrt{f\lambda}}(\varphi E_1)^V,F_{2m+j}=\frac{1}{\sqrt{f}}(\varphi E_j)^V\Big\}_{i=\overline{1,2m},j=\overline{2,2m}} \tag{4.9}$$

is a local orthonormal frame on TM.

Lemma 4.1 *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\tilde{g})$ its tangent bundle equipped with the vertical rescaled berger deformation metric and $(F_a)_{a=\overline{1,4m}}$ be a local orthonormal frame on $(TM,\tilde{g})$ defined by (4.9), then the sectional curvatures $\widetilde{K}$ satisfy the following equations*

$$\widetilde{K}(F_i,F_j)=K(E_i,E_j)-\frac{3f}{4}\|R(E_i,E_j)u\|^2,$$
$$\widetilde{K}(F_i,F_{2m+1})=\frac{E_i(f)^2}{4f^2}-\frac{1}{2f}g(\nabla_{E_i}grad\,f,E_i),$$
$$\widetilde{K}(F_i,F_{2m+l})=\frac{f}{4}\|R(u,E_l)E_i\|^2+\frac{E_i(f)^2}{4f^2}-\frac{1}{2f}g(\nabla_{E_i}grad\,f,E_i),$$
$$\widetilde{K}(F_{2m+k},F_{2m+1})=\frac{\delta^2\big(g(E_k,\varphi E_k)g(E_1,\varphi E_1)-g(E_k,\varphi E_1)^2\big)}{f\lambda^2}-\frac{\|grad\,f\|^2}{4f^2},$$
$$\widetilde{K}(F_{2m+k},F_{2m+l})=\frac{\delta^2\big(g(E_k,\varphi E_k)g(E_l,\varphi E_l)-g(E_k,\varphi E_l)^2\big)}{f\lambda}-\frac{\|grad\,f\|^2}{4f^2},$$

for $i,j=\overline{1,2m}$ and $k,l=\overline{2,2m}$, where K is a sectional curvature of (M^{2m},φ,g).

Proof. The results comes directly from Theorem 4.2 and formula (4.9).

We now consider the scalar curvature $\widetilde{\sigma}$ of $(TM, \tilde{g})$, with standard calculations we have the following result.

Theorem 4.3 *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled berger deformation metric. If σ (resp., $\widetilde{\sigma}$) denote the scalar curvature of (M^{2m}, φ, g) (resp., $(TM, \tilde{g})$), then we have*

$$\begin{aligned}\widetilde{\sigma} &= \sigma - \frac{f}{4}\sum_{i,j=1}^{2m}\|R(E_i,E_j)u\|^2 - \frac{2m(2m-3)}{4f^2}\|grad\, f\|^2 - \frac{2m}{f}\Delta(f)\\ &\quad + \frac{2\delta^2}{f\lambda^2}\Big(g(E_1,\varphi E_1)A - 1 + g(E_1,\varphi E_1)^2\Big) + \frac{\delta^2}{f\lambda}\Big(A^2 - 2m + 2 - g(E_1,\varphi E_1)^2\Big).\end{aligned}$$

where $A = \sum_{i=2}^{2m} g(E_i, \varphi E_i)$, $(E_i)_{i=\overline{1,2m}}$ be a local orthonormal frame on M and $\Delta(f)$ is the Laplacian of f.

Proof. Let $(F_a)_{a=\overline{1,4m}}$ be a local orthonormal frame on $(TM, \tilde{g})$ defined by (4.9). Using Theorem 4.2 and definition of scalar curvature, we have.

$$\begin{aligned}\widetilde{\sigma} &= \sum_{\substack{i,j=1\\ i\neq j}}^{2m}\widetilde{K}(F_i,F_j) + 2\sum_{i,j=1}^{2m}\widetilde{K}(F_i,F_{2m+j}) + \sum_{\substack{i,j=1\\ i\neq j}}^{2m}\widetilde{K}(F_{2m+i},F_{2m+j})\\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{2m}\widetilde{K}(F_i,F_j) + 2\sum_{i=1}^{2m}\widetilde{K}(F_i,F_{2m+1}) + 2\sum_{i=1,j=2}^{2m}\widetilde{K}(F_i,F_{2m+j})\\
&\quad +2\sum_{i=2}^{2m}\widetilde{K}(F_{2m+i},F_{2m+1}) + \sum_{\substack{i,j=2\\ i\neq j}}^{2m}\widetilde{K}(F_{2m+i},F_{2m+j})\\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{2m}\Big(K(E_i,E_j) - \frac{3f}{4}\|R(E_i,E_j)u\|^2\Big)\\
&\quad +2\sum_{i=1}^{2m}\Big(\frac{E_i(f)^2}{4f^2} - \frac{1}{2f}g(\nabla_{E_i}grad\, f, E_i)\Big)\\
&\quad +2\sum_{i=1,j=2}^{2m}\Big(\frac{f}{4}\|R(u,E_j)E_i\|^2 + \frac{E_i(f)^2}{4f^2} - \frac{1}{2f}g(\nabla_{E_i}grad\, f, E_i)\Big)\\
&\quad +2\sum_{i=2}^{2m}\Big(\frac{\delta^2\big(g(E_i,\varphi E_i)g(E_1,\varphi E_1) - g(E_i,\varphi E_1)^2\big)}{f\lambda^2} - \frac{\|grad\, f\|^2}{4f^2}\Big)\\
&\quad +\sum_{\substack{i,j=2\\ i\neq j}}^{2m}\Big(\frac{\delta^2\big(g(E_i,\varphi E_i)g(E_j,\varphi E_j) - g(E_i,\varphi E_j)^2\big)}{f\lambda} - \frac{\|grad\, f\|^2}{4f^2}\Big)\end{aligned}$$

$$
\begin{aligned}
\widetilde{\sigma} &= \sigma - \frac{3f}{4}\sum_{\substack{i,j=1\\ i\neq j}}^{2m}\|R(E_i,E_j)u\|^2 + \frac{\|grad\,f\|^2}{2f^2} - \frac{\Delta(f)}{f}\\
&+\frac{2f}{4}\sum_{i,j=1}^{2m}\|R(u,E_j)E_i\|^2 + \frac{(2m-1)\|grad\,f\|^2}{2f^2} - \frac{(2m-1)\Delta(f)}{f}\\
&+\frac{2\delta^2}{f\lambda^2}g(E_1,\varphi E_1)\sum_{i=2}^{2m}g(E_i,\varphi E_i) - \frac{2\delta^2}{f\lambda^2}\sum_{i=2}^{n}g(E_i,\varphi E_1)^2\\
&-\frac{2(2m-1)\|grad\,f\|^2}{4f^2} + \frac{\delta^2}{f\lambda}\sum_{i=2}^{2m}\sum_{j=2}^{2m}g(E_i,\varphi E_i)g(E_j,\varphi E_j)\\
&-\frac{\delta^2}{f\lambda}\sum_{i=2}^{2m}\sum_{j=2}^{2m}g(E_i,\varphi E_j)^2 - \frac{(2m-1)(2m-2)\|grad\,f\|^2}{4f^2}.
\end{aligned}
$$

In order to simplify this last expression, we put $A = \sum_{i=2}^{2m} g(E_i,\varphi E_i)$,

and we have $\sum_{i,j=1}^{m}\|R(\tilde{p},E_j)E_i\|^2 = \sum_{i,j=1}^{m}\|R(E_i,E_j)\tilde{p}\|^2$, see [5, 15], then

$$
\begin{aligned}
\widetilde{\sigma} &= \sigma - \frac{f}{4}\sum_{i,j=1}^{2m}\|R(E_i,E_j)u\|^2 - \frac{2m(2m-3)}{4f^2}\|grad\,f\|^2 - \frac{2m}{f}\Delta(f)\\
&+\frac{2\delta^2}{f\lambda^2}\Big(g(E_1,\varphi E_1)A - 1 + g(E_1,\varphi E_1)^2\Big) + \frac{\delta^2}{f\lambda}\Big(A^2 - 2m + 2 - g(E_1,\varphi E_1)^2\Big).
\end{aligned}
$$

Proposition 4.3 *Let (M^{2m},φ,g) be an anti-paraKähler manifold of constant sectional curvature κ and $(TM,\tilde{g})$ its tangent bundle equipped with the vertical rescaled berger deformation metric. If $\widetilde{\sigma}$ denote the scalar curvature of $(TM,\tilde{g})$, then we have*

$$
\begin{aligned}
\widetilde{\sigma} &= (2m-1)\kappa\Big(2m - \frac{\kappa f r^2}{2}\Big) - \frac{2m(2m-3)}{4f^2}\|grad\,f\|^2 - \frac{2m}{f}\Delta(f)\\
&+\frac{2\delta^2}{f\lambda^2}\Big(g(E_1,\varphi E_1)A - 1 + g(E_1,\varphi E_1)^2\Big)\\
&+\frac{\delta^2}{f\lambda}\Big(A^2 - 2m + 2 - g(E_1,\varphi E_1)^2\Big).
\end{aligned}
$$

Proof. Taking account that $\sigma = 2m(2m-1)\kappa$ and for all $X,Y,Z \in \Im^1_0(M)$,

$$R(X,Y)Z = \kappa\big(g(Y,Z)X - g(X,Z)Y\big),$$

then we obtain

$$\sum_{i,j=1}^{m}\|R(E_i,E_j)u\|^2 = 2\kappa^2(2m-1)r^2.$$

This completes the proof.

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